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ALGORITHMIC ASPECTS OF BRANCHED COVERINGS IV/V. EXPANDING MAPS

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ABSTRACT. Thurston maps are branched self-coverings of the sphere whose critical points have finite forward orbits. We give combinatorial and algebraic characterizations of Thurston maps that are isotopic to expanding maps as *Levy-free* maps (maps without Levy obstruction) and as maps with *contracting biset*.

We prove that every Thurston map decomposes along a unique minimal multicurve into homeomorphisms and Levy-free maps, and this decomposition is algorithmically computable. Each of these pieces admits a geometric structure.

We apply these results to matings of postcritically finite polynomials, extending a criterion by Mary Rees and Tan Lei: they are expanding if and only if they do not admit a cycle of periodic rays.

1. Introduction

Let $f\colon (S^2, A) \circlearrowleft$ be a branched covering of the sphere with finite, forward-invariant set A containing f's critical values, from now on called a *Thurston map*. A celebrated theorem by Thurston [8] gives a topological criterion for f to be isotopic to a rational map, for an appropriate complex structure on (S^2, A). One of the virtues of rational maps, following from Schwartz's lemma, is that they are expanding for the hyperbolic metric of curvature -1 associated with the complex structure.

In this article, following the announcement in [2], we give a criterion for f to be isotopic to an expanding map, namely for there to exist a metric on (S^2, A) that is expanded by a map isotopic to f. It will turn out that the metric may, for free, be required to be Riemannian of pinched negative curvature.

Some care is needed to define expanding maps with periodic critical points. Consider a noninvertible map $f\colon (S^2, A) \circlearrowleft$. Let $A^\infty \subseteq A$ denote the forward orbit of the periodic critical points of f. The map f is *metrically expanding* if there exists a subset $A' \subseteq A^\infty$ and a metric on $S^2 \setminus A'$ that is expanded by f, and such that at all $a \in A'$ the first return map of f is locally conjugate to $z \mapsto z^{\deg_a(f^n)}$. In other words, the points in A' are cusps or, equivalently at infinite distance, in the metric.

We call f *Böttcher expanding* if $A' = A^\infty$. This definition is designed to generalize the class of rational maps. Indeed, every postcritically finite rational map $f\colon (\widehat{\mathbb{C}}, A) \circlearrowleft$ is Böttcher expanding by considering the hyperbolic (or Euclidean if $|A| = 2$) metric of $(\widehat{\mathbb{C}}, \mathrm{ord})$ for an appropriate orbifold structure $\mathrm{ord}\colon A \to \mathbb{N} \cup \{\infty\}$.

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We call f *topologically expanding* if there exists a compact retract $\mathcal{M} \subset S^2 \setminus A'$ and a finite open covering $\mathcal{M} = \bigcup \mathcal{U}_i$ such that connected components of $f^{-n}(\mathcal{U}_i)$ get arbitrarily small as $n \to \infty$ and such that $S^2 \setminus \mathcal{M}$ is in the immediate attracting basin of A'; see [1]. If $A^\infty = \emptyset$, Böttcher expanding maps are the everywhere-expanding maps considered, e.g., in [6, 11].

An obstruction to topological expansion is the existence of a *Levy cycle*. This is an essential simple closed curve on $S^2 \setminus A$ that is isotopic to some iterated preimage of itself. We shall see that it is the only obstruction.

We recall briefly the algebraic encoding of branched coverings: given $f\colon (S^2, A) \circlearrowleft$, set $G := \pi_1(S^2 \setminus A, *)$ and define

$$B(f) := \{\gamma\colon [0,1] \to \mathcal{M} \mid \gamma(0) = f(\gamma(1)) = *\} \,/\, \text{homotopy}.$$

This is a set with commuting left and right G-actions; see §3 to which we refer for the definition of *contracting* bisets. Two branched self-coverings $f_0\colon (S^2, A_0) \circlearrowleft$ and $f_1\colon (S^2, A_1) \circlearrowleft$ are *combinatorially equivalent* if there is a path $(f_t\colon (S^2, A_t) \circlearrowleft)_{t\in[0,1]}$ of branched self-coverings joining them; this happens precisely when the bisets $B(f_0)$ and $B(f_1)$ are isomorphic in a suitably defined sense (see [14] and [4]). The main result of this part is the following criterion; equivalence of (2) and (3) was known in the case $A^\infty = \emptyset$ from [12, Theorem 4].

Theorem A (= Theorem 4.4). *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map, not doubly covered by a torus endomorphism. The following are equivalent:*

1. *f is combinatorially equivalent to a Böttcher expanding map, for an appropriate metric on the sphere;*
2. *f is combinatorially equivalent to a topologically expanding map;*
3. *$B(f)$ is an orbisphere contracting biset;*
4. *f is noninvertible and admits no Levy cycle.*

Furthermore, if these properties hold, the metric in (1) *may be assumed to be Riemannian of pinched negative curvature.*

Haïssinsky and Pilgrim ask in [12] whether every everywhere-expanding map is isotopic to a map that is smooth except at a finite set of points. By Theorem A, a combinatorial equivalence class contains a smooth Böttcher expanding map if and only if it is Levy free. We therefore answer positively their question in the case $A^\infty = \emptyset$, because then everywhere-expanding = Böttcher expanding ⇒ Levy-free ⇒ smooth Böttcher expanding = everywhere-expanding smooth except at A.

1.1. Geometric maps and decidability.

Let us define a {GTor/2} map as a non-invertible self-map of the sphere S^2 that is a quotient of a torus endomorphism $Mz+v\colon \mathbb{R}^2 \circlearrowleft$ by the involution $z \mapsto -z$ such that the eigenvalues of M are different from ± 1. Let us call a Thurston map *geometric* if it is either Böttcher expanding or {GTor/2}.

Recall from [4] that $R(f, A, \mathscr{C})$ denotes the small return maps of the decomposition of a Thurston map f under an invariant multicurve $\mathscr{C}$. The *canonical Levy obstruction* $\mathscr{C}_{\text{Levy}}$ of a Thurston map $f\colon (S^2, A) \circlearrowleft$ is a minimal f-invariant multicurve all of whose small Thurston maps are either homeomorphisms or admit no Levy cycle. It is unique by Proposition 2.7. The *Levy decomposition* of f (and equivalently of its biset) is its decomposition (as a graph of bisets) along the canonical Levy obstruction. It was proven in [20, Main Theorem II] that every Levy-free map that is doubly covered by a torus endomorphism is in {GTor/2}. Combined with Theorem A, this implies the following corollary.

Corollary B. *Let $f\colon (S^2,A) \circlearrowleft$ be a Thurston map. Then every map in $R(f,A,\mathscr{C}_{Levy})$ is either geometric or a homeomorphism.* □

The following consequences are essential for the decidability of combinatorial equivalence of Thurston maps.

Corollary C (= Algorithms 5.4 and 5.5)**.** *There is an algorithm that, given a Thurston map by its biset, decides whether it is geometric.*

As a consequence we have the following.

Corollary D (= Algorithm 5.6)**.** *Let f be a Thurston map. Then its Levy decomposition is symbolically computable.*

There may exist expanding maps in the combinatorial equivalence class of a Thurston map that are not Böttcher expanding. However, every expanding map is a quotient of a Böttcher expanding map by Theorem A combined with the following.

Proposition 1.1 (= Proposition 4.18)**.** *Let $f,g\colon (S^2,A) \circlearrowleft$ be isotopic Thurston maps, let $\mathcal{F}(f),\mathcal{F}(g)$ be their respective Fatou sets (see* §4.2*), and assume $A\cap(\mathcal{F}(g)\setminus \mathcal{F}(f)) = \emptyset$. Then there is a semiconjugacy from f to g, defined by collapsing to points those components of $\mathcal{F}(f)$ that are attracted towards $A\cap(\mathcal{F}(f)\setminus\mathcal{F}(g))$ under f.*

We will show in [5] that the semiconjugacy is unique.

We deduce the following extension of a classical result for rational maps (see, e.g., [8, Corollary 3.4(b)]) to Böttcher expanding maps.

Corollary 1.2 (= Lemma 4.15)**.** *Let f,g be Böttcher expanding Thurston maps. Then f and g are combinatorially equivalent if and only if they are conjugate.*

We also characterize maps (such as rational maps with Julia set a Sierpiński carpet) that are isotopic to an everywhere-expanding map. A *Levy arc* for a Thurston map $f\colon (S^2,A) \circlearrowleft$ is a nontrivial path with endpoints in A that is isotopic to an iterated lift of itself.

Proposition 1.3 (= Corollary 4.17 with $A = A'\cap\mathcal{F}(f)$)**.** *Consider a Thurston map f that is not doubly covered by a torus endomorphism. Then f is isotopic to an everywhere-expanding map if and only if f admits no Levy obstruction or Levy arc.*

1.2. **Matings and amalgams.** We finally apply Theorem A to the study of matings, and more generally to amalgams of expanding maps. We state the results for matings in this introduction, while §6 will discuss the general case of amalgams.

