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# RANDOM GEOMETRIC GRAPHS AND ISOMETRIES OF NORMED SPACES

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ABSTRACT. Given a countable dense subset $S$ of a finite-dimensional normed space $X$, and $0 < p < 1$, we form a random graph on $S$ by joining, independently and with probability $p$, each pair of points at distance less than 1. We say that $S$ is *Rado* if any two such random graphs are (almost surely) isomorphic.

Bonato and Janssen showed that in $\ell_\infty^d$ almost all $S$ are Rado. Our main aim in this paper is to show that $\ell_\infty^d$ is the unique normed space with this property: indeed, in every other space almost all sets $S$ are non-Rado. We also determine which spaces admit some Rado set: this turns out to be the spaces that have an $\ell_\infty$ direct summand. These results answer questions of Bonato and Janssen.

A key role is played by the determination of which finite-dimensional normed spaces have the property that every bijective step-isometry (meaning that the integer part of distances is preserved) is in fact an isometry. This result may be of independent interest.

## 1. INTRODUCTION

In [2] Bonato and Janssen introduced a new random geometric graph model, defined as follows. Let $V$ be a finite-dimensional normed space, and let $S$ be a fixed countable dense subset of $V$. Let $\widehat{G} = \widehat{G}(V, S)$ be the unit radius graph on $S$; that is, $x, y \in S$ are joined if $\|x - y\| < 1$. Form $G = G_p(V, S)$ by taking a random subgraph of $\widehat{G}(V, S)$ in which each edge is chosen independently with probability $p$, and let $\mathcal{G}_p(V, S)$ be the probability space of such graphs.

Motivated by the existence of the Rado graph, the unique infinite graph in the Erdős–Rényi random graph model, Bonato and Janssen asked when the random graph in their model is almost surely unique up to isomorphism. We say a set $S$ is *Rado* if the resulting graph is almost surely unique up to isomorphism, and we say it is *strongly non-Rado* if any two such graphs are almost surely not isomorphic. (Rather surprisingly, there are sets that are neither Rado nor strongly non-Rado; see Theorem 2 below.)

Bonato and Janssen proved that, for $V = \ell_\infty^d$ (the normed space on $\mathbb{R}^d$ with norm defined by $\|(x_1, x_2, \ldots, x_d)\| = \max_i |x_i|$), almost all countable dense sets are Rado.

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(The exact definition of 'almost all' for countable dense sets is a little subtle and we discuss it at the end of section 2, but, for now, we remark that the only property of almost all that we require is that almost all sets contain no integer distances, and no integer distances (or coincidences) in projections on to natural subspaces, such as the coordinate axes.)

In the same paper, Bonato and Janssen proved that all countable dense sets in the Euclidean plane are strongly non-Rado. Subsequently in [3], they showed that almost all countable dense sets in the plane with the hexagonal norm are strongly non-Rado, and in [4] they showed that, for $\mathbb{R}^2$ with any norm that is strictly convex or has a polygonal unit ball (apart from a parallelogram), there are no Rado sets. They asked which normed spaces contain a Rado set.

Our first result implies that $\ell_\infty^d$ is the only space for which almost all countable dense sets are Rado.

**Theorem 1.** *Let $V$ be a finite-dimensional normed space not isometric to $\ell_\infty^d$. Then, for any $0 < p < 1$, almost every countable dense set $S$ is strongly non-Rado.*

Theorem 1 shows what happens for 'typical' countable dense sets $S$, but leaves open the possibility of exceptional cases. Our second result, Theorem 2 below, is a refinement of Theorem 1 that answers the question of Bonato and Janssen, and, in fact, describes the precise situation in each normed space.

Before stating the theorem, we need the following fact about finite-dimensional normed spaces, which roughly says that any such space contains a unique maximal $\ell_\infty^d$ subspace embedded in an $\ell_\infty$ fashion. The precise statement (Proposition 14) is that, for any finite-dimensional normed space $V$, there exists a unique maximal subspace $W$ isometric to $\ell_\infty^d$ for some $d$, such that there is a subspace $U$ with $V = U \oplus W$ and $\|u + w\| = \max(\|u\|, \|w\|)$ for all $u \in U$ and $w \in W$. We prove this result in section 3. This decomposition is useful since, in essence, the complicated behaviour can only occur on the $\ell_\infty^d$ part. We call this decomposition the *$\ell_\infty$-decomposition* and write it as $V = (U \oplus \ell_\infty^d)_\infty$.

We are now ready to state the main result of the paper.

**Theorem 2.** *Let $V$ be a normed space with $\ell_\infty$-decomposition $(U \oplus \ell_\infty^d)_\infty$ as above, and let $0 < p < 1$. Then*

- (i) *If $V = \ell_\infty^d$ (equivalently, if $U$ is trivial), then almost all countable dense sets $S$ are Rado, but there exist countable dense sets which are strongly non-Rado. Additionally, there exist countable dense sets $S$ for which the probability that two graphs $G, G' \in \mathcal{G}_p(V, S)$ are isomorphic lies strictly between $0$ and $1$.*
- (ii) *If $d = 0$ (that is, if $V = U$), then all countable dense sets $S$ are strongly non-Rado.*
- (iii) *If $d > 0$ and $U \neq \{0\}$, then almost all countable dense sets $S$ are strongly non-Rado, but there exist countable dense sets $S$ which are Rado. Additionally, there exist countable dense sets for which the probability that two graphs $G, G' \in \mathcal{G}_p(V, S)$ are isomorphic lies strictly between $0$ and $1$.*

As we mentioned above, the typical case in (i) was proved by Bonato and Janssen. In fact they proved more: they showed that the graph is independent of $S$. More precisely, they showed that for almost all countable dense sets $S$ and $S'$ and for any $p, p' \in (0, 1)$, two graphs $G \in \mathcal{G}_p(\ell_\infty^d, S)$ and $G' \in \mathcal{G}_{p'}(\ell_\infty^d, S')$ are almost surely isomorphic. Of course, Theorem 2 shows that this does not hold for other normed spaces, as parts (ii) and (iii) show that, for almost all sets $S$, the probability that $G$ is isomorphic to any particular graph is zero.

We shall make use of a key lemma of Bonato and Janssen that shows that any graph isomorphism must induce an approximate isometric action on $S$.

**Definition.** Let $A \subseteq V$. A *step-isometry* on $A$ is a bijective function $f\colon A \to A$ such that, for all $x, y \in A$,

$$\lfloor \|x-y\| \rfloor = \lfloor \|f(x)-f(y)\| \rfloor .$$

We remark that Bonato and Janssen's definition was slightly different: they did not require the function to be a bijection. However, all our maps will be bijective, and many of the results we state only hold for bijective step-isometries, so we use the above definition. Note that we use 'isometry' to mean any distance-preserving map; in particular, it need not be surjective.

Bonato and Janssen [2] proved the following lemma.

**Lemma 3** (Bonato and Janssen [2])**.** *Suppose $G \in \mathcal{G}_p(V,S)$. Then, almost surely, for every pair of points $x, y \in S$ and every $k \in \mathbb{N}$ with $k \geqslant 2$ we have $\|x-y\| < k$ if and only if $d_G(x,y) \leqslant k$.*

*In particular, for almost all graphs $G, G'$ in $\mathcal{G}_p(V,S)$, every function $f\colon S \to S$ inducing an isomorphism of the graphs is a step-isometry on $S$.*

To see why this is true, first note that it is immediate that the existence of a path of length $k$ implies that the norm distance is less than $k$. For the converse they use the countable dense property to construct infinitely many edge disjoint paths of length $k$ between $x$ and $y$ in $\widehat{G}$. Each of these has a positive chance of occurring in $G$ so, almost surely, one of them does.

The second part now follows since an isomorphism between any two graphs satisfying the first part must be a step-isometry. (The case of $\|x-y\| < 1$ requires a small additional check.)

This result shows that a natural step towards characterising the possible graph isomorphisms is to characterise all the step-isometries and, indeed, this will form the bulk of this paper. As we shall prove (Proposition 24), any step-isometry of $S$ extends to a step-isometry of $V$ itself. Thus, we want to characterise the step-isometries of $V$.

Observe that a step-isometry on $V$ need not be an isometry. Indeed, consider the following example on $\mathbb{R}$. Let $g\colon [0,1) \to [0,1)$ be any increasing bijection. Now define $f(x) = \lfloor x \rfloor + g(x - \lfloor x \rfloor)$. It is easy to see that this is a step-isometry but not an isometry (unless $g$ is the identity function).

This example extends naturally to $\ell_\infty^d$: we can do the above independently in each coordinate. However, the following result shows this is essentially the only example.

We need one piece of notation first. If $V = U \oplus W$ is a vector space and $f\colon V \to V$, then we say $f$ *factorises over the decomposition* if there exist $f_U\colon U \to U$ and $f_W\colon W \to W$ such that $f(u+w) = f_U(u) + f_W(w)$ for all $u \in U$ and $w \in W$. We write $f = f_U \oplus f_W$.

**Theorem 4.** *Let $V$ be a finite-dimensional normed space with $\ell_\infty$-decomposition $V = (U \oplus \ell_\infty^d)_\infty$, and let $f\colon V \to V$ be a step-isometry. Then $f$ factorises over the decomposition as $f = f_U \oplus f_{\ell_\infty^d}$, where $f_U$ is a bijective isometry of $U$ and $f_{\ell_\infty^d}$ is a step-isometry of $\ell_\infty^d$.*

Thus, to obtain a full characterisation of the step-isometries of $V$, we need to classify the step-isometries of $\ell_\infty^d$. The following result does exactly that.

**Theorem 5.** *Let $f$ be a step-isometry of $\ell_\infty^d$. Then there exists a permutation $\sigma$ of $[d]$, and $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d) \in \{-1, +1\}^d$, and, for each $i$, an increasing bijection $g_i\colon [0,1) \to [0,1)$, such that*

$$f\left(\sum_{i=1}^{d} \lambda_i e_i\right) - f(0) = \sum_{i=1}^{d} \left(g_i\left(\lambda_i - \lfloor \lambda_i \rfloor\right) + \lfloor \lambda_i \rfloor\right) \varepsilon_i e_{\sigma(i)},$$

*where $e_1, e_2, \dots, e_d$ is the standard basis of $\ell_\infty^d$.*

Having established these two theorems, as we shall see, it is relatively straightforward to prove Theorems 1 and 2.

The layout of this paper is as follows. In the next section we introduce some standard definitions and notation, and then in section 3 we prove the existence and uniqueness of the $\ell_\infty$-decomposition together with some simple facts about it that will be useful later. In section 4 we prove that any step-isometry on a dense subset can be extended to a step-isometry on the whole space.

In sections 5–11 we prove Theorems 4 and 5. The proofs of these are quite lengthy, and we break them down as follows. Sections 5 and 6 show that any step-isometry is an isometry on the set of finite sums of extreme points of the unit ball of $V$ and that we can compose the step-isometry with an isometry so that the combination fixes all these finite sums. Then sections 7 and 8 show that any step-isometry that fixes these finite sums actually preserves many directions and that this implies it must fix a particular subspace. Finally, section 9 shows that this particular subspace is the non-$\ell_\infty$-component of the $\ell_\infty$-decomposition, and sections 10 and 11 put these facts together to complete the proofs of Theorems 4 and 5.

Parts of the proof of Theorem 2 rely on the back and forth method; as we use this several times, we abstract it out into section 12. Then in section 13 we use Theorem 4 to prove Theorem 2. We conclude with a brief discussion of some other exceptional cases and some open problems.

Throughout the paper we use standard results and notation from graph theory (see, e.g., [1] or [6]) and functional analysis (see, e.g., [8]).

## 2. Normed space preliminaries

Throughout this paper we will be working exclusively in finite-dimensional normed spaces, and we shall frequently make use of properties particular to such spaces, such as the compactness of the unit ball and the fact that a linear injection from the space to itself is necessarily a bijection.

Before stating any of the results that we need, we introduce some very basic notation. Given a normed space $V$, we write $B(x, r)$ for the closed ball of radius $r$ about $x$ and, on the few occasions we need it, $B^\circ(x, r)$ for the open ball.

In many cases the normed space will decompose naturally into subspaces, $V = U \oplus W$. Given a vector $v = u + w$, with $u \in U$ and $w \in W$, we call $u$ the *$U$-component of $v$*. In most cases we use the 'additive' notation $u + w$ for vectors, and $V = U \oplus W$ for subspaces. However, in some cases it will be easier to think of a vector $v \in V$ as the ordered pair $(u, w)$ and the space as $V = U \times W$, and we will occasionally use this alternative notation.

Much of our work will be on (not necessarily linear) functions mapping the vector space $V$ to itself. One key tool that we shall use several times is the Mazur–Ulam Theorem (see, e.g., [7]). This states that any isometry is 'affine'; that is, a translation of a linear map. More formally:

**Theorem 6** (Mazur–Ulam Theorem [9])**.** *Let $X$ and $Y$ be normed spaces, and let $f\colon X \to Y$ be a surjective isometry. Then the map $\hat{f}\colon X \to Y$, given by $\hat{f}(x) = f(x) - f(0)$, is linear.*

Since we are concerned only with finite-dimensional normed spaces in this paper, it is worth noting the following folklore result which shows that the Mazur–Ulam Theorem has a particularly simple form in this setting.