Let $p_+(z) = z^d + \cdots$ and $p_-(z) = z^d + \cdots$ be two postcritically finite monic polynomials of the same degree. Denote by $\overline{\mathbb{C}}$ the compactification of $\mathbb{C}$ by a circle at infinity $\{\infty\exp(2\pi i\theta)\}$, and consider the sphere

$$\mathbb{S} := (\overline{\mathbb{C}}\times\{\pm1\})\,/\,\{(\infty\exp(2\pi i\theta),+1)\sim(\infty\exp(-2\pi i\theta),-1)\}.$$

(Note the reversed orientation between the two copies of $\overline{\mathbb{C}}$.) The *formal mating*

$$p_+ \uplus p_- : \mathbb{S} \circlearrowleft, \quad (z,\varepsilon)\mapsto(p_\varepsilon(z),\varepsilon) \tag{1}$$

is the branched covering of $\mathbb{S}$ acting as p_+ on its northern hemisphere, as p_- on its southern hemisphere, and as z^d on the common equator $\{\infty\exp(2i\pi\theta)\}$. The maps p_+, p_- glue continuously by Lemma 4.7.

We recall the definition of *external rays* associated to the polynomials $p_\pm$. For a polynomial p, the *filled-in Julia set* $\mathcal{K}(p)$ of p is

$$\mathcal{K}(p) = \{z \in \mathbb{C} \mid p^n(z) \not\to \infty \text{ as } n \to \infty\}.$$

Assume that $\mathcal{K}(p)$ is connected. Then there exists a unique holomorphic isomorphism $\phi_p\colon \widehat{\mathbb{C}} \setminus \mathcal{K}(p) \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ satisfying $\phi_p(p(z)) = \phi_p(z)^d$ and $\phi_p(\infty) = \infty$ and $\phi_p'(\infty) = 1$. It is called a *Böttcher coördinate* and conjugates p to z^d in a neighbourhood of ∞. For $\theta \in \mathbb{R}/\mathbb{Z}$, the associated *external ray* is

$$R_p(\theta) = \{\phi_p^{-1}(re^{2i\pi\theta}) \mid r > 1\}.$$

Let Σ denote the quotient of $\mathbb{S}$ in which each $\overline{(R_{p_\varepsilon}(\theta), \varepsilon)}$ has been identified to one point for each $\theta \in \mathbb{R}/\mathbb{Z}$ and each $\varepsilon \in \{\pm 1\}$. Note that Σ is a quotient of $\mathcal{K}(p_+) \sqcup \mathcal{K}(p_-)$ and need not be a Hausdorff space. A classical criterion (due to Moore) determines when Σ is homeomorphic to S^2. If this occurs, p_+ and p_- are said to be *topologically mateable*, and the map induced by $p_+ \uplus p_-$ on Σ is called the *topological mating* of p_+ and p_- and is denoted $p_+ \amalg p_- : \Sigma \circlearrowleft$.

Definition 1.4. Let p_+, p_- be two monic postcritically finite polynomials of the same degree d. We say that p_+, p_- have a *pinching cycle of periodic angles* if there are angles $\phi_0, \phi_1, \dots, \phi_{2n-1} \in \mathbb{Q}/\mathbb{Z}$ with denominators coprime to d such that for all $\varepsilon = \pm 1$ and all $i = 0, \dots, 2n-1$, with indices treated modulo $2n$, the rays $R_{p_\varepsilon}(\varepsilon\phi_{2i})$ and $R_{p_\varepsilon}(\varepsilon\phi_{2i+\varepsilon})$ land together. △

We give a computable criterion for two hyperbolic polynomials to be mateable, which extends a well-known criterion "two quadratic polynomials are geometrically mateable if and only if they do not belong to conjugate primary limbs in the Mandelbrot set" due to Mary Rees and Tan Lei; see [22] and [7, Theorem 2.1].

Theorem E. *Let p_+, p_- be two monic hyperbolic postcritically finite polynomials. Then the following are equivalent:*

(1) *$p_+ \uplus p_-$ is combinatorially equivalent to an expanding map;*
(2) *$p_+ \amalg p_-$ is a sphere map (necessarily conjugate to any expanding map in* (1)*);*
(3) *p_+, p_- do not have a pinching cycle of periodic angles.*

To be more precise, the criterion due to Mary Rees and Tan Lei relies on the fact that, in degree 2, every Thurston obstruction is a Levy obstruction, so every expanding map is automatically conjugate to a rational map. In degree ≥ 3 there are topological matings that are not conjugate to rational maps: the example in [21] is precisely such a mating with an obstruction but no Levy obstruction, and it is isotopic to an expanding map.

Furthermore, in degree 2 every decomposition of a Thurston map along a Levy cycle has a fixed sphere or cylinder which maps to itself by a homeomorphism cyclically permuting the boundary components (namely, there exists a "good Levy cycle"). This implies that obstructed maps have a pinching cycle of periodic angles with $n = 2$. In Example 6.7, we show that this does not hold in higher degree.

1.3. **Notation.** Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map with an invariant multicurve $\mathscr{C}$. Recall that by $R(f, A, \mathscr{C})$ we denote the return maps induced by f on $S^2 \setminus \mathscr{C}$; see [4, §4.6].

We introduce the following notation. By default, curves and multicurves are considered up to isotopy rel the marked points; we use the terminology "equal" to mean that. In particular, a cycle of curves is really a sequence of curves that are mapped cyclically to each other, up to isotopy. If we want to insist that curves are equal and not just isotopic, we add the adjective "solid"; thus a solid cycle of curves is a sequence of curves mapped cyclically to each other, "on the nose".

We reserve the letters '$\mathscr{C}$' for invariant multicurves and 'C' for cycles of curves, or more generally for subsets of invariant multicurves.

2. Multicurves and the Levy decomposition

Let A be a finite subset of the topological sphere S^2 and consider simple closed curves on $S^2 \setminus A$. Recall that such a curve is *trivial* if it bounds a disc in $S^2 \setminus A$ and is *peripheral* if it may be homotoped into arbitrarily small neighbourhoods of A; otherwise, it is *essential.* A *multicurve* is a collection of mutually nonintersecting nonhomotopic essential simple closed curves. Following Harvey [13], we denote by $\mathcal{C}(S^2 \setminus A)$ the flag complex whose vertices are isotopy classes of essential curves, and a collection of curves belongs to a simplex if they have disjoint representatives; so multicurves on $S^2 \setminus A$ are naturally identified with simplices in $\mathcal{C}(S^2 \setminus A)$. (The empty multicurve corresponds to the empty simplex.)

Given two simple closed curves γ_1 and γ_2 on $S^2 \setminus A$, their *geometric intersection number* is defined as

$$i(\gamma_1, \gamma_2) = \min_{\gamma_1', \gamma_2'} \#(\gamma_1' \cap \gamma_2'),$$

with the minimum ranging over all curves γ_1' isotopic to γ_1 and γ_2' isotopic to γ_2. The simple closed curves γ_1 and γ_2 are in *minimal position* if $i(\gamma_1, \gamma_2) = \#(\gamma_1 \cap \gamma_2)$.

We say that two simple closed curves γ_1 and γ_2 *cross* if $i(\gamma_1, \gamma_2) > 0$. Clearly, if γ_1 and γ_2 are isotopic or one of them is inessential, then $i(\gamma_1, \gamma_2) = 0$. Two multicurves $\mathscr{C}_1$ and $\mathscr{C}_2$ *cross* if there are $\gamma_1 \in \mathscr{C}_1$ and $\gamma_2 \in \mathscr{C}_2$ that cross.

Proposition 2.1 (The Bigon criterion, [9, Proposition 1.3])**.** *Two transverse simple closed curves on a surface S are in minimal position if and only if the two arcs between any pair of intersection points never bound an embedded disc in S.* □

2.1. **Levy, anti-Levy, Cantor, and anti-Cantor multicurves.** Consider a Thurston map $f\colon (S^2, A) \circlearrowleft$. We construct the following directed graph: its vertex set is the set of essential simple closed curves on $S^2 \setminus A$, namely the vertex set of the curve complex $\mathcal{C}(S^2 \setminus A)$. For every simple closed curve γ and for every component δ of $f^{-1}(\gamma)$, we put an edge from γ to δ labeled $\deg(f{\restriction}_\delta)$. Note that the operation f^{-1} induces a map on the simplices of $\mathcal{C}(S^2 \setminus A)$, but not a simplicial map.

A multicurve $\mathscr{C} \in \mathcal{C}(S^2 \setminus A)$ is *invariant* if $f^{-1}(\mathscr{C}) = \mathscr{C}$. Given a multicurve $\mathscr{C}_0$ with $\mathscr{C}_0 \subseteq f^{-1}(\mathscr{C}_0)$, there is a unique invariant multicurve $\mathscr{C}$ *generated* by $\mathscr{C}_0$, namely the intersection of all invariant multicurves containing $\mathscr{C}_0$. The invariant multicurve $\mathscr{C}$ may readily be computed by considering $\mathscr{C}_0, f^{-1}(\mathscr{C}_0), f^{-2}(\mathscr{C}_0), \dots$; this is an ascending sequence of multicurves, and each multicurve contains at most $\#A - 3$ curves, so the sequence must stabilize.

Let $\mathscr{C}$ be an invariant multicurve and consider the corresponding directed subgraph of $\mathcal{C}(S^2 \setminus A)$ spanned by $\mathscr{C}$. A *strongly connected component* is a maximal

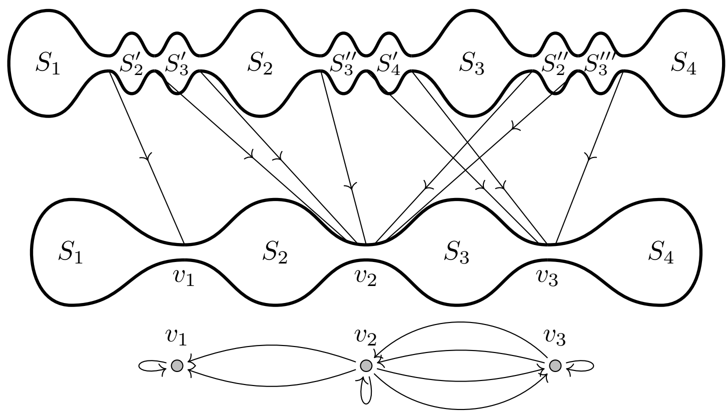


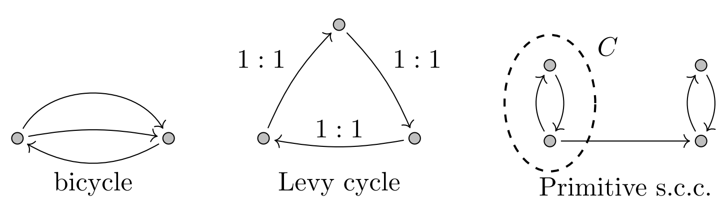
FIGURE 1. A bicycle $\{v_2, v_3\}$ generates a Cantor multicurve $\{v_1, v_2, v_3\}$. The action of the map f is indicated on the preimages of $\{v_1, v_2, v_3\}$. If annuli are mapped by degree 1, then it is also a Levy cycle. Trivial spheres are omitted on the top sphere. The graph below is the corresponding portion of the graph on $\mathcal{C}(S^2 \setminus A)$.

subgraph spanned by a subset $C \subseteq \mathscr{C}$ such that, for every $\gamma, \delta \in C$, there exists a nontrivial path from γ to δ in C. Note that singletons with no loop are never strongly connected components.

Strongly connected components are partially ordered: $C \prec D$ if there is a path from a curve in C to a curve in D. Consider a strongly connected component C. We call C *primitive in* $\mathscr{C}$ if it is minimal for $\prec$. We call C a *bicycle* if for every $\gamma, \delta \in C$ there exists $n \in \mathbb{N}$ such that at least two paths of length n join γ to δ in C, and a *unicycle* otherwise; see Figure 1 for an illustration.

We remark that bicycles contain at least two cycles, so the number of paths of length n grows exponentially in n. On the other hand, every unicycle is an actual *periodic cycle*, namely it can be written as $C = (\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0)$ in such a manner that γ_{i+1} has an f-preimage γ'_i isotopic to γ_i. If in a periodic cycle C the γ'_i may be chosen so that f maps each γ'_i to γ_{i+1} by degree 1, then C is called a *Levy cycle*.

A periodic cycle $C = (\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0)$ is a *solid periodic cycle* if f maps γ_i onto γ_{i+1} for all $i = 0, \dots, n-1$; if f maps every γ_i to γ_{i+1} by degree 1, then C is called a *solid Levy cycle.* Since the critical values of f are assumed to belong to A, the restrictions $f{\restriction_{\gamma_i}} : \gamma_i \to \gamma_{i+1}$ are all homeomorphisms. Note that a periodic cycle may be isotopic to more than one solid periodic cycle, possibly some solid Levy and some solid non-Levy cycles.



We remark that every invariant multicurve is generated by its primitive unicycles and bicycles, and that if C is a strongly connected component of an invariant multicurve $\mathscr{C}$ and C has a curve in common with an invariant multicurve $\mathscr{D}$, then C is also a strongly connected component in $\mathscr{D}$; and it is a bicycle in $\mathscr{C}$ if and only if it is a bicycle in $\mathscr{D}$. However, C could be primitive in $\mathscr{C}$ but not in $\mathscr{D}$.

We will sometimes speak of a strongly connected component without reference to an invaraint multicurve containing it. We will also say that a strongly connected component C is *primitive* if it is primitive in any invariant multicurve containing C.

Definition 2.2 (Types of invariant multicurves)**.** Let $\mathscr{C}$ be an invariant multicurve. Then $\mathscr{C}$ is

Cantor if it is generated by its bicycles;
anti-Cantor if $\mathscr{C}$ does not contain any bicycle;
Levy if it is generated by its Levy cycles;
anti-Levy if $\mathscr{C}$ does not contain any Levy cycle. △

Proposition 2.3. *Suppose $f\colon (S^2, A) \circlearrowleft$ is a Thurston map with an invariant multicurve $\mathscr{C}$. Then:*

(1) *there is a unique maximal invariant Cantor submulticurve $\mathscr{C}_{Cantor} \subseteq \mathscr{C}$ such that the restrictions of $\mathscr{C}$ to pieces in $S^2 \setminus \mathscr{C}_{Cantor}$ are anti-Cantor invariant multicurves of return maps in $R(f, A, \mathscr{C}_{Cantor})$;*
(2) *there is a unique maximal invariant Levy submulticurve $\mathscr{C}_{Levy} \subseteq \mathscr{C}$ such that the restrictions of $\mathscr{C}$ to pieces in $S^2 \setminus \mathscr{C}_{Levy}$ are anti-Levy invariant multicurves of return maps in $R(f, A, \mathscr{C}_{Levy})$.*

Proof. The multicurve $\mathscr{C}_{\mathrm{Cantor}}$ is generated by all the bicycles in $\mathscr{C}$, while the multicurve $\mathscr{C}_{\mathrm{Levy}}$ is generated by all the Levy cycles in $\mathscr{C}$. □

2.2. **Crossings of Levy cycles.** We now show that Levy cycles cross invariant multicurves in a quite restricted way. We will need the following technical properties. See [21, §3] for related results; in particular statement (2) is essentially [21, Corollary 3.10].

Proposition 2.4. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. Then:*

(1) *if C is a periodic cycle, then there is a homeomorphism $h\colon (S^2, A) \circlearrowleft$ isotopic to the identity rel A such that C is a solid periodic cycle of the map $h \circ f$, and is Levy for $h \circ f$ if it was Levy for f;*
(2) *if a periodic cycle C crosses a Levy cycle, then C is a periodic primitive unicycle. A strictly preperiodic curve does not cross a Levy cycle;*
(3) *if L is a Levy cycle and C is a periodic cycle crossing L such that C and L are in minimal position, then there is a homeomorphism $h\colon (S^2, A) \circlearrowleft$ isotopic to the identity rel A such that C and L are solid curve cycles of the map $h \circ f$.*

We remark that the last statement cannot be improved much. Indeed, there is an example, due to Wittner [23], of the mating of the airplane and rabbit polynomials, which may be decomposed in two manners as a mating; in other words, the map admits two “equators” (invariant curves mapped $d : 1$ to themselves). It is impossible to make both equators simultaneously solidly periodic and in minimal position; worse, if they are both made solidly periodic, then they must have infinitely many crossings. We recall the following.

Lemma 2.5 (The Alexander method, [9, Proposition 2.8]). *A collection of pairwise nonisotopic essential curves $\{\gamma_i\}_i$ can be simultaneously isotopically moved into $\{\gamma'_i\}_i$ if* (1) *all curves in $\{\gamma_i\}_i$ are pairwise in minimal position,* (2) *all curves in $\{\gamma'_i\}_i$ are pairwise in minimal position,* (3) *every γ_i is isotopic to the corresponding γ'_i, and* (4) *for pairwise different i, j, k at least one of $i(\gamma_i, \gamma_j)$, $i(\gamma_i, \gamma_k)$, and $i(\gamma_j, \gamma_k)$ is* 0. ☐

Proof of Proposition 2.4. We begin with (1). Write $C = (\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0)$. For every i choose a component γ'_i of $f^{-1}(\gamma_{i+1})$ that is isotopic to γ_i, mapping by degree 1 if C is a Levy cycle. Note that the γ'_i are disjoint. Any isotopy moving all γ_i to γ'_i satisfies the claim.