**Corollary 7.** *Suppose $V$ is a finite-dimensional normed space and that $f\colon V \to V$ is an isometry. Then $f$ is an affine bijection.*

*Proof.* By the Mazur–Ulam Theorem it suffices to show that $f$ is surjective. First, observe that, by translating $f$ if necessary, we way assume that $f(0) = 0$.

We claim that $f(V)$ is closed. Indeed, if a sequence $f(x_n)$ tends to $y$, then $f(x_n)$ is Cauchy. This implies that, since $f$ is an isometry, the sequence $(x_n)$ is Cauchy, and thus converges to some point, $x$ say. But then $f(x) = y$, which completes the proof of the claim.

Now, suppose, for a contradiction, that there is some point $x \notin f(V)$. By the claim, $f(V)$ is closed, so there exists $\varepsilon > 0$ such that the open ball $B^\circ(x, \varepsilon)$ is disjoint from $f(V)$.

Trivially, this implies that, for any $n \geqslant 1$, $f^n(x) \notin B^\circ(x, \varepsilon)$ or, equivalently, that $\|f^n(x) - x\| > \varepsilon$. Since $f$ is an isometry, this shows that, for any $n > m \geqslant 1$, we have $\|f^n(x) - f^m(x)\| = \|f^{n-m}(x) - x\| > \varepsilon$. Therefore, the sequence $x, f(x), f^2(x) \cdots$ is $\varepsilon$-separated. But these terms all have norm $\|x\|$ (since $f(0) = 0$), so this contradicts the compactness of the closed ball $B(0, \|x\|)$. ☐

Much of our work will concern properties of the closed unit ball $B = B(0, 1)$, and we recall some simple facts and notation related to $B$.

The ball $B$ is a convex compact set, and the norm is determined by $B$. An *extreme point* of $B$ is a point $x$ such that if $y, z \in B$ and $x$ is a convex combination of $y, z$, then $y = z = x$ (for general background on extreme points, see, e.g., [11]). We write $\mathrm{Ext}(B)$ for the extreme points of $B$. The set $B$ is the convex hull of its extreme points; that is, $\mathrm{conv}(\mathrm{Ext}(B)) = B$. Since $B$ is not contained in any proper subspace, we see that the vectors in $\mathrm{Ext}(B)$ span all of $V$. For any set of vectors $A$ we use $\langle A \rangle$ to denote the span of the vectors in $A$.

It will be useful to work with finite sums of extreme points. Thus, we let $\Lambda$ be the 'lattice' generated by the extreme points of the unit ball $B$; that is, all points of the form $\sum_i \lambda_i x_i$ with $\lambda_i \in \mathbb{Z}$ and $x_i \in \mathrm{Ext}(B)$. Note that $\Lambda$ need not be discrete.

We start with a simple lemma that shows that $\Lambda$ is not too sparse.

**Lemma 8.** *Let $V$ be a finite-dimensional normed space, and let $v \in V$. Then there exists $x \in \Lambda$ such that $\|x - v\| \leqslant \dim V/2$.*

*Proof.* As noted above, the extreme points of $B$ span $V$, so let $x_1, x_2, \ldots x_d$, where $d = \dim V$, be any minimal spanning set of extreme points of $B$. Note that $\|x_i\| = 1$ for all $i$.

We can write $v = \sum_{i=1}^d a_i x_i$. For each $i$ let $\lambda_i$ be $a_i$ rounded to the nearest integer (so $\lambda_i = \lfloor a_i + 1/2 \rfloor$). Then

$$\|v - x\| = \left\| \sum_{i=1}^{d} (a_i - \lambda_i) x_i \right\| \leqslant \sum_{i=1}^{d} |a_i - \lambda_i| \|x_i\| \leqslant d/2,$$

as claimed. □

We remark that it is easy to see that this bound is obtained for the space $\ell_1^d$ (the space $\mathbb{R}^d$ with norm $\|(x_1, x_2, \ldots, x_d)\| = \sum_{i=1}^d |x_i|$).

Since the set $\Lambda$ need not be discrete, we will often work with its closure $\overline{\Lambda}$ which has a relatively simple form.

**Lemma 9.** *Let $V$ be a $d$-dimensional normed space. Then there is a basis $e_1, e_2, \ldots, e_d$ of unit vectors in $V$ and an $r$ with $0 \leqslant r \leqslant d$ such that*

$$\overline{\Lambda} = \sum_{i \leqslant r} \mathbb{R} e_i \oplus \sum_{i > r} \mathbb{Z} e_i.$$

*Proof.* As remarked above, the extreme points of $B$ span $V$, so $\Lambda$ spans $V$. Hence $\overline{\Lambda}$ is a closed additive subgroup of $V \equiv \mathbb{R}^d$, so it must have the form specified (see, e.g., [5]). □

The following subspace will be important later.

**Definition.** We call the subspace $\sum_{i<r} \mathbb{R} e_i$ in the decomposition given by Lemma 9 the *continuous subspace* of $\overline{\Lambda}$, and we usually denote it $U_0$.

We make the following simple observation for future reference.

**Corollary 10.** *The extreme points of the unit ball $B$ are covered by finitely many cosets of the continuous subspace $U_0$.* □

We conclude this section with a brief discussion of the meaning of 'almost all' for countable dense sets. Before doing this, we remark that, for our purposes, all we need is the following: if $V = (U \oplus \ell_\infty^d)_\infty$, then, for almost all sets $S$, no two points of $S$ have the same $U$-component, nor do they differ by an integer in any coordinate direction in their $\ell_\infty^d$-component. This obviously holds for any sensible definition of 'almost all.'

Indeed, there are several possible definitions in the literature, any of which would be suitable. One such possibility is to take any distribution on $\mathbb{R}^d$ with a strictly positive density function, and let $S$ be the set formed by taking countably many independent samples from it. Another would be to take the union of countably many density-1 Poisson Processes. (There are also rather less intuitive possibilities—for example, taking $S$ to be the set of all local minima of a Brownian motion on $\mathbb{R}^d$.) In fact Tsirelson [12] showed that these all give the same resulting measure. See that paper for a thorough discussion of the whole area.

## 3. The $\ell_\infty$-decomposition

In this section we prove the existence of the $\ell_\infty$-decomposition mentioned in the Introduction.

**Definition.** A unit vector $v$ is an *$\ell_\infty$-direction* if there exists a subspace $U$ of $V$ such that $V = (\langle v\rangle \oplus U)_\infty$; that is, if $\|\alpha v + u\| = \max(|\alpha|, \|u\|)$ for all $\alpha \in \mathbb{R}$ and $u \in U$. We call $U$ the subspace corresponding to $v$. Note, we view $v$ and $-v$ as the *same* $\ell_\infty$-direction.

This definition is useful since, in any decomposition of $V$ as $(U \oplus \ell_\infty^d)_\infty$, then each basis vector of the $\ell_\infty^d$ is an $\ell_\infty$-direction; see Proposition 14 for a formal proof.

**Lemma 11.** *Suppose that $v$ is an $\ell_\infty$-direction. Then the corresponding subspace $U$ is unique.*

*Proof.* Fix a corresponding subspace $U$. Suppose $u' \in V$ is any vector satisfying $\|\alpha v + u'\| = \max(|\alpha|, \|u'\|)$ for all $\alpha \in \mathbb{R}$. We can write $u' = \beta v + u$ for some $\beta \in \mathbb{R}$ and $u \in U$. By the definition of an $\ell_\infty$-direction, $\|u'\| \geqslant \|u\|$. Let $\gamma = \|u'\|$. By our assumption on $u'$, we have $\|u' + \gamma v\| = \|u' - \gamma v\|$ so $\|u + (\beta+\gamma)v\| = \|u + (\beta-\gamma)v\|$. Since $\gamma \geqslant \|u\|$, this implies $\beta = 0$ and, hence, $u' \in U$. □

**Lemma 12.** *Suppose that $v_1$ and $v_2$ are distinct $\ell_\infty$-directions with corresponding subspaces $U_1$ and $U_2$. Then $v_2 \in U_1$.*

*Proof.* First, we claim that, for any vector $v'$, the line $\{v' + \lambda v_2 : \lambda \in \mathbb{R}\}$ either contains a nontrivial interval of vectors of minimal norm (among points on the line) or contains 0. Indeed, this line contains a point, say $u'$ of $U_2$. Thus, we can write the line as $\{u' + \lambda v_2 : \lambda \in \mathbb{R}\}$. Since $\|u' + \lambda v_2\| = \max(|\lambda|, \|u'\|)$, we see that if $u' = 0$ we have the latter case; and if $\|u'\| > 0$, all vectors in the set $\{u' + \lambda v_2 : |\lambda| \leqslant \|u'\|\}$ have minimal norm. The claim follows.

We can write $v_2 = \alpha v_1 + \beta u_1$ with $u_1 \in U_1$ and $\|u_1\| = 1$. If $\alpha = 0$, then $v_2$ is in $U_1$, as claimed; if $\beta = 0$, then $v_2 = \pm v_1$ so $v_2$ is the same $\ell_\infty$-direction as $v_1$, contradicting the assumption that $v_1$ and $v_2$ are distinct $\ell_\infty$-directions.

Thus, we assume $\alpha, \beta \neq 0$ and, by negating either or both of $v_1$ and $u_1$, we may assume $\alpha, \beta > 0$. Consider the set of vectors

$$\{v_1 - u_1 + \lambda v_2 : \lambda \in \mathbb{R}\}.$$

Since

$$\|v_1 - u_1 + \lambda v_2\| = \|(1 + \lambda\alpha)v_1 - (1 - \lambda\beta)u_1\| = \max(|1 + \lambda\alpha|, |1 - \lambda\beta\|),$$

we see that $\lambda = 0$ gives the unique vector of minimal norm in this set and that this vector has norm 1 which contradicts the above claim that, whenever the minimum norm on the line is not zero, there must be an interval of minimal norm. □

The next lemma shows that any set of $\ell_\infty$-directions combine to give an $\ell_\infty$ subspace of $V$.

**Lemma 13.** *Suppose that $v_1, v_2, \ldots, v_k$ are any (distinct) $\ell_\infty$-directions with corresponding subspaces $U_1, U_2, \ldots, U_k$. Then*

$$V = \left(\langle v_1\rangle \oplus \langle v_2\rangle \oplus \cdots \oplus \langle v_k\rangle \oplus \bigcap_{i=1}^{k} U_i\right)_\infty .$$

*Proof.* First we show inductively that we can write any vector $v$ as $\sum_{i=1}^{j} \lambda_i v_i + w_j$ where $w_j \in \bigcap_{i=1}^{j} U_i$. For $j = 1$ it is just the definition of an $\ell_\infty$-direction. Suppose it holds for $j$. Then since $v_{j+1}$ is an $\ell_\infty$-direction, we can write $w_j = \lambda_{j+1} v_{j+1} + w_{j+1}$

for some $w_{j+1} \in U_{j+1}$. Since, for each $1 \leqslant i \leqslant j$, $w_j \in U_i$ and $v_{j+1} \in U_i$, we see that $w_{j+1} \in U_i$. Hence $w_{j+1} \in \bigcap_{i=1}^{j+1} U_i$, and the induction is complete.

Next we show that the sum

$$\langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \cdots \oplus \langle v_k \rangle \oplus \bigcap_{i=1}^{k} U_i$$

is direct. Suppose that $u \in \bigcap_{i=1}^{k} U_i$, that $u + \sum_{i=1}^{k} \lambda_i v_i = 0$ is a nontrivial linear relation, and that $\lambda_j \neq 0$. By Lemma 12, $v_i \in U_j$ for all $i \neq j$, and obviously $u \in U_j$. Hence $v_j = \frac{1}{\lambda_j}\left(-u - \sum_{i \neq j} \lambda_i v_i\right) \in U_j$, which is a contradiction.

To complete the proof, observe that, by applying the $\ell_\infty$-direction property inductively, we have

$$\left\| \sum_{i=1}^{j} \lambda_i v_i + u \right\| = \max(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_j|, \|u\|)$$

for any $j$, $\lambda_i \in \mathbb{R}$, and $u \in \bigcap_{i=1}^{j} U_i$. Taking $j = k$ gives the result. □

Thus we see that the $\ell_\infty$-decomposition is unique in the strongest possible sense; namely, that the $\ell_\infty^d$-component is the space spanned by *all* the $\ell_\infty$-directions. We sum this up in the following proposition.