Let us move to the second claim. Assume that $C = (\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0)$ crosses a Levy cycle L. By part (1) we may assume that L is a solid Levy cycle.

Put γ_0 in minimal position with respect to L and denote by $\#(\gamma_0 \cap L)$ the total number of crossings of γ_0 with L. Since L is a solid Levy cycle we have

$$\#(f^{-m}(\gamma_0) \cap L) = \#(\gamma_0 \cap L)$$

for every $m \geq 0$. If m is a multiple of n, then $f^{-m}(\gamma_0)$ contains at least one component γ'_0 isotopic to γ_0. By minimality,

$$\#(\gamma'_0 \cap L) \geq \#(\gamma_0 \cap L).$$

We conclude that for every $m \geq 0$ there is exactly one component in $f^{-m}(\gamma_0)$ that crosses L. This component is necessarily isotopic to γ_{-m}, with subscripts computed modulo n. Claim (2) follows from the observation that if γ crosses a Levy cycle L, is periodic and is a preimage of some γ', then γ' crosses L.

Let us prove claim (3). Write $L = (\lambda_0, \dots, \lambda_p = \lambda_0)$ and $C = (\gamma_0, \dots, \gamma_q = \gamma_0)$. By part (1) we may assume that L is a solid Levy cycle. By part (2), there is a unique component γ'_i of $f^{-1}(\gamma_{i+1})$ that is isotopic to γ_i. It follows from the above discussion that

$$C' = (\gamma'_0, \gamma'_1, \dots, \gamma'_q)$$

is also in minimal position with respect to C. It follows from the Alexander method, Lemma 2.5, that there is an isotopy moving every γ'_i into γ_i while fixing every λ_i. ☐

Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map, and let $\mathscr{C}$ be an invariant multicurve. The components of $S^2 \setminus \mathscr{C}$ can be compactified to *small spheres* by shrinking each boundary component to a point, and f induces *small maps* between the small spheres, well defined up to isotopy. A periodic small sphere S_0 gives rise to a *small Thurston cycle* of maps $S_0 \to S_1 \to \cdots \to S_0$ (see [4, Definition 4.9]), which is a *small homeomorphism cycle* if all the small maps are homeomorphisms.

The next result states that two Levy cycles can be joined so as to give a finer decomposition, with additional small homeomorphism maps. Its content is nontrivial only if the Levy cycles intersect.

Corollary 2.6. *Let C_1 and C_2 be two Levy cycles. Then a small neighbourhood of their union is a small homeomorphism cycle.*

More precisely, assume that C_1 and C_2 are in minimal position. Denote by $\mathscr{C}$ the invariant multicurve generated by the boundary of a small neighbourhood of $C_1 \cup C_2$ in S^2. Then the small spheres of $(S^2, A) \setminus \mathscr{C}$ that intersect $C_1 \cup C_2$ form a small homeomorphism cycle.

Proof. By Proposition 2.4(3) we may assume that C_1 and C_2 are solid Levy cycles in minimal position.

Let $\mathscr{C}_0$ be the boundary of a small neighbourhood N of $C_1 \cup C_2$ in S^2. By the Bigon criterion, Proposition 2.1, all curves in $\mathscr{C}_0$ are nontrivial. For every $\gamma \in \mathscr{C}_0$, its image $f(\gamma)$ belongs to $\mathscr{C}_0$ and the restriction $f{\restriction_\gamma} \colon \gamma \to f(\gamma)$ has degree 1. Since f is a covering away from A, it extends to a homeomorphism on N. Up to isotopy, we may suppose that N is invariant.

Since $\mathscr{C}$ does not contain the peripheral or trivial curves in $\mathscr{C}_0$, we should extend $f\colon N \to N$ to all connected components of $S^2 \setminus N$ that contain at most one marked point.

By passing to an iterate of f to lighten notation, we may assume that f preserves each component of ∂N. Let D be a disc in $S^2 \setminus N$, and assume that $f\colon D \to f(D)$ has degree at least 2. Since f preserves ∂D and is a homeomorphism on N, the image $f(D)$ contains D. Likewise, $f^{-1}(D) \cap D$ contains a component, say E, whose boundary contains ∂D. We get a map $f\colon E \to D$ of degree at least 2; so D contains at least two critical values, and then it is essential.

It follows that f extends to a homeomorphism on the union of N and the inessential discs touching it. $\square$

2.3. **The Levy decomposition.** A Thurston map $f\colon (S^2, A) \circlearrowleft$ is called *Levy-free* if f does not admit a Levy cycle and the degree of f is at least 2. Here we characterize the multicurves along which f decomposes into Levy-free maps.

We say that an invariant Levy multicurve $\mathscr{C}$ is *complete* if every small Thurston map in $R(f, A, \mathscr{C})$ is either Levy-free or a homeomorphism.

Proposition 2.7. *Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be complete invariant Levy multicurves of a Thurston map $f\colon (S^2, A) \circlearrowleft$. Then:*

1. *if a periodic curve $\gamma_1 \in \mathscr{C}_1$ crosses a curve $\gamma_2 \in \mathscr{C}_2$, then γ_1 and γ_2 belong to primitive Levy unicycles;*
2. *the Levy-free maps in $R(f, A, \mathscr{C}_1)$ and in $R(f, A, \mathscr{C}_2)$ are the same;*
3. *the multicurve $\mathscr{C}_1 \cap \mathscr{C}_2$ is a complete invariant Levy multicurve.*

It follows that there is a unique minimal complete invariant Levy multicurve, which is called the *canonical Levy obstruction of f* and written $\mathscr{C}_{f,\mathrm{Levy}}$. Any other invariant complete Levy multicurve $\mathscr{C}$ contains $\mathscr{C}_{f,\mathrm{Levy}}$ as a submulticurve.

Proof of Proposition 2.7.

(1) By the definition of a Levy multicurve, for every $\gamma_2 \in \mathscr{C}_2$ there is a Levy cycle C_2 such that γ_2 is an iterated preimage of a curve in C_2. Consider $\gamma_1 \in \mathscr{C}_1$. Then γ_1 crosses C_2 because γ_1 is periodic. It follows from Proposition 2.4(2) that γ_1 belongs to a primitive Levy unicycle and by symmetry the same is true for γ_2.

(2) Consider a Levy-free cycle $f^p\colon S_0 \to S_1 \to \cdots \to S_p = S_0$ in $R(f, A, \mathscr{C}_1)$. We show that $\mathscr{C}_2$ intersects none of the $S_1, S_2, \dots, S_p$; this implies that $\bigsqcup_i S_i$ is contained in a Levy-free cycle $S'_0 \to \cdots \to S'_{p'} = S'_0$ of small spheres of $R(f, A, \mathscr{C}_2)$, and symmetrically $\bigsqcup_j S'_j$ is contained in a Levy-free cycle of small spheres of $R(f, A, \mathscr{C}_1)$, so $\bigsqcup_i S_i$ and $\bigsqcup_j S'_j$ are isotopic in (S^2, A)

Assume therefore for contradiction that $\mathscr{C}_2$ intersects some small sphere S_i. If this intersection is entirely contained in S_i, it will generate a Levy cycle in $\bigsqcup_i S_i$, contradicting the assumption that $\bigsqcup_i S_i$ is Levy-free; therefore $\mathscr{C}_2$ crosses $\bigsqcup_i \partial S_i$.

There is then a periodic curve in $\bigsqcup_i \partial S_i$ crossing $\mathscr{C}_2$. Choose a curve cycle $C_1 \subseteq \bigsqcup_i \partial S_i$ and a curve cycle $C_2 \subseteq \mathscr{C}_2$ such that C_1 crosses C_2. By part (1) of the proposition, C_1 and C_2 are anti-Cantor Levy cycles. By Corollary 2.6, there is a small homeomorphism cycle $\{S'_i\}_i$ containing $C_1 \cup C_2$. All curves in $\bigsqcup_i \partial S'_i$ belong to Levy cycles.

If $\bigsqcup_i S'_i$ contains (up to isotopy) the union $\bigsqcup_i S_i$, then we have a contradiction because the degree of f on $\bigsqcup_i S_i$ is at least 2, while it is 1 on $\bigsqcup_i S'_i$.

We now show that we can always reduce to this case. If $\bigsqcup_i S'_i$ does not contain $\bigsqcup_i S_i$, then there is a curve cycle in $\bigsqcup_i \partial S'_i$ crossing at least one curve in $\bigsqcup_i \partial S_i$. This implies that there is a Levy cycle in $\bigsqcup_i \partial S'_i$ crossing a Levy cycle in $\bigsqcup_i \partial S_i$. Again invoking Corollary 2.6, we can enlarge $\bigsqcup_i S'_i$. Repeating, we may enlarge $\bigsqcup_i S'_i$ so that it contains $\bigsqcup_i S_i$.

Finally, (3) follows formally from (2). □

Definition 2.8 (Levy decomposition)**.** The Levy decomposition of a Thurston map $f\colon (S^2, A) \circlearrowleft$ is the decomposition of f along the canonical Levy obstruction $\mathscr{C}_{f,\mathrm{Levy}}$. △

We may understand the Levy decomposition of a Thurston map $f\colon (S^2, A) \circlearrowleft$ as follows, if we consider more general subsets of S^2 on which f acts as a homeomorphism. Let us call a "Levy kernel" a subset $L \subseteq S^2$ together with a partition $L = \bigsqcup_{i\in I} S_i$ and a map $f\colon I \circlearrowleft$ such that each S_i is either an essential simple closed curve or an essential small sphere, and it is considered up to isotopy; we require that every S_i be isotopic to a degree-1 preimage of $S_{f(i)}$ and that if S_i is a curve, then it is not homotopic to any (boundary) curve in $\bigsqcup_{j\neq i} \partial S_j$. (The last condition replaces the "nonhomotopic" condition in the definition of a multicurve.) There is a natural order on Levy kernels, given by inclusion up to isotopy.

We may think about a Levy kernel as a subset of a sphere on which f has degree one. Corollary 2.6 states that if two Levy kernels L_1, L_2 intersect, then we can construct a bigger Levy kernel $\widetilde{L}$ that contains both L_1 and L_2. Therefore, there exists a maximal Levy kernel, and its boundary generates the Levy decomposition.

3. Self-similar groups and automata

We recall basic notions about self-similar groups; for a reference see [16].

3.1. **Contracting bisets.** Let G be a group. Recall that a *G-G-biset* is a set B endowed with commuting left and right G-actions. Such a biset B is called *left-free* if the left G-action is free, i.e., has trivial stabilizers. A *basis* is a choice of one element per left G-orbit: a subset $X \subseteq B$ such that $B = \bigsqcup_{x\in X} Gx$. We therefore have a bijection $G \times X \leftrightarrow B$, and using it we may write the right G-action as a map $\Phi\colon X \times G \to G \times X$, determining the structure of B.

Important examples of bisets come from dynamics: let $f\colon \mathcal{M}' \to \mathcal{M}$ be a partial self-covering of a topological space $\mathcal{M}$, defined on a subset $\mathcal{M}' \subseteq \mathcal{M}$. Fix a basepoint $* \in \mathcal{M}$, and set $G := \pi_1(\mathcal{M}, *)$. Set

$$B(f) := \{\gamma\colon [0,1] \to \mathcal{M} \mid \gamma(0) = f(\gamma(1)) = *\} \,/\, \text{homotopy}, \tag{2}$$

with left G-action given by preconcatenation and right G-action given by postconcatenation of the unique f-lift making the resulting path continuous. A basis of B consists of, for every $z \in f^{-1}(*)$, a choice of path in $\mathcal{M}$ from $*$ to z.

Of particular interest, for us, is when $f\colon (S^2, A) \circlearrowleft$ is a Thurston map, with $\mathcal{M} := S^2 \setminus A$ and $\mathcal{M}' := S^2 \setminus f^{-1}(A)$. Recall that two Thurston maps $f_0\colon (S^2, A_0) \circlearrowleft$ and $f_1\colon (S^2, A_1) \circlearrowleft$ are called *combinatorially equivalent* if there is a path $(f_t\colon (S^2, A_t) \circlearrowleft)_{t\in[0,1]}$ of Thurston maps joining them. Kameyama proved in [14], in another language, that f_0, f_1 are combinatorially equivalent if and only $B(f_0)$ and $B(f_1)$ are *conjugate*: setting $G_i = \pi_1(S^2 \setminus A_i, *_i)$ for $i = 0, 1$, there exists a homeomorphism $\phi\colon S^2 \setminus A_0 \to S^2 \setminus A_1$ and a bijection $\beta\colon B(f_0) \to B(f_1)$ with $g \cdot b \cdot h = \phi_*(g) \cdot \beta(b) \cdot \phi_*(h)$ for all $g, h \in G_0, b \in B(f_0)$. See [4] for details.

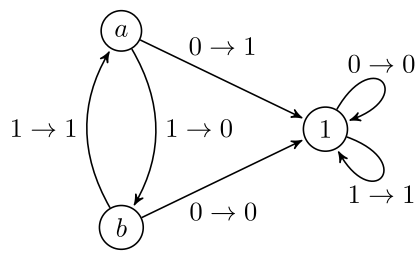
Bisets may be composed: the product of two G-G-bisets B, C is $B \otimes_G C := (B \times C)/\{(bg, c) = (b, gc)\}$, and it is related to the composition of partial coverings: we have a natural isomorphism $B(g \circ f) \cong B(f) \otimes B(g)$. If B, C are left-free with respective bases S, T, then $B \otimes C$ is left-free with basis $S \times T$.

Definition 3.1 ([16, Definition 2.11.8])**.** Let B be a G-G-biset. It is called *contracting* if for some basis $X \subseteq B$ there exists a finite subset $N \subseteq G$ with the following property: for every $g \in G$ and every n large enough we have the inclusion $X^n g \subseteq N X^n$ in $B^{\otimes n}$. △

Recall from [16, Proposition 2.11.6] that if B is contracting for some basis X, then it is contracting for every basis, possibly with a different N. The set N in Definition 3.1 is certainly not unique; but for every basis X there exists a minimal such N, written $N(B, X)$ and called the *nucleus* of (B, X).

Recall also from [16, Proposition 2.11.3] that if G is finitely generated, X is a basis of B, and B is right transitive, then G is generated by $N(B, X)$. These hypotheses are satisfied by bisets of Thurston maps; see [4, Definition 2.6]. It is then convenient to express the structure of B by a *Mealy automaton*: it is a finite directed labeled graph with vertex set $N(B, X)$, with labels in $X \times X$ on edges, and with an edge labeled "$x \to y$" from $g \in N(B, X)$ to $h \in N(B, X)$ whenever the equality $xg = hy$ holds in B. In fact, one may consider the graph $\mathfrak{G}$ with vertex set G and an edge from $g \in G$ to $h \in G$ labeled "$x \to y$" whenever $xg = hy$ holds in B, and then $N(B, X)$ is precisely the forward attractor of $\mathfrak{G}$: an equivalent formulation of Definition 3.1 is that every infinite path in $\mathfrak{G}$ eventually reaches $N(B, X)$, where it stays. Here is an example of an automaton, to which we shall return:

(3)



In this automaton, we have $X = \{0, 1\}$ and $G = \langle a, b\rangle$. The biset structure is determined by the equations

$$0 \cdot a = 1, \quad 1 \cdot a = b \cdot 0, \quad 0 \cdot b = 0, \quad 1 \cdot b = a \cdot 1.$$

The reader may check that this is the biset $B(f)$ as defined in (2) for the partial self-covering $f(z) = z^2 - 1$ of $\widehat{\mathbb{C}} \setminus \{0, -1, \infty\}$.

Proposition 3.2. *Let G be a finitely generated group with solvable word problem, and let B be a computable left-free biset (namely, for a basis X the structure map*

$X \times G \to G \times X$ is computable). Then it is semidecidable *whether B is contracting: there is an algorithm that either runs forever (if G is not contracting) or returns $N(B, X)$ (if G is contracting).*

Proof. Assume that B is contracting, with nucleus $N(B, X)$. Denote by $\mathscr{P}_f(G)$ the set of finite subsets of G, and define the self-map $\phi\colon \mathscr{P}_f(G) \circlearrowleft$ by

$$\phi(A) = \{h \in G \mid \text{ there exist } x, y \in X \text{ with } hy \in xA\}.$$

Clearly ϕ is computable and $\phi(N(B, X)) = N(B, X)$. For $A \subseteq G$ finite, set $\psi(A) := \bigcup_{k \geq 0} \phi^k(A)$. The sequence $(\bigcup_{k=0}^n \phi^k(A))_n$ is ascending and eventually all $\phi^k(A)$ are contained in $N(B, X)$, so $\psi\colon \mathscr{P}_f(G) \circlearrowleft$ is computable. Again for $A \subseteq G$ finite, set $\omega(A) := \bigcap_{n \geq 0} \phi^n(\psi(A))$. The sequence $(\phi^n(\psi(A)))_n$ is a decreasing subsequence of the finite set $\psi(A)$, so ω is also computable.