**Proposition 14.** *Suppose $V$ is a finite-dimensional normed space. Then there is a unique maximal space $W$ isometric to $\ell_\infty^d$, for some $d$, with the property that there is a subspace $U$ with $V = U \oplus W$ and $\|u + w\| = \max(\|u\|, \|w\|)$, for any $u \in U$ and $w \in W$.*

*Moreover, if $v_1, v_2, \ldots, v_d$ are all the $\ell_\infty$-directions with corresponding subspaces $U_1, U_2, \ldots, U_d$, then $W = \langle v_1, v_2, \ldots, v_d \rangle$ and $U = \bigcap_{i=1}^{d} U_i$.*

*Proof.* As in the statement of the proposition, let $v_1, v_2, \ldots, v_d$ be all the $\ell_\infty$-directions, let $W = \langle v_1, v_2, \ldots, v_d \rangle$, and let $U = \bigcap_{i=1}^{d} U_i$, where $U_i$ is the corresponding subspace to $v_i$. By Lemma 13 $V = U \oplus W$, and for any $u \in U$ and $w = \sum_{i=1}^{d} \lambda_i v_i \in W$, we have $\|w\| = \max(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_d|)$, so $W$ is isometric to $\ell_\infty^d$ and, by Lemma 13 again,

$$\|u + w\| = \max(\|u\|, |\lambda_1|, |\lambda_2|, \ldots, |\lambda_d|) = \max(\|u\|, \|w\|),$$

as required.

To complete the proof, suppose that $W'$ is any subspace isometric to $\ell_\infty^{d'}$ for some $d'$ and that $U'$ is a subspace with the property that $V = U' \oplus W'$ and $\|u' + w'\| = \max(\|u'\|, \|w'\|)$ for any $u' \in U'$ and $w' \in W'$. Let $e_1, e_2, \ldots, e_{d'}$ be the natural basis of $W'$ viewed as $\ell_\infty^{d'}$. We see that, for any $\lambda_1, \lambda_2, \ldots, \lambda_{d'}$ and any $u' \in U'$,

$$\begin{aligned} \|u' + \sum_{i=1}^{d'} \lambda_i e_i\| &= \max\left(\|u'\|, \|\sum_{i=1}^{d'} \lambda_i e_i\|\right) \\ &= \max\left(\|u'\|, |\lambda_1|, |\lambda_2|, \ldots, |\lambda_{d'}|\right) \\ &= \max\left(|\lambda_1|, \|u + \sum_{i=2}^{d'} \lambda_i e_i\|\right), \end{aligned}$$

so, in particular, $e_1$ is an $\ell_\infty$-direction with the corresponding subspace $U' \oplus \langle e_2, e_3, \ldots, e_{d'}\rangle$. Thus $e_1$ is one of the $v_i$ or $-v_i$ and, in particular, $e_1 \in W$. Since this is true for each $e_i$, $1 \leqslant i \leqslant d'$, we see that $W' \subseteq W$. □

**Corollary 15.** *Let $Q$ be a linear isometry of a finite-dimensional normed spaced $V$ with $\ell_\infty$-decomposition $U \oplus \ell_\infty^d$. Then $Q$ factorises over the decomposition as $Q_U \oplus Q_{\ell_\infty^d}$ and each factor is an isometry.*

We remark that there are direct proofs of this result, based on Proposition 14; our proof, whilst a little longer, will be useful for the next result.

*Proof.* First, observe that, since $Q$ is linear, factorising over the decomposition is the same as saying $Q(U) \subseteq U$ and $Q(\ell_\infty^d) \subseteq \ell_\infty^d$, and this is what we shall show.

Suppose $v_1, v_2, \ldots, v_d$ are the $\ell_\infty$-directions with corresponding subspaces $U_1, U_2, \ldots U_d$. Let $v_i' = Q(v_i)$ and $U_i' = Q(U_i)$ for each $i$. We claim that $v_i'$ is an $\ell_\infty$-direction with subspace $U_i'$. Indeed, given $v' \in V$, let $v = Q^{-1}(v')$. Since $v_i$ is an $\ell_\infty$-direction, we can write $v = \alpha v_i + u_i$ for some $u_i \in U_i$, and we have $\|v\| = \max(|\alpha|, \|u_i\|)$. Since $Q$ is linear and writing $u_i'$ for $Q(u_i)$, this implies that $v' = Q(v) = Q(\alpha v_i + u_i) = \alpha v_i' + u_i'$ with $u_i' \in U_i'$. Since $Q$ is an isometry, we have

$$\|v'\| = \|v\| = \max(|\alpha|, \|u_i\|) = \max(|\alpha|, \|u_i'\|),$$

as claimed.

Thus $Q$ permutes the $\ell_\infty$-directions (possibly negating some of them) and, in particular, maps $\ell_\infty^d = \langle v_1, v_2, \ldots, v_d\rangle$ to itself. Also, $Q$ permutes the corresponding subspaces so $U = \bigcap_{i=1}^d U_i$ is also mapped to itself. As observed above, this shows that $Q$ factorises as $Q|_U \oplus Q|_{\ell_\infty^d}$ and, since the factors are just the restrictions of $Q$ to $U$ and $\ell_\infty^d$, respectively, we see that each factor is an isometry. □

The proof of Corollary 15 actually describes what the isometries of $\ell_\infty^d$ are.

**Corollary 16.** *Suppose that $f$ is an (bijective) isometry of $\ell_\infty^d$. Then there is a permutation $\sigma$ of $[d]$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d) \in \{-1, +1\}^d$ such that $f$ is the linear map that sends each basis vector $e_i$ to $\varepsilon_i e_{\sigma(i)}$, combined with a translation.*

*Proof.* Define $\hat{f}$ by $\hat{f}(x) = f(x) - f(0)$. By the Mazur–Ulam Theorem $\hat{f}$ is linear. The proof of Corollary 15 shows that $\hat{f}$ permutes the basis vectors of $\ell_\infty^d$ (which are obviously the $\ell_\infty$-directions), possibly changing the sign. The result follows. □

## 4. Extending step-isometries from $S$ to $V$

Suppose that $f$ is a step-isometry on a dense set $S$ in $V$. In this section we show that $f$ extends to a continuous step-isometry $\bar{f} \colon V \to V$.

As one would expect, we shall define $\bar{f}$ in terms of sequences in $S$. We start by proving some simple results about such sequences.

**Lemma 17.** *Suppose $f$ is a step-isometry on $S$, that $(x_n)$ is a sequence in $S$ converging to $x$, and that $f(x_n)$ converges to $x'$. Then, for any $y \in S$ and $k \in \mathbb{N}$ which satisfy $\|x - y\| < k$, we have $\|x' - f(y)\| \leqslant k$.*

*Proof.* Suppose $\|x - y\| < k$. Then, for all sufficiently large $n$, $\|x_n - y\| < k$. Thus, since $f$ is a step isometry, $\|f(x_n) - f(y)\| < k$. Hence $\|x' - f(y)\| \leqslant k$. □

**Lemma 18.** *Suppose $f$ is a step-isometry on $S$, that $(x_n),(y_n)$ are sequences in $S$ converging to $x$ and $y$, respectively, and that $f(x_n), f(y_n)$ converge to $x'$ and $y'$, respectively. Then, for any $k \in \mathbb{N}$, $\|x-y\| < 3k$ if and only if $\|x'-y'\| < 3k$.*

*In particular, for any $k \in \mathbb{N}$, if $y \in S$, then $\|x-y\| < 3k$ if and only if $\|x' - f(y)\| < 3k$.*

*Proof.* Suppose that $\|x-y\| < 3k$. Since $S$ is dense, we can pick $s,t \in S$ such that $\|x-s\| < k$, $\|s-t\| < k$ and $\|t-y\| < k$. By Lemma 17 $\|x' - f(s)\| \leqslant k$ and $\|f(t) - y'\| \leqslant k$. Also, since $f$ is a step-isometry on $S$, $\|f(s) - f(t)\| < k$. Hence, by the triangle inequality, $\|x'-y'\| < 3k$.

We obtain the converse by applying the above to $f^{-1}$, which is also a step-isometry on $S$.

The final part follows by taking the sequence $(y_n)$ to be the constant sequence $y$. □

**Lemma 19.** *Suppose $f$ is a step-isometry on $S$, that $(x_n),(y_n)$ are two sequences in $S$ converging to $x$, and that $f(x_n), f(y_n)$ converge to $x'$ and $y'$, respectively. Then $x' = y'$.*

*Proof.* Suppose that $x' \neq y'$. Then the set

$$\{v \in V : \|x' - v\| < 3 \text{ and } \|y' - v\| > 3\}$$

is open and nonempty. Since $S$ is dense in $V$, there exists $z' \in S$ with $\|x' - z'\| < 3$ and $\|y' - z'\| > 3$. Let $z = f^{-1}(z')$. Then, Lemma 18 applied to the sequences $(x_n)$ and $(y_n)$ implies $\|x-z\| < 3$ and $\|x-z\| \geqslant 3$, which is a contradiction. □

**Lemma 20.** *Suppose that $(x_n)$ is a sequence in $S$ that converges in $V$. Then $f(x_n)$ is a convergent sequence.*

*Proof.* Since $(x_n)$ is convergent, there is an $m$ such that, for all $n > m$, we have $\|x_n - x_m\| < 1$. Hence, since $f$ is a step-isometry, $\|f(x_n) - f(x_m)\| < 1$ for all $n > m$; in particular, $f(x_n)$ is a bounded sequence. Thus, since $V$ is finite dimensional, there is a subsequence $(x_{n_i})$ such that $f(x_{n_i})$ converges to some value $x'$, say.

Suppose that $f(x_n)$ does not converge to $x'$. Then there exists a subsequence bounded away from $x'$. As above we can take a further subsequence which converges and is bounded away from $x'$; in particular, it must converge to some value $x'' \neq x'$. But this contradicts Lemma 19. □

**Corollary 21.** *Suppose $f$ is a step-isometry on $S$. Then there is a unique continuous function $\bar{f}\colon V \to V$ that extends $f$.*

*Proof.* For any $x \in V$ define $\bar{f}(x)$ as follows. Choose a sequence $(x_n)$ in $S$ converging to $x$, and let $\bar{f}(x) = \lim_{n\to\infty} f(x_n)$. This limit exists by Lemma 20 and the function is well-defined by Lemma 19.

Finally, it is easy to see that $\bar{f}$ is continuous. Indeed, suppose that $(x_n)$ is a sequence in $V$ converging to $x$, say. By the definition of $\bar{f}$ we can pick a sequence $(x'_n)$ in $S$ such that $\|x_n - x'_n\| < 1/n$ and $\|f(x'_n) - \bar{f}(x_n)\| < 1/n$ for all $n$. Then $x'_n \to x$, so, since $\bar{f}$ is well-defined, $f(x'_n) \to \bar{f}(x)$ and, thus, $\bar{f}(x_n) \to \bar{f}(x)$, as required. □

**Corollary 22.** *Any step-isometry on $V$ is continuous.*

*Proof.* This follows immediately from Corollary 21 by taking the dense set $S$ to be the whole of $V$. $\square$

Lemma 18 shows that $\bar{f}$ is a *scaled* step-isometry; that is, a step isometry in the norm $\frac{1}{3}\|\cdot\|$. Whilst that would be sufficient for our needs, $\bar{f}$ is actually a step isometry in the original norm, and we prove that next. We start with the following trivial fact.

**Lemma 23.** *Suppose that $S$ is a dense set in $V$ and that $x, y \in V$. Then there exist sequences $(x_n), (y_n)$ of points in $S$ converging to $x$ and $y$, respectively, such that $\|x_n - y_n\| > \|x - y\|$ for all $n$. Similarly, providing $x \neq y$, we may choose such sequences $(x_n), (y_n)$ such that $\|x_n - y_n\| < \|x - y\|$ for all $n$.*

*Proof.* Let $\varepsilon > 0$. Let $r = \|x-y\|$. The point $y' = x+(1+\varepsilon)(y-x)$ has $\|y-y'\| = \varepsilon r$ and $\|x - y'\| = (1+\varepsilon)r$. Let $x''$ be any point of $S$ in the set $B(x, \varepsilon r/2)$ and $y''$ any point of $S$ in $B(y', \varepsilon r/2)$. By the triangle inequality, we have $\|x - x''\| < \varepsilon r/2$, $\|y - y''\| < 3\varepsilon r/2$ and, also, $\|x'' - y''\| > r$.

We get the required sequence by setting $x_n, y_n$ to be the points $x'', y''$ given by the above argument when $\varepsilon = 1/n$.

The second inequality is very similar, but this time we choose $y' = x+ (1-\varepsilon)(y-x)$. $\square$

**Proposition 24.** *The function $\bar{f}$ defined above is a step-isometry. Moreover, $\bar{f}$ preserves integer distances.*

*Proof.* Suppose $x$ and $y$ have $\|x - y\| \geqslant k$ for some $k \in \mathbb{Z}$. Then by Lemma 23 we can find sequences $(x_n)$ and $(y_n)$ in $S$ that converge to $x$ and $y$, respectively, and have $\|x_n - y_n\| > k$. Hence, since $f$ is a step-isometry, $\|\bar{f}(x) - \bar{f}(y)\| = \lim_{n\to\infty} \|f(x_n) - f(y_n)\| \geqslant k$.

Similarly, if $x$ and $y$ have $\|x-y\| \leqslant k$, then, by taking sequences with $\|x_n-y_n\| < k$, we see that $\|\bar{f}(x) - \bar{f}(y)\| \leqslant k$.

This shows that if $\|x - y\| \in (k, k+1)$, then $\|\bar{f}(x) - \bar{f}(y)\| \in [k, k+1]$. Also, if $\|x - y\| = k$, then $\|\bar{f}(x) - \bar{f}(y)\| = k$; that is, $\bar{f}$ preserves integer distances.