Let S be a finite generating set for G, and assume $1 \in S = S^{-1}$. Set $N_0 := \{1\}$, and for $n \geq 1$ set $N_n := \omega(N_{n-1}S)$. Then $(N_n)_n$ is an increasing subsequence of $N(B, X)$, so it stabilizes, and its limit $\bigcup_{n \geq 0} N_n$ is computable and equals $N(B, X)$. □

3.2. **Limit spaces.** Let B be a contracting G-G-biset, and let X be a basis of B. Define a relation on X^∞, called *asymptotic equivalence*, by

$$\begin{aligned}&(x_1 x_2 \dots) \sim (y_1 y_2 \dots)\\ \iff &\exists (g_0, g_1, g_2, \dots) \in G^\infty \text{ with } \#\{g_n\} < \infty \text{ and } x_n g_n = g_{n-1} y_n \text{ for all } n \geq 1.\end{aligned}$$

More precisely, one says in this case that $x_1 x_2 \cdots$ and $y_1 y_2 \cdots$ are *g_0-equivalent*. The *limit space* of B is the quotient

$$\mathcal{J}(B) := X^\infty / {\sim}.$$

More precisely, it is a topological orbispace, with at class $[x_1 x_2 \dots] \in \mathcal{J}(B)$ the isotropy group $\{g_0 \in G \mid x_1 x_2 \dots \text{ is } g_0\text{-equivalent to itself}\}$.

By [16, Theorem 3.6.3], we have $x_1 x_2 \cdots \sim y_1 y_2 \dots$ if and only if there exists a left-infinite path in the Mealy automaton of B, with labels $\dots, x_2 \to y_2, x_1 \to y_1$ on its arrows. These sequences are g_0-equivalent for g_0 the terminal vertex of the path. In particular, $\sim$-equivalence classes have cardinality at most $\# N(B, X)$. The topological (orbi)space $\mathcal{J}(B)$ is compact, metrizable, and of finite topological dimension. For example, (3) gives $x_1 \cdots x_n 0(11)^\infty \sim x_1 \cdots x_n 1(10)^\infty$ for all $x_1, \dots, x_n \in X = \{0, 1\}$.

Denote by $s\colon X^\infty \circlearrowleft$ the shift map $x_1 x_2 x_3 \cdots \mapsto x_2 x_3 \cdots$. Clearly the asymptotic equivalence is invariant under s, so s induces a self-map $s\colon \mathcal{J}(B) \circlearrowleft$. By [16, Corollary 3.6.7], the dynamical system $(\mathcal{J}(B), s)$ is independent, up to topological conjugacy, of the choice of X. Note that s induces a partial self-covering of $\mathcal{J}(B)$ if the orbispace structure of $\mathcal{J}(B)$ is taken into account.

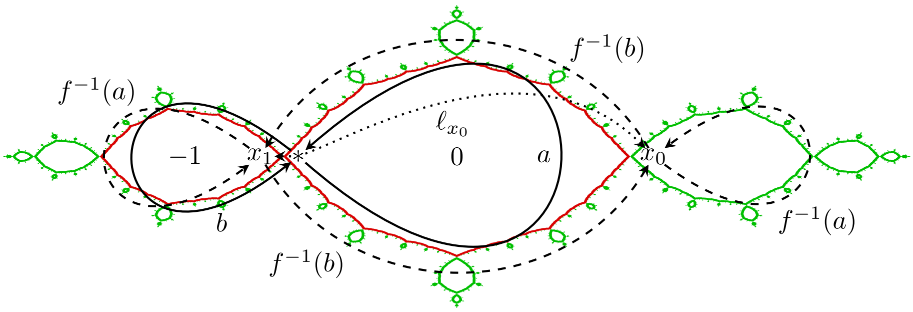
Let $f\colon \mathcal{M}' \to \mathcal{M}$ be a partial self-covering as above, and assume that $\mathcal{M}$ has a complete length metric that is expanded by f. The *Julia set* of f is defined as the accumulation set of backward iterates of a generic point: fix $z \in \mathcal{M}$, and define

$$\mathcal{J}(f) := \overline{\bigcap_{n \geq 0} \bigcup_{m \geq n} f^{-m}(z)}, \tag{4}$$

a definition that does not depend on the choice of z.

By [16, Theorem 5.5.3] the biset $B(f)$ defined in (2) is contracting and the dynamical systems $(\mathcal{J}(f), f)$ and $(\mathcal{J}(B(f)), s)$ are conjugate.

The following image shows the Julia set of $f(z) = z^2 - 1$, the loops $a, b \in \pi_1(\widehat{\mathbb{C}} \setminus \{0, -1, \infty\}, *)$, and the basis $\{\ell_{x_0}, \ell_{x_1}\}$ that were used to compute the automaton (3) (ℓ_{x_1} is so short that it is not visible):



3.3. **Orbisphere contracting bisets.** We slightly modify the definition of "contracting" for sphere bisets because of the orbisphere structures. Let ${}_GB_G$ be a sphere biset with $G = \pi_1(S^2 \setminus A, *)$. Recall from [4, Equation (35)] that there is a minimal orbisphere structure ord_B given by B. We call an orbisphere structure $\mathrm{ord}\colon A \to \{2, 3, \dots, \infty\}$ *bounded* if $\mathrm{ord}(a) = \infty \Leftrightarrow \mathrm{ord}_B(a) = \infty$ and $\mathrm{ord}(a) \deg_a(B) \mid \mathrm{ord}(B_*(a))$ for all $a \in A$. Let $\overline{G}$ denote the quotient orbisphere group $G/\langle \gamma_a^{\mathrm{ord}(a)} : a \in A\rangle^G$, with γ_a representing a small loop around puncture a. Then we call B an *orbisphere contracting* biset if $\overline{G} \otimes_G B \otimes_G \overline{G}$ is contracting for some bounded orbisphere structure on (S^2, A).

4. Expanding nontorus maps

Our purpose is, in this section, to endow the sphere (S^2, A) with a smooth metric that is expanded by a self-map $f\colon (S^2, A) \circlearrowleft$. We recall that by $A^\infty \subset A$ we denote the forward orbit of periodic critical points of f. A *nontorus* map is a map that is not doubly covered by a torus endomorphism.

Definition 4.1 (Metrically expanding maps)**.** Let us consider a Thurston map $f\colon (S^2, A) \circlearrowleft$ and let A' be a forward-invariant subset of A^∞. We say that f is *metrically expanding* if there exists a length metric μ on $S^2 \setminus A^\infty$ such that

(1) for every nontrivial rectifiable curve $\gamma\colon [0,1] \to S^2 \setminus A'$ the length of any lift of γ under f is strictly less than the length of γ

and (2) at all $a \in A'$ the first return map of f is locally conjugate to $z \mapsto z^{\deg_a(f^n)}$.

If $A' = A^\infty$, then $f\colon (S^2, A) \circlearrowleft$ is called a *Böttcher expanding* map. △

If $\mu = \mathbf{d}s$ is a Riemannian orbifold metric on (S^2, A) (i.e., μ is a smooth metric on $S^2 \setminus A'$ with possible cone singularities in $A \setminus A'$), then condition (1) may be replaced by $f^*\mathbf{d}s < \mathbf{d}s$.

Let us now define a more general notion of topological expansion. Consider first a covering map $f\colon \mathcal{M}' \to \mathcal{M}$ between compact topological orbispaces, with $\mathcal{M}' \subseteq \mathcal{M}$. We call f *topologically expanding* if there exists a finite covering by connected open sets $\mathcal{M}' = \bigcup \mathcal{U}_i$ such that connected components of $f^{-n}(\mathcal{U}_i)$ get arbitrarily small as $n \to \infty$, in the sense that for every finite open covering $\mathcal{M} = \bigcup \mathcal{V}_j$ there exists $n \in \mathbb{N}$ such that every connected component of every $f^{-n}(\mathcal{U}_i)$ is contained in some $\mathcal{V}_j$. Equivalently, the diameter of connected components of $f^{-n}(\mathcal{U}_i)$ tends to 0 with respect to any metric on $\mathcal{M}'$.

Definition 4.2 (Topological expanding maps)**.** Let us consider a Thurston map $f\colon (S^2, A) \circlearrowleft$ and let A' be a forward-invariant subset of A^∞. We call f *topologically expanding* if there exist $\mathcal{M}' \subseteq \mathcal{M} \subseteq S^2$ compact with a topologically expanding orbifold covering map $f\colon (\mathcal{M}', A) \to (\mathcal{M}, A)$ such that every connected component $\mathcal{U}$ of $S^2 \setminus \mathcal{M}$ is a disk containing a unique point $a \in A'$, and $\mathcal{U}$ is attracted to a, and the first return of f is locally conjugate to $z \mapsto z^{\deg_a(f^n)}$ at a.