Observe that $f^{-1}$ is also a step-isometry on $S$, so it extends to $\overline{f^{-1}}$ a step-isometry on $V$. Since we have $\overline{f^{-1}} \circ \bar{f} = \bar{f} \circ \overline{f^{-1}} = \mathrm{id}$ on $S$, and $\bar{f}$ and $\overline{f^{-1}}$ are both continuous, we see that $\bar{f}^{-1} = \overline{f^{-1}}$. Thus, if $\|\bar{f}(x) - \bar{f}(y)\| = k$, then $\|x - y\| = k$, and the result follows. $\square$

**Corollary 25.** *Suppose $f$ is a step-isometry on $V$. Then $f$ preserves integer distances. Moreover, for any integer $k$ and $x \in V$, we have $f(B(x,k)) = B(f(x), k)$.*

*Proof.* For the first part, take $S = V$ in Proposition 24. By the definition of a step-isometry, $f$ maps the open ball $B^\circ(x,k)$ to $B^\circ(f(x),k)$ so, since it and its inverse preserve integer distances, the second part follows. $\square$

## 5. Extreme points

For this section we assume $f$ is a (necessarily continuous by Corollary 22) step-isometry on all of $V$ that fixes 0. The assumption that 0 is fixed makes the results simpler to state and this case is sufficient for our needs.

Our aim in this section is to prove that $f$ maps the extreme points of the unit ball to themselves, and that restricted to these extreme points it is an isometry.

First we characterise the extreme points of $B$ in a purely norm/metric way.

**Lemma 26.** *Suppose that $x$ is an extreme point of $B = B(0,1)$ and $n \in \mathbb{N}$. Then $B(0,1) \cap B(nx, n-1) = \{x\}$.*

*Proof.* Suppose that $y \in B$ and $\|nx - y\| \leqslant n-1$. Then $\|y\| \leqslant 1$ and $\|\frac{n}{n-1}x - \frac{1}{n-1}y\| \leqslant 1$. Since

$$x = \frac{n-1}{n}\left(\frac{n}{n-1}x - \frac{1}{n-1}y\right) + \frac{1}{n}y$$

and $x$ is an extreme point of $B$, we see that $y = x$. □

**Lemma 27.** *A point $x$ in the unit ball $B = B(0,1)$ is an extreme point if and only if there exists a point $z$ such that $B(z,1) \cap B(0,1) = \{x\}$.*

*Proof.* If $x$ is an extreme point, then the point $z = 2x$ is such a point by Lemma 26.

Now suppose that $z$ is a point such that $B(z,1) \cap B(0,1) = \{x\}$. Let $y = z - x$. Then $\|y\| \leqslant 1$. Hence, the point $y$ is in $B(0,1)$ and $B(z,1)$. Thus, since $x$ is the unique point in the intersection, $y = x$, so $z = 2x$.

Now suppose that $x = \frac{1}{2}(y + w)$ for some $y, w \in B$. Then $2x - y = w \in B$, so $y \in B(0,1) \cap B(2x,1)$. Using the fact that $x$ is the unique point in this intersection again, we have $y = w = x$, and we see that $x$ is an extreme point of $B$. □

We use this characterisation of the extreme points to show that $f$ maps them among themselves.

**Corollary 28.** *The extreme points of the unit ball map to themselves under $f$.*

*Proof.* Lemma 27 characterises the extreme points by their integer distance properties. These are preserved by the step-isometry so the extreme points must be. Indeed, suppose $x$ is an extreme point of $B$. Then by Lemma 26 the point $2x$ has the property that $B(0,1) \cap B(2x,1) = \{x\}$. Hence, by Corollary 25, $B(0,1) \cap B(f(2x),1)$ must be the single point $f(x)$. Thus, by Lemma 27, $f(x)$ is an extreme point of $B$. □

The final aim in this section is to show that $f$ restricted to the extreme points of $B$ is an isometry.

**Lemma 29.** *Suppose that $n \in \mathbb{N}$ and that $x$ is an extreme point of $B$. Then $f(nx) = nf(x)$.*

*Proof.* Obviously, $f$ is also a step-isometry in the norm $\frac{1}{n}\|\cdot\|$ which has unit ball $nB$. Thus, since $nx$ is an extreme point of $nB$, it must map to a point $ny$ which is an extreme point of $nB$ and, thus, $y$ is an extreme point of $B$. We need to show that $f(x) = y$.

By Lemma 26, $B(0,1) \cap B(nx, n-1) = \{x\}$. Hence, by Corollary 25, $B(f(0),1) \cap B(f(nx), n-1) = \{f(x)\}$. Since $f(0) = 0$ and $f(nx) = ny$, Lemma 26 again shows that

$$B(f(0),1) \cap B(f(nx), n-1) = B(0,1) \cap B(ny, n-1) = \{y\},$$

and, thus, $f(x) = y$, as required. □

The next lemma provides a useful criterion for certain distances to be preserved.

**Lemma 30.** *Suppose $x, y \in V$ have the property that $f(nx) = nf(x)$ and $f(ny) = nf(y)$ for any $n \in \mathbb{N}$. Then $\|x - y\| = \|f(x) - f(y)\|$.*

*Proof.* By hypothesis, for any $n \in \mathbb{N}$,

$$\|f(x) - f(y)\| = \frac{1}{n}\|f(nx) - f(ny)\|.$$

Also, since $f$ is a step-isometry,

$$\lfloor \|nx - ny\| \rfloor = \lfloor \|f(nx) - f(ny)\| \rfloor,$$

in particular

$$\big| \|f(nx) - f(ny)\| - \|nx - ny\| \big| < 1.$$

Hence,

$$\|f(x) - f(y)\| = \lim_{n\to\infty} \frac{1}{n}\|f(nx) - f(ny)\| = \lim_{n\to\infty} \frac{1}{n}\|nx - ny\| = \|x - y\|. \quad \square$$

**Proposition 31.** *The function $f$ is an isometry on the extreme points of $B$.*

*Proof.* Suppose $x$ and $y$ are extreme points of $B$. We know that they map to extreme points. By Lemma 29 we know that $f(nx) = nf(x)$ and $f(ny) = nf(y)$ for all $n \in \mathbb{N}$. Hence, by Lemma 30, $\|x - y\| = \|f(x) - f(y)\|$. Since this is true for all $x, y \in \mathrm{Ext}(B)$, $f$ is an isometry on $\mathrm{Ext}(B)$. $\square$

## 6. The lattice generated by the extreme points

Throughout this section we assume that $f$ is a (continuous) step-isometry of $V$ that fixes 0. In the previous section we showed that $f$ maps the extreme points of $B$ to themselves. Obviously, the same argument shows that $f$ maps the extreme points of $B(y, 1)$ to extreme points of $B(f(y), 1)$. We start this section by showing that this mapping is the 'same' mapping.

**Lemma 32.** *Suppose $x$ is an extreme point of $B$. Then for any $y \in V$, we have $f(y + x) = f(y) + f(x)$.*

*Proof.* The point $y+x$ is an extreme point of $B(y, 1)$ so, by Corollary 28, $f(y+x) = f(y) + z$ for some extreme point $z \in B$ and, by Lemma 29, $f(y + nx) = f(y) + nz$ for all $n \in \mathbb{N}$. Now the pairs of points $nx$ and $y + nx$ are each $\|y\|$ apart: in particular, these distances are bounded. Thus, since $f$ is a step-isometry, the same is true of the pairs $f(nx) = nf(x)$ and $f(y + nx) = f(y) + nz$. Hence, $z = f(x)$, as claimed. $\square$

**Corollary 33.** *For any extreme point $x$ of $B$, we have $f(-x) = -f(x)$.*

*Proof.* This is instant from Lemma 32. Indeed

$$0 = f(0) = f(x + (-x)) = f(x) + f(-x). \quad \square$$

Next we show that $f$ behaves well on the lattice $\Lambda$. (Recall from section 2 that $\Lambda$ denotes the lattice generated by the extreme points of $B$.)

**Corollary 34.** *The function $f$ maps $\Lambda$ to itself with*

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i f(x_i)$$

*for any $\lambda_i \in \mathbb{Z}$ and $x_i \in \mathrm{Ext}(B)$. Moreover, for any $x \in \Lambda$ and $y \in V$, we have $f(y + x) = f(y) + f(x)$.*

*Proof.* Both parts follow by applying Lemma 32 and Corollary 33 repeatedly. $\square$

**Lemma 35.** *The function $f$ restricted to $\Lambda$ is an isometry.*

*Proof.* By Corollary 34

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i f(x_i).$$

In particular for any $n \in \mathbb{N}$ and $x \in \Lambda$, we have $f(nx) = nf(x)$. Thus Lemma 30 shows that, for any $x, y \in \Lambda$, we have $\|x - y\| = \|f(x) - f(y)\|$; that is, $f$ is an isometry on $\Lambda$. □

Of course this isometry extends from $\Lambda$ to $\overline{\Lambda}$.

**Corollary 36.** *$f$ restricted to the closure $\overline{\Lambda}$ of $\Lambda$ is an additive isometry.*

*Proof.* $f$ is continuous and is an additive isometry on $\Lambda$. □

Our final aim in this section is to show that there exists an isometry $Q$ of $V$ such that $Q \circ f$ fixes $\Lambda$ pointwise. Obviously, $Q \circ f$ is also a step-isometry, so in our later arguments we are able to reduce to the case when $f$ fixes $\Lambda$.

**Lemma 37.** *There exists a unique linear isometry $\hat{f}\colon V \to V$ such that $\hat{f}$ and $f$ agree on $\overline{\Lambda}$.*

*Proof.* First, define $\hat{f}$ on $\mathbb{Q}\Lambda$ by $f(qv) = qf(v)$, where $q \in \mathbb{Q}$ and $v \in \Lambda$. This is well-defined and linear since $f$ is additive on $\Lambda$. Since $f$ is an isometry on $\Lambda$, $\hat{f}$ is an isometry on $\mathbb{Q}\Lambda$. Now, since $\Lambda$ is spanning, $\mathbb{Q}\Lambda$ is dense in $V$, and thus $\hat{f}$ extends to a linear isometry on $V$.

The uniqueness is trivial since $\Lambda$ is spanning. □

**Corollary 38.** *There exists an isometry $Q$ of $V$ such that $Q \circ f$ fixes $\Lambda$ pointwise.*

*Proof.* Let $Q$ be the isometry extending $f^{-1}$, as guaranteed by the previous lemma. Then $Q \circ f$ fixes $\Lambda$ pointwise. □

## 7. Extreme lines and preserved directions

In this section we assume that $f$ is a step-isometry of $V$ that fixes $\Lambda$ pointwise, and so, in particular, $f(0) = 0$.

Our aim in this section is to show that many directions are unchanged or *preserved.*

**Definition.** A *preserved direction* is a vector $x$ such that, for all $\alpha \in \mathbb{R}$ and for all $y \in V$, the vector $f(y + \alpha x) - f(y)$ is a multiple of $x$.

In particular, since we are assuming $f(0) = 0$, for any preserved direction $x$, $f(x)$ is a multiple of $x$.

Preserved directions turn out to be closely related to extreme lines, which are a standard generalisation of the notion of extreme points.

**Definition.** Suppose $A$ is a convex body. An *extreme line* of $A$ is a line segment $[x, y]$ in $A$ such that, for all $z \in [x, y]$, if $z$ is a convex combination of $s, t \in A$, then $s, t \in [x, y]$.

*Remark.* Obviously, if $[x, y]$ is an extreme line, then $x$ and $y$ are extreme points of $A$.

Just as extreme points are characterised by the intersection properties of balls, so are extreme lines.

**Lemma 39.** *Suppose* $[x, y]$ *is an extreme line of the unit ball* $B = B(0, 1)$*. Then*

$$[x, y] = B(0, 1) \cap B(x + y, 1).$$

*Proof.* Since $x, y \in B$, we have $x, y \in B(x+y, 1)$, so $x, y \in B \cap B(x+y, 1)$. Hence, by convexity, $[x, y] \subseteq B \cap B(x + y, 1)$.

Suppose that $z \in B \cap B(x + y, 1)$. Then $z \in B$ and $x + y - z \in B$. Thus

$$\frac{x + y - z}{2} + \frac{z}{2} = \frac{x + y}{2}$$

is a point in $[x, y]$ that is a convex combination of points in $B$. Since $[x, y]$ is an extreme line, this implies that $z \in [x, y]$. □

We will be interested in the directions of the extreme lines rather than the lines themselves. Thus we make the following definition.

**Definition.** Suppose $B$ is the unit ball of a normed space $V$. An *extreme line direction* is any nonzero multiple of the vector $x - y$ where $[x, y]$ is an extreme line in $B$.

*Remark.* We view extreme line directions that are (nonzero) multiples of each other as the *same* extreme line direction.

The key result for preserved directions is that all extreme line directions are preserved directions.

**Proposition 40.** *Suppose* $B$ *is the unit ball and* $[x, y]$ *is an extreme line. Then* $x - y$ *is a preserved direction.*

*Proof.* Suppose $v_1, v_2 \in V$ satisfy $v_2 = v_1 + \alpha(y - x)$ for some $\alpha > 0$. Let $n = \lceil \alpha \rceil$ and $u = v_1 - nx$. Then we have $v_1, v_2 \in u + [nx, ny]$.

Now, by Lemma 39, for any point $z \in u + [nx, ny]$, we have $z \in B(u, n) \cap B(u + nx + ny, n)$. Hence, since $f$ is a step-isometry, $f(z) \in B(f(u), n) \cap B(f(u+nx+ny), n)$. Since $nx+ny \in \Lambda$, by Corollary 34, we have $f(u+nx+ny) = f(u) + nx + ny$. Thus,

$$f(z) \in B(f(u), n) \cap B(f(u) + nx + ny, n) = f(u) + [nx, ny]$$

by Lemma 39 again. In particular, both $f(v_1)$ and $f(v_2)$ lie in $f(u)+[nx, ny]$. Thus

$$f(v_2) - f(v_1) = \beta(x - y)$$

for some $\beta$, as claimed. □

*Remark.* The map $f$ need not preserve the directions of the extreme points: indeed, consider the $\ell_\infty^2$ case where $f$ can treat each coordinate separately and, thus, need not preserve the line $y = x$ through the extreme point $(1, 1)$.

## 8. Strongly fixed subspaces

In this section we assume that $f$ is a step-isometry of $V$ that fixes $\Lambda$ pointwise.

**Definition.** We say a subspace $U$ of $V$ is *strongly fixed* if, for all $u \in U$ and $v \in V$, we have $f(u + v) = u + f(v)$.

*Remark.* It is immediate from the definition that if $U$ and $U'$ are strongly fixed subspaces, then $U + U'$ is a strongly fixed subspace.

We have seen (Corollary 34) that $f(u+v) = u + f(v)$ for all $u \in \overline{\Lambda}$ and $v \in V$. Hence, the continuous subspace $U_0$ of $\overline{\Lambda}$ is a strongly fixed subspace. Our aim in the next two sections is to show that the whole of $U$ in the $\ell_\infty$-decomposition of $V$ is strongly fixed; in this section we show that a 'large' subspace is strongly fixed. Then, in the next section, we show that what is left is essentially an $\ell_\infty$ subspace—in particular, that it is spanned by $\ell_\infty$-directions.

**Lemma 41.** *Suppose $x_1, x_2, \dots, x_k$ is a linearly independent set of preserved directions. Then*

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) = \sum_{i=1}^{k} f(\lambda_i x_i)$$

*for any $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$.*

*Proof.* We prove this by induction on $k$. It is trivial for $k = 1$.

Suppose it is true for $k-1$; that is, suppose

$$f\left(\sum_{i=1}^{k-1} \lambda_i x_i\right) = \sum_{i=1}^{k-1} f(\lambda_i x_i)$$

for all $\lambda_1, \lambda_2, \dots, \lambda_{k-1} \in \mathbb{R}$.

Since $\sum_{i=1}^{k} \lambda_i x_i - \sum_{i=1}^{k-1} \lambda_i x_i = \lambda_k x_k$, which is a preserved direction, we see that

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) = f\left(\sum_{i=1}^{k-1} \lambda_i x_i\right) + \mu_k x_k = \sum_{i=1}^{k-1} f(\lambda_i x_i) + \mu_k x_k$$

for some $\mu_k$.

Similarly, by applying the induction hypothesis to the last $k-1$ summands rather than the first, we see that

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) = f\left(\sum_{i=2}^{k} \lambda_i x_i\right) + \mu_1 x_1 = \sum_{i=2}^{k} f(\lambda_i x_i) + \mu_1 x_1.$$

The $x_i$ are preserved directions so $f(\lambda_i x_i)$ is a multiple of $x_i$ for each $i$. Thus, since the $x_i$ are linearly independent, we see that $\mu_k x_k = f(\lambda_k x_k)$ as required. □

**Lemma 42.** *Suppose $x_1, x_2, \dots, x_k$ form a minimal linearly dependent set of preserved directions and that $k \geqslant 3$. Then $\langle x_1, x_2, \dots, x_k \rangle$ is a strongly fixed subspace.*

*Proof.* Suppose that $\sum_{i=1}^{k} \lambda_i x_i = 0$ is a nontrivial linear dependence. Since the $x_i$ form a minimal linear dependent set all the $\lambda_i$ are nonzero. Thus we may assume $\lambda_1 = 1$.

We start by showing that for any $m \in \mathbb{N}$, we have $f(mx_1) = mf(x_1)$. We prove this by induction. The case $m = 1$ is trivial, so suppose that $f((m-1)x_1) = (m-1)f(x_1)$. We have

$$\begin{aligned}
f(mx_1) &= f\left((m-1)x_1 - \sum_{i=2}^{k} \lambda_i x_i\right) \\
&= f\left((m-1)x_1 - \sum_{i=2}^{k-1} \lambda_i x_i\right) + Cx_k \qquad \text{(for some } C\text{)} \\
&= f((m-1)x_1) + f\left(\sum_{i=2}^{k-1} -\lambda_i x_i\right) + Cx_k \\
&= f((m-1)x_1) + f\left(\sum_{i=2}^{k} -\lambda_i x_i\right) + C'x_k \qquad \text{(for some } C'\text{)} \\
&= (m-1)f(x_1) + f(x_1) + C'x_k \\
&= mf(x_1) + C'x_k,
\end{aligned}$$

where the second line follows since $x_k$ is a preserved direction; the third line by Lemma 41 twice, since $x_1, \ldots, x_{k-1}$ are linearly independent; the fourth since $x_k$ is a preserved direction; and the fifth by the inductive hypothesis.

But since $x_1$ is a preserved direction and $x_1, x_k$ are linearly independent, $C' = 0$, and the induction is complete.

Obviously, $\alpha x_1$ is also a preserved direction for any $\alpha \neq 0$, so the above shows that $f(\alpha x_1) = \alpha f(x_1)$ for all $\alpha \in \mathbb{Q}$ with $\alpha > 0$. Since $f$ is continuous, this means that $f(\alpha x_1) = \alpha f(x_1)$ for all $\alpha > 0$.

Now, for any $\alpha > 0$, by Lemma 8 there is a point $y \in \Lambda$ with $\|\alpha x_1 - y\| \leqslant \dim V/2$. Thus, since $f$ is a step-isometry, we have

$$\|f(\alpha x_1) - f(y)\| \leqslant \|\alpha x_1 - y\| + 1 \leqslant \dim V/2 + 1.$$

But $f$ fixes $\Lambda$ pointwise so $f(y) = y$ and, thus, $\|f(\alpha x_1) - \alpha x_1\|$ is bounded independently of $\alpha$. Since $\|f(\alpha x_1) - \alpha x_1\| = \alpha \|f(x_1) - x_1\|$, this implies that $f(x_1) = x_1$ and, thus, that $f(\alpha x_1) = \alpha x_1$ for all $\alpha > 0$. The same argument applied to $-x_1$—obviously also a preserved direction—shows that $f(-\alpha x_1) = -\alpha x_1$. This shows that $f$ is the identity on $\langle x_1 \rangle$.

We have shown that $f$ fixed $\langle x_1 \rangle$ pointwise, but we want to show more: that $f$ strongly fixes $\langle x_1 \rangle$. For any $v \in V$, the function $g$ defined by $g(x) = f(x+v) - f(v)$ is also a step-isometry and, by Corollary 34, fixes $\Lambda$. Moreover, $g$ also preserves the directions $x_i$. Thus, by the above argument $g$ is the identity on $\langle x_1 \rangle$. Hence $f(v + \alpha x_1) = f(v) + \alpha x_1$ for all $\alpha \in \mathbb{R}$; that is, $\langle x_1 \rangle$ is a strongly fixed subspace.

Since this is true for each $x_i$, we see that $\langle x_1, x_2, \ldots x_k \rangle$ is a strongly fixed subspace. □

The previous lemmas show that the span of linearly dependent preserved directions is strongly fixed. Of course, we also know that the continuous subspace $U_0$ of $\overline{\Lambda}$ is strongly fixed. Thus we make the following definition to cover the largest subspace that we know (so far) is strongly fixed. Later, we shall show that this is the non-$\ell_\infty^d$-component of the $\ell_\infty$-decomposition.

Before stating the main definition, we need a little more notation. Suppose that $W$ is any subspace of $V$ and $x_1, x_2, \ldots, x_k$ are vectors in $V$. A linear combination of the $x_i$ over $W$ is any sum of the form $w + \sum_i \lambda_i x_i$, where $w \in W$; the *span of the $x_i$ over $W$* is $\langle W, x_1, x_2, \ldots, x_k \rangle$; the $x_i$ are *linearly independent over $W$* if $\sum_i \lambda x_i \in W$ implies that $\lambda_i = 0$ for all $i$.

**Definition.** Suppose that $V$ is a normed space with unit ball $B$, that $U$ is a subspace, and $x_i$, $i \in I$ are the extreme line directions in $U$. Then $U$ is *well-spanned* if

(1) it contains the continuous subspace $U_0$ of $\Lambda$,
(2) the $x_i$ span $U$ over $U_0$,
(3) every $x_i \in U \setminus U_0$ can be written as a linear combination of the other $x_j$ over $U_0$.

First, we show that there is a unique maximal well-spanned subspace and then that any step-isometry that pointwise fixes $\Lambda$ strongly fixes this subspace.

**Lemma 43.** *Suppose that $V$ is a normed space with unit ball $B$. Then there is a unique maximal well-spanned subspace $U$. Moreover, the extreme line directions outside $U$ are linearly independent over $U$.*

*Proof.* Obviously $U_0$ is well-spanned. Now suppose that $U$ and $U'$ are well-spanned subspaces of $V$. We show that $U + U'$ is also well-spanned. Indeed, it is immediate that (1) and (2) of the definition hold. To show condition (3), suppose that $x$ is an extreme line direction in $(U + U') \setminus U_0$. If $x$ is in $U$, then, since condition (3) holds in $U$, $x$ can be written as the required linear combination, and similary if $x \in U'$. On the other hand, if $x \notin U \cup U'$, then, since condition (2) holds in $U$ and $U'$, $x$ can be written as a linear combination (over $U_0$) of the extreme line directions inside each of these spaces (which cannot include $x$ as $x$ is not contained in $U$ or $U'$). Thus $U + U'$ is well-spanned. It follows that there is a unique maximal well-spanned subspace.

To prove the second part, let $U$ be the maximal well-spanned subspace and let $x_i$, $i \in I$, be the extreme line directions in $U$. Suppose that $y_1, y_2, \ldots, y_l \in V \setminus U$ is a minimal linearly dependent set of extreme line directions over $U$. Then, since the $x_i$ span $U$ over $U_0$, we see that, for each $j$, $y_j$ can be written as a linear combination of the $\{x_i : i \in I\} \cup \{y_i : i \neq j\}$ over $U_0$. Hence $U + \langle y_1, y_2, \ldots, y_\ell \rangle$ is a well-spanned subspace contradicting the maximality of $U$. $\square$

**Corollary 44.** *Suppose that $V$ is a normed space with maximal well-spanned subspace $U$ and that $f$ is a step-isometry fixing $\Lambda$. Then $U$ is a strongly fixed subspace.*

*Proof.* We have seen that $f$ strongly fixes $U_0$. Consider any extreme line direction $v$ in $U$. Then $v$ occurs in a minimal linear relation with other extreme line directions in $U$ over $U_0$. Since, by Proposition 40, extreme line directions are preserved directions of $f$, Lemma 42 shows that $f$ is strongly fixed on the span of these directions and, in particular, on $\langle v \rangle$. Since this is true for every extreme line direction in $U$ and these directions span $U$ over $U_0$, we see that $U$ is strongly fixed. $\square$

## 9. The complement of the maximal well-spanned subspace

In this section we prove that $V = (U \oplus \langle v_1 \rangle \oplus \langle v_2 \rangle \cdots \langle v_k \rangle)_\infty$, where $U$ is the maximal well-spanned subspace and $v_1 \cdots v_k$ are extreme line directions outside of $U$, and, thus, we deduce that $U$ is the non-$\ell_\infty^d$-component in the $\ell_\infty$-decomposition.

We start by showing that, unless $U = V$, there is an extreme line direction outside of $U$. Since we use induction, it is convenient to prove a (stronger) result for a general convex set rather than just for the unit ball of the normed space.

**Lemma 45.** *Suppose $U$ is a codimension 1 subspace of $V$, that $v \in V \setminus U$, and that $U_i = U + \lambda_i v$, $1 \leqslant i \leqslant k$, are distinct cosets of $U$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_k$. Further, suppose that, for each $i$, $A_i$ is a (nonempty) compact convex subset of $U_i$ and that, for some $s < k$, $x \in A_s$ is an extreme point of $A = \mathrm{conv}(\bigcup_i A_i)$. Then there exists $t > s$ and $y \in A_t$ such that $[x, y]$ is an extreme line of $A$.*

*Proof.* We prove this by induction on the dimension of $V$. If $\dim V = 1$ it is trivial: $V = \mathbb{R}$ and each $A_i$ is a single point. Since $s < k$ and $x$ is an extreme point, we must have $x \in A_1$, and the line from $x$ to the point in $A_k$ is the required extreme line. Thus suppose that the result holds for all spaces of dimension less than $\dim V$.