If $A' = A^\infty$, then $f\colon (S^2, A) \circlearrowleft$ is called a *Böttcher topologically expanding* map. △

Proposition 4.3. *A metrically expanding map is topologically expanding.*

Proof. Let $f\colon (S^2, A) \circlearrowleft$ be metrically expanding. For each point $a \in A'$ choose an open neighbourhood $\mathcal{U}_a \ni a$ such that $f(\mathcal{U}_a)$ is compactly contained in $\mathcal{U}_{f(a)}$. Set $\mathcal{M} = S^2 \setminus \bigcup \mathcal{U}_a$ and $\mathcal{M}' = f^{-1}(\mathcal{M})$.

Note that points in A^∞ are at infinite distance from each other, so $(S^2 \setminus A^\infty, \mu)$ is a complete, locally compact metric space. Then Condition 1 of Definition 4.1 and the Hopf-Rinow Theorem imply that points away from A' are repelled by f: given two points at small distance δ from each other, their images may be joined by a geodesic, which must have length $> \delta$. □

The goal of this section is to prove the following criterion.

Theorem 4.4 (Expansion criterion)**.** *The following are equivalent, for a combinatorial equivalence class $\mathscr{F}$ of Thurston maps:*

1. *$\mathscr{F}$ contains a metrically Böttcher expanding map;*
2. *$\mathscr{F}$ contains a topologically expanding map;*
3. *$B(f)$ is an orbisphere contracting biset for every $f \in \mathscr{F}$;*
4. *$\mathscr{F}$ does not admit a Levy cycle, and if $\mathscr{F}$ is doubly covered by a torus endomorphism $Mz + v\colon \mathbb{R}/\mathbb{Z} \circlearrowleft$, then both eigenvalues of M have absolute value greater than* 1.

Furthermore, if any of these properties hold, then the expanded metric may be assumed to be Riemannian of pinched negative curvature.

We will prove Theorem 4.4 for maps not doubly covered by torus endomorphisms. The remaining case follows from [12, Theorem 4] or from [20]. The hardest implication in the proof is (4)⇒(1).

Proof of Theorem 4.4, (1)⇒(2)⇒(3)⇒(4). The implication (1)⇒(2) follows from Proposition 4.3. By [1, Proposition 6.4], the biset of a topologically expanding map is contracting; this is (2)⇒(3) with slight adjustments to sphere maps.

Next consider a combinatorial equivalence class $\mathscr{F} = [f]$ admitting a Levy cycle $(\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0)$. Write $G = \pi_1(S^2 \setminus A, *)$, consider the G-G-biset $B(f)$, and choose a basis X for it. The assumption that $(\gamma_i)_i$ is a Levy cycle means that there exist basis elements $x_0, x_1, \dots, x_n = x_0 \in X$ with $x_i\gamma_{i+1} = \gamma_i' x_i$ and γ_i' conjugate to γ_i for all $i \in \mathbb{Z}/n$. In particular, for every $j \in \mathbb{Z}$ there is a conjugate of γ_0^j in the nucleus of $(B(f), X)$. Now γ_0 has infinite order in G, because it is not peripheral. It follows that the nucleus of $(B(f), X)$ is infinite, so $B(f)$ is not orbisphere contracting. □

Let us outline the proof of Theorem 4.4(4)⇒(1); the details of the proof are given in §4.4, after some preparation in §§4.1–4.3.

We wish to prove that a Levy-free nontorus Thurston map f admits an expanding metric. We do so by explicitly constructing the metric adapted to f.

We consider the *decomposition* of S^2 into *small spheres* along the canonical obstruction $\mathscr{C}_f$. The map f restricts to maps between the small spheres, well-defined up to isotopy, and the *small Thurston maps*—the return maps to small spheres—are combinatorially equivalent to rational maps.

We first isotope the periodic small spheres into complex spheres in such a manner that the small Thurston maps are rational. We put the hyperbolic metric on these periodic small spheres and pull it back to preperiodic small spheres.

It remains to attach the small spheres together. They are spheres with cusps; some of the cusps correspond to the marked set A, and some to $\mathscr{C}_f$. Cut the cusps corresponding to $\mathscr{C}_f$ along a very small horocycle and connect the small spheres by very long and thin cylinders along the combinatorics of the original decomposition. We have constructed a space X with a piecewise-smooth non-positively curved metric.

Define a self-map $F\colon X \circlearrowleft$ as follows: away from the truncated cusps, apply the original map f. Subdivide the long cylinders into long “annuli” and short “annular spheres”. Map the annular spheres to the small spheres they originally mapped to, and map the annuli affinely to each other.

The map F is expanding: on periodic small spheres, because it is modelled on rational maps; on preperiodic small spheres, too; on annular and trivial small spheres, because they are contained in thin cylinders; and on annuli, because of properties of the canonical obstruction—it contains neither Levy cycles nor primitive unicycles. $\square$

4.1. **Conformal metrics.** Recall first that every Riemannian metric s on a surface (for example a sphere) admits local isothermal coördinates, i.e., there is a local chart $\mathcal{U}$ where $\mathbf{d}s$ takes form $\rho(z)|\mathbf{d}z|$ on the tangent space of $\mathcal{U}$; the function $\rho\colon \mathcal{U} \to \mathbb{R}_+$ should be smooth. A metric in this form is called *conformal.* The Gaussian curvature $\kappa\colon \mathcal{U} \to \mathbb{R}$ is given by

$$\kappa(z) = \frac{-\Delta(\log \rho(z))}{\rho(z)^2},$$

by an easy calculation (see, e.g., [10, page 77]). We note for future reference the following simple calculation: if $\rho(z) = \sigma(|z|)$ is rotationally invariant around $0 \in \mathcal{U}$ in the chart z, then the Gaussian curvature may be computed as

$$\kappa(z) = -\frac{\log(\sigma)'' + \log(\sigma)'/|z|}{\sigma(|z|)^2}. \tag{5}$$

We shall consider a conformal metric s on an orbisphere (S^2, A). This means that in suitable coördinates we have $2\mathbf{d}s = \rho(z)|\mathbf{d}z|$ with $\rho\colon S^2 \setminus A \to \mathbb{R}_+$ that has a continuous extension $\rho\colon S^2 \to \mathbb{R}_+ \cup \{+\infty\}$ such that if for $a \in A$

- $\rho(a) < +\infty$, then ρ is smooth at a (i.e., a is a usual point);
- $\rho(a) = +\infty$ but a is at finite distance from points in S^2, then (S^2, s) around a is a quotient of a chart $\mathcal{U}$ endowed with a conformal metric under a finite group of isometries; the point a is called a *cone singularity.*

If $\rho(a) = +\infty$ and a is at infinite distance from points in S^2, then a is called a *cusp.*

4.2. **Fatou and Julia sets.** We adapt the definition of Julia sets from (4) to expanding Thurston maps. We recall some well-known facts and include their proofs for convenience.

Definition 4.5. Let $f\colon (S^2,A) \circlearrowleft$ be an expanding Thurston map. Its *Julia set* $\mathcal{J}(f)$ is the closure of the set of repelling periodic points, namely the closure of the set of points $z \in S^2$ with $f^n(z) = z$ for some $n > 0$ but admitting no neighbourhood $\mathcal{U} \ni z$ with $f^n(\mathcal{U})$ compactly contained in $\mathcal{U}$.

The *Fatou set* $\mathcal{F}(f)$ is the locus of continuity of forward orbits, namely the set of $z \in S^2$ at which the orbit map $S^2 \to (S^2)^\infty, z \mapsto (z, f(z), f^2(z), \dots)$ is continuous in supremum norm (of any metric on S^2 realizing its topology). △

Lemma 4.6. *$S^2 = \mathcal{J}(f) \sqcup \mathcal{F}(f)$. Moreover, in the notation of Definition 4.2 the Julia set $\mathcal{J}(f)$ is the set of points in $\mathcal{M}'$ that do not escape $\mathcal{M}'$ under iteration of f.*

Proof. By definition, every point z escaping $\mathcal{M}'$ is in the attracting basin of A', so z has a stable orbit and $z \in \mathcal{F}(f)$. Conversely, suppose that z does not escape $\mathcal{M}'$. Fix a metric on S^2 realizing its topology and consider $\varepsilon > 0$ such that for every $\mathcal{V} \subset \mathcal{M}$ with diameter less than ε the components of $f^{-n}(\mathcal{V})$ get arbitrarily small as $n \to \infty$. Choose a large $n \in \mathbb{N}$ and consider the ε-neighbourhood $\mathcal{V} \subset \mathcal{M}$ of $f^n(z)$. The pullback of $\mathcal{V}$ along the orbit of z is a small (since n is large) neighbourhood $\mathcal{V}'$ of z; so there are points close to z that have orbits at least ε-away from the orbit of z. This shows that $z \notin \mathcal{F}(f)$.