Let $H_0$ be a codimension 1 tangent hyperplane at $x$ to $A$, and let $h_0$ be a corresponding linear functional; that is, such that $H_0 = \{y \in V : h_0(y) = h_0(x)\}$. We may assume that $h_0(y) \leqslant h_0(x)$ for all $y \in \bigcup_i A_i$.

Let $q$ be the linear functional on $V$ defined, for any $u \in U$ and $\lambda$, by $q(u+\lambda v) = \lambda$. By hypothesis $q(U_i) = \lambda_i$ is increasing with $i$. Consider the family of hyperplanes $H_\alpha$ through $x$ given by the functionals $h_\alpha = h_0 + \alpha q$; that is, $H_\alpha = \{y \in V : h_\alpha(y) = h_\alpha(x)\}$. Let $H_\alpha^- = \{y \in V : h_\alpha(y) \leqslant h_\alpha(x)\}$. Note that, $A_i \subseteq H_0^-$ for all $i$.

For each $i > s$, the function $\alpha_i(y) = (h_0(x) - h_0(y))/(\lambda_i - \lambda_s)$ is continuous and nonnegative on the compact set $A_i$ and so attains an absolute minimum $\alpha_i^* \geq 0$. Set $\beta = \min_{i>s}\{\alpha_i^*\} \geq 0$. Then, by the choice of $\beta$, for every $i > s$ and $y \in A_i$, we have $h_\beta(y) \leq h_\beta(x)$. Additionally, for every $i \leqslant s$, and $y \in A_i$, since $\beta \geqslant 0$ and $\lambda_i \leqslant \lambda_s$, we also have $h_\beta(y) \leq h_\beta(x)$. Thus, $\bigcup_i A_i \subseteq H_\beta^-$, so $H_\beta$ is a tangent hyperplane to $A$ at $x$.

Furthermore, since all the minimums $\alpha_i^*$ were attained in the choice of $\beta$, there is at least one $j > s$ and $y \in A_j$ with $h_\beta(y) = h_\beta(x)$, and so $H_\beta \cap (\bigcup_{i>s} A_i) \neq \emptyset$.

Let $H = H_\beta$ and $H^- = H_\beta^-$, and, for each $i$, let $A_i' = A_i \cap H$. Note some of the $A_i$ may be empty and we ignore these sets. Let

$$A' = \mathrm{conv}(\bigcup_i A_i') = \mathrm{conv}(\bigcup_i A_i \cap H) = \mathrm{conv}(\bigcup_i A_i) \cap H = A \cap H,$$

where the third equality follows since $\bigcup_i A_i \subset H^-$. Now each $A_i'$ lies in $U_i \cap H$ which are cosets of $U \cap H$ which is codimension 1. Obviously the $A_i'$ are compact convex subsets. Also $x \in A_s'$ and, since $A' \subset A$, we see that $x$ is an extreme point of $A'$. Finally, by our choice of $H$, at least one of the $A_{s'}'$ for $s' > s$ is nonempty. Hence the $A_i'$ satisfy the induction hypothesis. Thus, there exists $y \in A_t'$ with $t > s$ such that $[x, y]$ is an extreme line of $A'$.

To complete the induction step and thus the proof, we show that $[x, y]$ is extreme line of $A$. Indeed, suppose $z \in [x, y]$ is a convex combination of $s, t \in A$. Since $[x, y] \subset A' \subset H$ and $A \subset H^-$, both $s, t$ must lie in $H$, and thus $s, t \in A'$. Since $[x, y]$ is an extreme line in $A'$, this shows that $s, t \in [x, y]$, and thus $[x, y]$ is an extreme line of $A$, as claimed. ◻

We use this result to deduce that there are many extreme line directions.

**Corollary 46.** *Suppose that $U$ is the maximal well-spanned subspace of $V$. Then the extreme line directions outside $U$ span $V$ over $U$.*

*Proof.* If $U = V$, then the statement is (rather vacuously) true, so assume $U \neq V$. Since $\mathrm{Ext}(B)$ spans $V$, there is an extreme point $x \notin U$. Let $y_i$, $i \in I$, be the endpoints of the extreme lines $[x, y_i]$ which have $x$ as the other endpoint. If $U$ together with the vectors $x - y_i$ span $V$, then the result holds, so suppose they do not.

Let $U'$ be a codimension 1 subspace containing $U$ and all the vectors $x - y_i$. Fix $v \in V \setminus U'$, and let $U'_1, U'_2, \ldots, U'_k$ be the cosets of $U'$ covering the extreme points of $B$, where $U'_i = U + \lambda_i v$ are such that the $\lambda_i$ are increasing. By Corollary 10, such a $k$ exists and, since $B$ is not contained in any codimension 1 affine hyperplane, $k \geqslant 2$.

By replacing $v$ with $-v$ (and thus reversing the order of the $U'_i$) if necessary, we may assume $x \in U'_s$ for some $s < k$. Now apply the previous lemma with $U'$, taking the set $A_i$ in $U'_i$ to be $B \cap U'_i$ for each $i$. Note that, since all the extreme points of $B$ are contained in $\bigcup_i A_i$, we have $\mathrm{conv}(\bigcup_i A_i) = B$.

This gives an extreme line $[x, y]$ of $B$ with $x - y$ not in $U'$, contradicting the choice of $U'$. □

**Lemma 47.** *Suppose that $v_1$ is an extreme line direction not in the maximal well-spanned subspace $U$. Then $v_1$ is an $\ell_\infty$-direction and $U$ is a subset of $U_1$, the corresponding subspace.*

*Proof.* Let $v_2, \ldots, v_k$ be the other extreme line directions outside of $U$. We may assume that they all, and $v_1$, have norm 1. By Lemma 43, the $v_i$ are linearly independent over $U$, and by Corollary 46 they span over $U$.

Let $U'$ be the subspace spanned by $U$ and $v_2, v_3, \ldots, v_k$. Since the $v_i$ are linearly independent and span over $U$, we see that $U'$ has codimension 1.

Suppose that $U'_1, U'_2, \ldots, U'_t$ are finitely many cosets (Corollary 10) of $U'$ that cover the extreme points of $B$. Our first step is to show that, from every extreme point of $B$, we can either add or subtract a multiple of $v_1$ and stay in $B$.

Write $U'_i = U' + \lambda_i v_1$, and we may assume that the $\lambda_i$ are increasing. Define $A_1, A_2, \ldots, A_t$ by $A_i = B \cap U'_i$. Note that $B = \mathrm{conv}(\bigcup_i A_i)$.

For any extreme point $x$ of $B$ in some $A_i$ with $i < t$, Lemma 45 shows that there exists $y$ in one of the $A_s$ with $s > i$ such that $x - y$ is an extreme line direction. Since $v_1$ is the only extreme line direction not in $U'$, we must have that $x - y$ is in the same direction as $v_1$. Thus, $y = x + \lambda v_1$ for some $\lambda$, and since $s > i$, we see $\lambda > 0$.

By applying Lemma 45 again, this time to the $A_i$ in reverse order, we see that any extreme point $x'$ of $B$ in any of the $A_i$ with $i > 1$ there is also a $y' \in A_s$ for some $s < i$ with $x' - y'$, an extreme line direction. Again $x' - y'$ must be the same direction as $v_1$; that is, $y' = x' + \lambda' v_1$. This time, since $s < i$, we see that $\lambda' < 0$.

Since the extreme points of $B$ span $V$ and $B$ is symmetric, we see that $\mathrm{Ext}(B)$ is not a subset of $U'$ or any single coset of $U'$, and thus $t \geqslant 2$. Hence, for any extreme point of $B$, at least one of the two cases above applies; thus, we have shown that from any extreme point of $B$, we can either add or subtract a multiple of $v_1$ and stay in $B$.

It now follows that $t = 2$; that is, the extreme points of $B$ are contained in two cosets of $U'$. Indeed, suppose $t \geqslant 3$. By applying the two cases above to any extreme point $x$ in $A_2$, we see that $x + \lambda v_1$ and $x + \lambda' v_1$ are both in $B$ for some $\lambda > 0$ and $\lambda' < 0$. But this contradicts $x$ being an extreme point of $B$.

Since $B$ is symmetric, we must have $U_1' = U' - \lambda v_1$ and $U_2' = U' + \lambda v_1$ for some $\lambda > 0$. Let $B_1 = A_1 + \lambda v_1$ and $B_2 = A_2 - \lambda v_1$ be the projections of $A_1, A_2$ onto $U'$.

We claim that $B_1 = B_2$. For a contradiction suppose there is a point in $B_2 \setminus B_1$. Then there must be an extreme point $z$ of $B_2$ in $B_2 \setminus B_1$. Obviously, $z' = z + \lambda v_1 \in A_2$, is an extreme point of $B$. However, since $z \notin B_1$, we see that we cannot add or subtract any multiple of $v_1$ to $z'$ and stay in $B$, which is a contradiction.

Now since $v_1 \in B$ (recall we assumed $\|v_1\| = 1$), we see $\lambda \geqslant 1$. Also for any $z \in A_1$, the vector $z + 2\lambda v_1 \in A_2$, so $z$ and $z + 2\lambda v_1$ are both in $B$; in particular $\lambda \leqslant 1$. Thus $\lambda = 1$.

Combining this, we see that $B = \operatorname{conv}(B_1 + v_1, B_1 - v_1)$. We use this to show that $v_1$ is an $\ell_\infty$-direction. Given any $v \in V$, write $v = \alpha v_1 + \beta u_1$ for some $\alpha, \beta \in \mathbb{R}$ and $u_1 \in U'$ with $\|u_1\| = 1$. Observe that the description of $B$ above shows that $B \cap U' = B_1$. Thus, since $\|u_1\| = 1$, we see that $u_1 \in B$ so $u_1 \in B_1$.

Now

$$\|v\| = \inf\{\lambda : v/\lambda \in B\} = \max(\alpha, \beta) = \max(\alpha, \|\beta u_1\|).$$

Since $U \subseteq U'$, the result follows. □

**Lemma 48.** *Suppose that $x$ is an $\ell_\infty$-direction with corresponding subspace $W$. Then $x$ is an extreme line direction and the maximal well-spanned subspace $U$ is contained in $W$.*

*Proof.* Let $B_W = B \cap W$ be the unit ball in $W$. We claim that $B = \operatorname{conv}(B_W + x, B_W - x)$. Suppose $v \in V$. Then, since $x$ is an $\ell_\infty$-direction, we can write $v = w + \lambda x$, and we have $\|v\| = \|w + \lambda x\| = \max(\|w\|, |\lambda|)$. This implies that, if $\|v\| \leqslant 1$, then $v$ is a convex combination of $w + x$ and $w - x$, for some $w \in B_W$, that is $B \subseteq \operatorname{conv}(B_W + x, B_W - x)$; conversely, it implies that $B_W + x \subset B$ and $B_W - x \subset B$, so $\operatorname{conv}(B_W + x, B_W - x) \subseteq B$. This completes the proof of the claim.

It is immediate that $x$ is an extreme line direction; indeed, for any $w \in \operatorname{Ext}(B_W)$, $[w - x, w + x]$ is an extreme line.

We also see that $\operatorname{Ext}(B) \subset (W + x) \cup (W - x)$, so, in particular, the continuous subspace $U_0$ is contained in $W$. Moreover, the only extreme line direction outside $W$ is $x$. Indeed, suppose $[y_1, y_2]$ is an extreme line. If $y_1, y_2$ are both contained in $W + x$ or both in $W - x$, then $y_2 - y_1 \in W$. Thus assume $y_1 \in W - x$ and $y_2 \in W + x$. Write $y_1 = z_1 - x$ and $y_2 = z_2 + x$, so $z_1, z_2 \in B_W$. The point $\frac{1}{2}(z_1 + z_2) \in [y_1, y_2]$ is a convex combination of $z_1 + x, z_2 - x$. Since $[y_1, y_2]$ is an extreme line, this shows that $z_1 + x, z_2 - x \in [y_1, y_2]$, and thus $z_1 = z_2$ and $y_2 - y_1 = 2x$. It follows that the extreme line $[y_1, y_2]$ has direction $x$.

In particular, this shows that $x$ is not a linear combination of other extreme line directions over $U_0$, which implies that $x$ is not in any well-spanned subspace. Moreover, since all other extreme line directions and the continuous subspace lie in $W$, we see that the maximal well-spanned subspace is contained in $W$. □

Finally, we show that the well-spanned subspace is actually the non-$\ell_\infty^d$-component in the $\ell_\infty$-decomposition.

**Proposition 49.** *Suppose that $V$ is a normed space with $\ell_\infty$-decomposition $(W \oplus \ell_\infty^d)_\infty$ and that $U$ is the maximal well-spanned subspace. Then $U = W$.*

*Proof.* Let $u_1, u_2, \ldots, u_k$ be the extreme line directions outside $U$, and let $U' = \langle u_1, u_2, \ldots, u_k \rangle$. Let $w_1, w_2, \ldots, w_d$ be all the $\ell_\infty$-directions, with corresponding subspaces $W_i$, and let $W' = \langle w_1, w_2, \ldots, w_d\rangle$ (so $W'$ is the $\ell_\infty^d$-component in the $\ell_\infty$-decomposition).

By Lemma 47, each $u_i$ is an $\ell_\infty$-direction, so $U' \subseteq W'$. Also, by Lemma 48, $U \subset W_i$ for each $i$, so $U \subset \bigcap_{i=1}^d W_i = W$ (Proposition 14). Since the sum $V = W \oplus W'$ is direct, we must have $U = W$ (and $U' = W'$). □

## 10. Proof of Theorem 4

Finally, we are in a position to prove Theorem 4. We prove it first for the case when $f$ fixes $\Lambda$ pointwise.

**Lemma 50.** *Suppose that $V$ is a normed space with $\ell_\infty$-decomposition $V = (U \oplus \ell_\infty^d)_\infty$ and that $f$ is a step-isometry fixing $\Lambda$ pointwise. Then $f$ factorises over the decomposition as $f_U \oplus f_{\ell_\infty^d}$, where $f_U$ is the identity on $U$ and $f_{\ell_\infty^d}$ is a step-isometry on $\ell_\infty^d$.*

*Proof.* Let $f_U$ be the identity on $U$, and define $f_{\ell_\infty^d} = f|_{\ell_\infty^d}$. We show that this is a factorisation of $f$ over the decomposition $U \oplus \ell_\infty^d$. By Proposition 49, $U$ is the maximal well-spanned subspace, so by Corollary 44 it is strongly fixed by $f$. Thus $f = f_U \oplus f_{\ell_\infty^d}$. Obviously, $f_U$ maps $U$ to itself, so it remains to show that $f_{\ell_\infty^d}$ maps $\ell_\infty^d$ to itself.

Let $v_1, v_2, \ldots, v_d$ be the $\ell_\infty$-directions (that is, the natural basis of the $\ell_\infty^d$-component). By Lemma 48, each $v_i$ is an extreme line direction, so by Proposition 40 it is a preserved direction. Suppose that $v = \sum_{i=1}^d \lambda_i v_i$. Then, inductively using the fact that each $v_i$ is a preserved direction, we have

$$f_{\ell_\infty^d}(v) = f(v) = f(\sum_{i=1}^d \lambda_i v_i) = \sum_{i=1}^d \lambda_i' v_i$$

for some $\lambda_i'$, and thus $f_{\ell_\infty^d}$ does map $\ell_\infty^d$ to itself.

It is easy to see that the factors in any factorisation of a bijection are also bijections. Thus, since $f_{\ell_\infty^d}$ is just the restriction of $f$ to $\ell_\infty^d$, we see that $f_{\ell_\infty^d}$ is a step-isometry as claimed. □

*Proof of Theorem* 4. We have that $f$ is any step-isometry on $V$. Define $\hat{f} = f - f(0)$. Then $\hat{f}$ is a step-isometry that fixes zero. By Corollary 38 there is a linear isometry $Q$ of $V$ such that $Q \circ \hat{f}$ is a step-isometry fixing $\Lambda$. Let $g = Q \circ \hat{f}$.

By Lemma 50, $g$ factorises over the $\ell_\infty$-decomposition $V = (U \oplus \ell_\infty^d)_\infty$ as $g_U \oplus g_{\ell_\infty^d}$, where $g_U$ is the identity on $U$ and $g_{\ell_\infty^d}$ is a step-isometry on $\ell_\infty^d$.

Obviously, $Q^{-1}$ is a linear isometry of $V$, so by Corollary 15 it factorises as $q_U \oplus q_{\ell_\infty^d}$ over $U \oplus \ell_\infty^d$ and is an isometry on each part. Note that $q_u$ and $q_{\ell_\infty^d}$ are both bijective (either immediate from linearity, or from Corollary 7).

Define $f_U = q_U \circ g_U$ and $f_{\ell_\infty^d} = q_{\ell_\infty^d} \circ g_{\ell_\infty^d}$. By definition, $f_U$ maps $U$ to itself isometrically and $f_{\ell_\infty^d}$ maps $\ell_\infty^d$ to itself as a step-isometry. Furthermore,

$$\begin{aligned} f(u+w) &= Q^{-1}(g(u+w)) = Q^{-1}(g_U(u) + g_{\ell_\infty^d}(w)) \\ &= q_u(g_U(u)) + q_{\ell_\infty^d}(g_{\ell_\infty^d}(w)) \\ &= f_u(u) + f_{\ell_\infty^d}(w). \end{aligned}$$

Hence, $f = f_U \oplus f_{\ell_\infty^d}$ is a factorisation of $f$ over $V = U \oplus \ell_\infty^d$. This completes the proof. □

## 11. Proof of Theorem 5

In this section we use the results we have proved to deduce Theorem 5. We prove it first for the case $d = 1$; that is, when $V = \mathbb{R}$.

**Lemma 51.** *Suppose $f$ is a step isometry of $\mathbb{R}$. Then there exists an isometry $Q$ of $\mathbb{R}$ and a continuous increasing bijection $g\colon [0,1) \to [0,1)$ such that*

$$Q \circ f(x) = \lfloor x \rfloor + g(x - \lfloor x \rfloor).$$

*Proof.* Trivially, the lattice $\Lambda$ generated by the unit ball is just the set $\mathbb{Z}$. Thus, by Corollary 38, there exists an isometry $Q$ such that $Q \circ f$ fixes $\mathbb{Z}$, and by Corollary 34

$$Q \circ f(x + k) = Q \circ f(x) + k \tag{1}$$

for any $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Let $\hat{f} = Q \circ f$. Since $\hat{f}$ is a step-isometry and fixes both 0 and 1, it must map $(0,1)$ to $(0,1)$, as must $\hat{f}^{-1}$. Hence, defining $g = \hat{f}|_{[0,1)}$, we see that $g$ maps $[0,1)$ to $[0,1)$ bijectively. From (1) we see that

$$Q \circ f(x) = \lfloor x \rfloor + g(x - \lfloor x \rfloor).$$

It is immediate that $g$ is continuous (it a restriction of the continuous function $\hat{f}$), so, to complete the proof, we just need to show that $g$ is increasing. Suppose that $0 \leqslant x < y < 1$. Pick $z \in (1+x, 1+y)$. We showed above that $\hat{f}$ maps $(0,1)$ to itself and similarly it also maps $(1,2)$ to itself; in particular, $\hat{f}(z) \in (1,2)$. Thus, since $\hat{f}$ is a step-isometry, we have $\hat{f}(z) > 1 + \hat{f}(x) = 1 + g(x)$ and $\hat{f}(z) < 1 + \hat{f}(y) = 1 + g(y)$, which shows $g(x) < g(y)$, as claimed. ☐

*Proof of Theorem* 5. By Corollary 38 there is an isometry $Q$ such that $Q \circ f$ is a step-isometry fixing $\Lambda$ the lattice generated by the extreme points of $B$.

By Proposition 40, the step-isometry $Q \circ f$ preserves extreme line directions. It is obvious that the points $\sum_{i=1}^{d} e_i$ and $\sum_{i=1}^{d} e_i - 2e_j$ are endpoints of an extreme line with direction $e_j$. Thus, each coordinate direction $e_j$ is preserved, and we see that $Q \circ f$ decomposes into independent actions on each coordinate direction. Each of these has the form specified by Lemma 51. Since $Q$ has the form given by Corollary 16, the result follows. ☐

## 12. The back and forth method in our setting

A standard technique for proving infinite graphs are isomorphic is the *back and forth method*. As we shall use it several times in the proof of Theorem 2 from Theorem 4, we collect precisely what we need here. We need the following notation. For graphs $G$ and $H$ we write $G \cong H$ to denote that $G$ and $H$ are isomorphic graphs. For any subset $S_0$ of $V(G)$, we write $G[S_0]$ for the (induced) subgraph of $G$ restricted to $S_0$.

**Lemma 52.** *Let $V = (U \oplus \mathbb{R})_\infty$, and let $S_U$ be a countable dense subset of $U$. Suppose that $S$ is a countable dense subset of $V$ such that, for each $s \in S_U$, $S \cap (\{s\} \times \mathbb{R})$ is dense in $\{s\} \times \mathbb{R}$, and no two points in $S$ differ by an integer in their $\mathbb{R}$-component. Then $S$ is Rado.*

*Further, suppose $S_0$ is any finite set of points in $V$ with no two points, one from $S$ and one from $S_0$, differing by an integer in their $\mathbb{R}$-components. Then, for two graphs $G, G'$ in $\mathcal{G}_p(S \cup S_0)$, we have*

$$\mathbb{P}(G \cong G' \mid G[S_0] = G'[S_0]) = 1.$$

*Remark.* Note, we do not require this to be the $\ell_\infty$-decomposition; for example, it also holds for $V = \ell_\infty^d = (\ell_\infty^{d-1} \oplus \mathbb{R})_\infty$ itself. Indeed, we do not even need $U$ to be nontrivial; that is, it holds when $V = \mathbb{R}$.

*Proof.* We start by showing that almost all graphs $G$ in $\mathcal{G}_p(V, S)$ have the following property $P$: for every point $s' \in S_U$, every open subset $A$ of $\mathbb{R}$, and every pair of disjoint finite sets $T_1, T_2 \subset S$ such that $\{s'\} \times A \subset \bigcap_{x \in T_1 \cup T_2} B^\circ(x, 1)$, there exist infinitely many $s \in (\{s'\} \times A) \cap S$ such that $st \in E(G)$ for all $t \in T_1$ and $st \notin E(G)$ for all $t \in T_2$.

It is obviously sufficient to prove the claim for all open sets in any base for $\mathbb{R}$. In particular, if we take a countable base, there are only countably many choices for $A$, $s'$, $T_1$, and $T_2$. For each choice there are infinitely many points in $(\{s'\} \times A) \cap S$. Since, each of these points has distance strictly less than one to each point of $T_1 \cup T_2$, each such point has a positive probability of having the required adjacency. Thus, almost surely, infinitely many of them do have the required adjacency. The claim follows.

To complete the proof, we show that if $G$ and $G'$ are two graphs in $\mathcal{G}_p(V, S)$ both having property $P$, then $G$ and $G'$ are isomorphic.

Indeed, we construct our isomorphism guaranteeing that it factorises over $U \oplus \mathbb{R}$ as $f_U \oplus f_\mathbb{R}$ and that $f_U$ is actually the identity on $U$. In other words $f(u + w) = u + f_\mathbb{R}(w)$. Further, we insist that $f_\mathbb{R}$ is monotone and satisfies

$$f_\mathbb{R} = \lfloor x \rfloor + f_\mathbb{R}(x - \lfloor x \rfloor) \tag{2}$$

(in fact this is essentially forced if $f_\mathbb{R}$ is to be a step-isometry).

For the rest of the proof fix an enumeration $s_1, s_2, s_3, \ldots$ of $S$. We use the back and forth method to construct the desired isomorphism. Start the process by mapping $s_1 = u_1 + w_1$ to itself. In particular, this defines $f_\mathbb{R}(w_1) = w_1$, so, by our requirement on $f_\mathbb{R}$, this defines $f_\mathbb{R}$ on $w_1 + \mathbb{Z}$ by $f_\mathbb{R}(w_1 + k) = w_1 + k$, for all $k \in \mathbb{Z}$.

Suppose that $v = u + w$ is the first point in the enumeration for which $f$ has not already been defined and that $v_i = u_i + w_i$, for $1 \leqslant i \leqslant n$, are the points for which $f$ has already been defined. Let $v_i' = f(v_i)$ for each $i$. Consider the set of points for which we have already defined $f_\mathbb{R}$, namely $\bigcup_i (w_i + \mathbb{Z})$. The point $w$ must lie between two consecutive points of this set, say $x$ and $y$. (It is not one of these points since we have assumed there are no two points differ by an integer in their $\mathbb{R}$-component.)

Let $A$ be the open interval $(f_\mathbb{R}(x), f_\mathbb{R}(y))$, and let $T$ be the subset of the $v_i'$ that have distance strictly less than one from any point (equivalently, all points) of $\{u\} \times A$, and partition $T$ into $T_1$ and $T_2$ according to whether $v_i$ is joined to $v$ in $G$ or not.

Since $G'$ has property $P$, there are infinitely many points $v' \in \{u\} \times A$ which are joined to everything in $T_1$ and nothing in $T_2$. Let $v'$ be any such point that has not already been used (i.e., not in $\{v_1', v_2', \ldots, v_n'\}$), and set $f(v) = v'$. Let $w' \in \mathbb{R}$ be the $W$-component of $v'$ (so $v' = u + w'$). By our requirement on $f_\mathbb{R}$ this defines $f_\mathbb{R}$ on $w + \mathbb{Z}$ by $f_\mathbb{R}(w + k) = w' + k$, for all $k \in \mathbb{Z}$. By our choice of $v'$ we see that $f_\mathbb{R}$ is still a monotone and increasing and satisfies (2).

We repeat this argument, but this time mapping from $G'$ to $G$. That is, we take the first point $v'$ in our enumeration of $S$ that is not one of the $v_i'$ and define $f^{-1}(v') = v$ for a suitable point $v$ found as above but working with $f^{-1}$.

Thus, since as we alternate back and forth, the process takes the first point not yet defined in $G$ or $G'$ at each stage, this process creates a bijection. Since we maintain the isomorphism and the step-isometry at each stage, this bijection is an isomorphism (and a step isometry), as claimed.

To prove the final part, just start the process with the map $f\colon S_0 \to S_0$ defined to be the identity which, since we are conditioning on $G[S_0] = G'[S_0]$, is an isomorphism. □

## 13. Proof of Theorems 1 and 2 from Theorem 4

In this section we prove Theorem 2 (which includes Theorem 1).

**Lemma 53.** *Let $V$ be a finite-dimensional normed space, and let $S$ be a countable dense set in $V$.*

1. *Suppose that there are only countably many step-isometries on $S$. Then $S$ is strongly non-Rado.*
2. *Instead, suppose that $S$ contains a subset $T$, which contains infinitely many pairs of points at distance less than one, and the step-isometries on $S$ induce only countably many distinct mappings of $T$. Then $S$ is strongly non-Rado.*

*Proof.* Obviously, the second statement gives the first statement, so it suffices to prove that.

Let $P$ be the property that, for every pair of points $x, y \in S$ and every $k \in \mathbb{N}$ with $k \geqslant 2$, we have $\|x - y\| < k$ if and only $d_G(x, y) \leqslant k$. Let $\mathcal{G}_0$ be the set of graphs for which property $P$ fails. By Lemma 3, $\mathcal{G}_0$ has measure zero. Any $G \notin \mathcal{G}_0$ can only be isomorphic to graphs in $\mathcal{G}_0$ or to a graph $f(G)$ where $f$ is step-isometry of $S$. Obviously, if $f$ is an isomorphism between $G$ and $G'$, then $f|_T$ is an isomorphism between $G[T]$ and $G'[f(T)]$. Since $T$ has infinitely many pairs of points at distance less than one, it has infinitely many potential edges, and the probability any particular mapping $f|_T$ is an isomorphism is zero. By hypothesis there are only countably many such mappings, so the probability that any such mapping is an isomorphism is zero.

We have shown that almost every graph $G$ is isomorphic to almost no other graphs. Thus, by Fubini's theorem, two independent random graphs are almost surely not isomorphic. (The event that two graphs are isomorphic, although not Borel, is product measurable because it is analytic; see e.g., [10].) □

Throughout the proof of Theorem 2 we shall use the $\ell_\infty$-decomposition. We make the following definition.

**Definition.** Suppose $V$ is a normed space with $\ell_\infty$-decomposition $V = (U \oplus \ell_\infty^d)_\infty$. Then, for any $u \in U$, the *fibre* over $u$ is the set $\{u + w : w \in \ell_\infty^d\}$.

*Proof of Theorem* 2(ii). Suppose $f$ is a step-isometry of $S$. By Proposition 24, $f$ extends to a step-isometry of $V$. Since $U = V$ in the $\ell_\infty$-decomposition, Theorem 4 shows that $f = f_U$ must be a (bijective) isometry on the whole of $V$. By the Mazur–Ulam Theorem this isometry is an affine map.

Let $S' \subset S$ be an affine basis of $V$ (an *affine basis* is a linear basis together with any one point not in its affine span). Then, the affine map $f$ is determined by its action on $S'$. Since $f$ maps $S$ to $S$, there are only countably many choices for the images of the points of $S'$. Hence, the number of such isometries is countable.

This shows that the number of step-isometries on $S$ is countable so, by Lemma 53, $S$ is strongly non-Rado. □

*Proof of Theorem* 2(iii). First suppose that no two (distinct) points $u+w, u'+w' \in S$ have $u = u'$ (that is, each fibre over $U$ contains zero or one point). Obviously, almost all countable dense sets have this property. Again, suppose that $f$ is a step-isometry of $S$. As before, it extends to a step-isometry of $V$. By Theorem 4, $f$ factorises as $f = f_U \oplus f_{\ell_\infty^d}$, where $f_U$ is a (bijective) isometry on $U$. Thus, by the Mazur–Ulam Theorem again, $f_U$ is an affine map.

Let $S' \subset S$ be a set of points $u_1 + w_1, u_2 + w_2, \ldots, u_k + w_k$, where $u_i \in U$ and $w_i \in \ell_\infty^d$ for each $i$, and $u_1, u_2, \ldots, u_k$ form an affine basis of $U$. The map $f_U$ is determined by its action on $u_1, u_2, \ldots, u_k$, so is determined by $f$'s action on $S'$. As in part (ii), $f$ maps $S$ to $S$ so there are only countably many choices for the images of the points of $S'$. Thus the number of possible $f_U$ is countable.

However, $f_U$ determines $f$ since, once we know the $U$-component of $f(s)$, the fact that $f(s) \in S$ determines the point uniquely (there may be no possible point, but that only helps us since it reduces the number of potential step-isometries). Hence, exactly as in the proof of part (ii), this means there are only countably many such step-isometries so, again by Lemma 53, $S$ is strongly non-Rado.

The fact that there are some sets $S$ that have atypical behaviour is immediate from Lemma 52. Indeed, write $V = (U' \oplus \mathbb{R})_\infty$, where $U' = (U \oplus \ell_\infty^{d-1})_\infty$, then any $S$ of the form required by that lemma is Rado. We remark that this construction also works in the case $V = \ell_\infty^d$, but it is not atypical there.

Since our construction of sets for which the probability the graphs are isomorphic has probability strictly between 0 and 1 works for both parts (i) and (iii) of the theorem, we defer it until after our proof of part (i). □

*Proof of Theorem* 2(i). The almost all statement of part (i) was proved by Bonato and Janssen. They showed that all countable dense sets that do not contain any two points differing by an integer in any coordinate are Rado. (In fact, they claimed the slightly stronger result that any set which does not contain two points an integer distance apart is Rado—but this is not true. Indeed, it is easy to construct counterexamples along the lines of the examples given in the next section.)

The following shows that there are countable dense sets $S$ which are strongly non-Rado. Let $S'$ be any countable dense set in $\mathbb{R}^{d-1}$. Let $S = S' \times \mathbb{Q}$ in $\mathbb{R}^d$, and fix $s' \in S'$. Suppose $f$ is a step-isometry mapping on $S$. As usual $f$ extends to a step-isometry of $V$. Consider the action of $f$ on the subset $T = \{s'\} \times (\mathbb{Z} \cup \mathbb{Z} + \frac{1}{2})$ of the fibre $\{s'\} \times \mathbb{Q}$. By Theorem 5 we see that this action is determined by the permutation $\sigma$ of the basis vectors, the vector $\varepsilon$ of signs, together with the images $f(s', 0)$ and $f(s', 1/2)$. Since $f(s', 0, ), f(s', 1/2) \in S$, there are only countably many choices for the step-isometry's action on $T$. Thus, since $T$ contains infinitely many pairs of points with distance less than one, Lemma 53 shows that $S$ is strongly non-Rado.

We deal with the case of sets where the probability that two graphs are isomorphic is strictly between zero and one in the following proposition. □

Finally, we complete the proof of Theorem 2 by proving that there exist sets which are neither Rado nor strongly non-Rado; that is, sets $S$ for which the probability two graphs are isomorphic lies strictly between zero and one.

**Proposition 54.** *Let $V = (U \oplus \mathbb{R})_\infty$. Then there exist countable dense sets $S$ such that the probability that two random graphs taken from $\mathcal{G}_p(V, S)$ are isomorphic lies strictly between zero and one.*

*Remark.* Again, we do not require this to be the $\ell_\infty$-decomposition; for example, it holds for $V = \ell_\infty^d = (\ell_\infty^{d-1} \oplus \mathbb{R})_\infty$ and for $V = \mathbb{R}$.

*Proof.* The key idea is to find a set $S$ with some finite subset $S_0$ such that all step-isometries map $S_0$ to $S_0$. If we do this, then an obvious necessary condition for two graphs $G$ and $G'$ to be isomorphic via a step-isometry is that $G[S_0]$ is isomorphic to $G'[S_0]$, which is an event with probability strictly between zero and one, provided $S_0$ contains at least one possible edge.

Of course, that is just a necessary condition; to find a set $S$ with the desired property, we wish to make this a sufficient condition for the existence of such an isomorphism.

One natural possibility is to let $S_0$ be two points that are the unique pair of points at unit distance in $S$. Since step-isometries preserve integer distances, any step-isometry must map $S_0$ to $S_0$. However, $S_0$ does not contain any potential edge. Instead, fix a unit vector $u$, and let $S_0 = \{0, u, 3u/2, 5u/2\}$. Provided $0, u$ and $3u/2, 5u/2$ are the only pairs of points at unit distance in $S$, then $S_0$ must map to itself. Moreover, $S_0$ contains a unique possible edge (that is a unique pair of vertices at distance strictly less than one)—that between the points $u$ and $3u/2$—and we see that any step-isometry must map these two points to themselves.

Having found our set $S_0$, we turn to defining $S$, which we do as in Lemma 52—we just add the requirements that no point of $S$ is at unit distance from any point in $S \cup S_0$.

As discussed above, all step isometries map the set $\{u, 3u/2\}$ to itself and, in particular, a necessary condition for $G$ and $G'$ to be isomorphic via a step-isometry is that they agree on the potential edge $u, 3u/2$. (As Lemma 3 shows that the probability two graphs are isomorphic via a function which is not a step-isometry is zero, we can ignore this possibility.)

Conversely, if they agree on this edge, then $G[S_0] = G'[S_0]$ so, by Lemma 52 they are almost surely isomorphic.

Thus, the probability that $G$ and $G'$ are isomorphic is the probability that they agree on the edge $u, 3u/2$ which is $p^2 + (1-p)^2$; in particular it is strictly between zero and one. □

## 14. Further results and open problems

We have not completely classified the behaviour of all countable dense sets in the cases (i) and (iii) above, and that is our main open question.

**Question 1.** Let $V$ be a normed space with $\ell_\infty$-decomposition $V = (U \oplus \ell_\infty^d)_\infty$ for some $d \geqslant 1$. Which countable dense sets are Rado?

It is easy to extend the argument for the typical case of part (iii) above to show that, in that setting, if each fibre over $U$ contains a discrete set (rather than just zero or one points as above), then the set is strongly non-Rado. Thus, the open cases include cases where a fibre is neither dense nor discrete.

However, since the behaviour when all fibres are discrete (strongly non-Rado) is different from the case when all fibres are dense (Rado—assuming some no integer difference conditions) it is unsurprising that sets with some fibres discrete and some fibres dense can give either behaviour. We briefly outline two sets which look very similar but have different behaviour. The examples we give are in in $V = (U \oplus \mathbb{R})_\infty$, but it is easy to generalise them to either $(U \oplus \ell_\infty^d)_\infty$ or (with slightly more effort along the lines of the proof of the atypical case of part (i) above) to $\ell_\infty^d$.

Let $S_U$ be a dense set in $U$ and let $S$ be a set which is dense in each fibre over $S_U$ and contains no two points differing by an integer in their $\mathbb{R}$-component. (So far this is exactly the set used in the atypical case of part (iii) above.)

Now let $T_U$ be an infinite 1-separated family in $U$ disjoint from $S_U$, and let $T$ be a set containing exactly one point from each fibre over $T_U$, such that no two points in $S \cup T$ differ by an integer in their $\mathbb{R}$-component.

We claim that, by choosing the single points in each fibre of $T$, we can ensure that $S \cup T$ is Rado, or that it is strongly non-Rado. Suppose that $T$ is the set $\{(t_1, r_1), (t_2, r_2), \dots\}$.

As usual, any step-isometry $f$ of $S \cup T$ extends to a step-isometry of $V$, which factorises as $f_U \oplus f_\mathbb{R}$, where $f_U$ is an isometry and $f_\mathbb{R}$ is a step-isometry. Obviously, $f_U$ maps $T$ to itself (as all other fibres contain either no points or infinitely many points). Thus, once we know the $U$-component of the image $f(t)$ of a point $t \in T$, we know its $\mathbb{R}$-component; that is, $f_U$ determines $f_\mathbb{R}(r_i)$ for each $i$. If the $r_i \mod 1$ are dense in $[0, 1]$, then, since $f_\mathbb{R}$ is a step-isometry, this determines $f_\mathbb{R}$ entirely. As in our proofs above there are only countably many step-isometries mapping $S \cup T$ to itself, so by Lemma 53 $S \cup T$ is strongly non-Rado.

On the other hand if the $r_n = n + 1/n$ and no point of $S$ has integer $\mathbb{R}$-component, then $S \cup T$ is Rado. Indeed, we construct our map fixing $U$ and use the back and forth argument as in Lemma 52, observing that the key property used there—that for every point $(u, w) \in S \cup T$ not yet mapped, the point $w$ lies in an open interval between consecutive previously defined points—still holds in this case.

The above discussion shows that the classification of exactly which countable dense sets give a unique graph will be rather complicated. Thus we have restricted ourselves to the typical case and have shown that the atypical cases can occur.

Finally, all our work in this paper has been finite-dimensional spaces without consideration for the infinite-dimensional setting. It would be interesting to know what happens there.

**Question 2.** Suppose that $V$ is an infinite-dimensional normed space and that $S$ is a countable dense subset. When is $S$ Rado?

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