Now choose a small closed topological disc $\mathcal{V}$ containing z. There is an $n \geq 1$ such that $f^n(\mathcal{V}) \supset \mathcal{V}$. Therefore, there is a periodic point in $\mathcal{V}$. This shows that $z \in \mathcal{J}(f)$. □

The Fatou set of f is open. Every periodic component of $\mathcal{F}(f)$ contains an attracting periodic point called its *center*; this point belongs to A'. By Lemma 4.6 every nonperiodic component of $\mathcal{F}(f)$ is preperiodic because it consists of points escaping to $S^2 \setminus \mathcal{M}$. We may now deduce that every component of $\mathcal{F}(f)$ is an open topological disc.

First consider a periodic connected component O of $\mathcal{F}(f)$ and let $a \in A' \cap O$ be its center. There is a conjugacy from O to the open disk $\mathbb{D} \subset \mathbb{C}$ such that the first return map $f^n\colon O \to O$ is conjugate to the map $z^{\deg_a(f^n)}$. We write $\deg_O(f) := \deg_a(f)$ and call the conjugacy $\phi_O\colon O \to \mathbb{D}$ a *Böttcher coördinate.* We may then determine coördinates on every Fatou component on the forward and backward orbit of O in such a manner that, for every Fatou component U, the restriction $f\restriction_U\colon U \to f(U)$ is conjugate to a monomial map by $\phi_{f(U)} \circ f\restriction_U = z^{\deg_U(f)} \circ \phi_U$.

We use Böttcher coördinates to define, in every Fatou component O, *internal rays* $R_{O,\theta} \subset O$ by

$$R_{O,\theta} = \phi_O^{-1}\{re^{2i\pi\theta} \mid r < 1\}.$$

These rays are mapped to each other by $f(R_{O,\theta}) = R_{f(O),\deg_O(f)\theta}$. The following statement follows immediately from the existence of Böttcher coördinates.

Lemma 4.7. *Let $f\colon (S^2,A) \circlearrowleft$ be a Böttcher expanding map, and let $a \in A$ be a degree-d attracting point. Let F denote its immediate basin of attraction; then F is a connected component of the Fatou set of f. Let $\overline{D}$ denote the compactification* *of $S^2 \setminus \{a\}$ by adding a circle of directions in replacement of a; then f extends continuously to a self-map of $\overline{D}$ such that the boundary circle is mapped to itself by $z \mapsto z^d$.* □

4.3. **Canonical obstructions and decompositions.** We shall make essential use of Pilgrim's *canonical decomposition.* Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map. Then there is an induced pullback map f^* on the Teichmüller space $\mathscr{T}_A$ of complex structures on (S^2, A); see [4, §8]. For a given complex structure η, the pullback $f^*\eta$ is defined by requiring that the map $f\colon (S^2, A, f^*\eta) \to (S^2, A, \eta)$ be holomorphic. The map f is combinatorially equivalent to a rational map if and only if f^* has a fixed point.

Let γ be an essential simple closed curve and let $\eta \in \mathscr{T}_A$ be a complex structure. The length $\langle \gamma, \eta \rangle$ of γ with respect to η is defined as the length of the unique geodesic in (S^2, A, η) that is homotopic to γ. This defines an analytic function $\langle \gamma, - \rangle \colon \mathscr{T}_A \to \mathbb{R}$.

Definition 4.8 (Canonical obstruction [18, Theorem 1.2])**.** Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map and consider $\eta \in T_A$.

The *canonical obstruction* $\mathscr{C}_f$ is the set of homotopy classes of essential simple closed curves γ such that $\langle \gamma, f^{n*}\eta \rangle$ tends to 0 as n tends to infinity. △

It follows from the following theorem that the definition of $\mathscr{C}_f$ does not depend on η. It was proved by Kevin Pilgrim that $\mathscr{C}_f$ is a multicurve.

Theorem 4.9 (Pilgrim, [17])**.** *If $\mathscr{C}_f$ is empty, the degree of f is at least* 2*, and the minimal orbisphere structure of f is hyperbolic, then f is combinatorially equivalent to a rational map.* □

For f a Thurston map, its *canonical decomposition* is the collection of spheres and annuli obtained by cutting f along the canonical obstruction $\mathscr{C}_f$. Recall that the *small Thurston maps* are the return maps of f to the small spheres in a decomposition.

Theorem 4.10 (Pilgrim, Selinger [19])**.** *Every small Thurston map in the canonical decomposition of f is either*

- *combinatorially equivalent to a rational non-Lattes postcritically finite map;*
- *double covered by a torus endomorphism;*

or • *a homeomorphism.*

Theorem 4.10 was conjectured by Kevin Pilgrim (who also proved a slightly weaker version of this theorem; see [18, page 13]) and was eventually proved by Nikita Selinger.

4.4. **Construction of the model.** Here we give the proof of the implication (4)⇒(1) by constructing a negatively curved Riemannian metric on $X \simeq S^2$ and an expanding map $F\colon X \circlearrowleft$ isotopic to f; see Figure 2 for an illustration of the construction.

4.4.1. *Setup.* The space X is constructed by *plumbing* between cusped spheres: we enlarge the cusps to make them almost cylindrical, and then truncate them and glue them on their common boundary. Three variables dictate the construction. In specifying what their range can be, we introduce the notation "$x \ggg y$" to mean

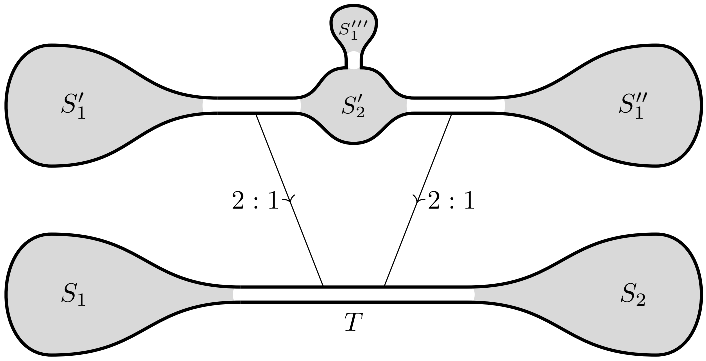


FIGURE 2. Illustration to the Proof of Theorem 4.4. The map f is indicated by the arrows and sends S_1', S_1'', S_1''' to S_1 and S_2' to S_2. We first define a metric on the periodic small spheres (S_1), then on the preperiodic small spheres (S_2), and finally on the annuli between them. This map could be Pilgrim's "blow-up an arc" map; see [2, §8.2.2].

that if one of x, y has already been specified, then the other one may be chosen arbitrarily as long as the ratio x/y is large enough.

First a parameter $w \lll 1$ is chosen; the perimeters of the "cylindrical parts" will lie between πw and $2\pi w$. Then a parameter $\ell \ggg 1/w$ is chosen; the cylindrical parts will all have length between ℓ and 2ℓ. Finally, a parameter $\epsilon \lll 1/\ell$ is chosen; it will be a final adjustment to the construction that makes the curvature bounded by $-\epsilon^2$ from above.

The map F is very close to a rational map on each small sphere and is very close to an affine map on each cylinder connecting small spheres. After the main part of the construction is carried out we obtain a metric μ that is weakly expanded by F (namely, F does not contract μ) and a certain iteration of F expands μ. In Lemma 4.14 we perturb μ infinitesimally to make F expanding.

4.4.2. *The canonical decomposition.* Throughout this section, we let $\mathscr{C} = \mathscr{C}_f$ denote the canonical obstruction of the Thurston map $f\colon (S^2, A) \circlearrowleft$ and we denote by $\mathscr{S}$ the collection of small spheres (components of $S^2 \setminus \mathscr{C}$) of the canonical decomposition so that

$$S^2 = \bigsqcup_{\gamma \in \mathscr{C}} \gamma \cup \bigsqcup_{S \in \mathscr{S}} S.$$

As in [4], for $S \in \mathscr{S}$ we denote by $\widehat{S}$ the corresponding topological sphere marked by the image of $A \cap S^2$ and the boundary curves. The map f induces a map $f\colon \mathscr{S} \circlearrowleft$, and for each $S \in \mathscr{S}$ a map $f\colon \widehat{S} \to \widehat{f(S)}$, well defined up to isotopy; see [4, Lemma 4.9].

Recall that we assumed that f is a nontorus map: a map that is not finitely covered by a torus endomorphism.

Lemma 4.11. *If $f\colon (S^2, A) \circlearrowleft$ is a Levy-free nontorus Thurston map and $\mathscr{C}_f$ is nonempty, then $\mathscr{C}_f$ is an anti-Levy Cantor multicurve, and all small Thurston maps in the canonical decomposition of f are equivalent to nontorus rational maps.*

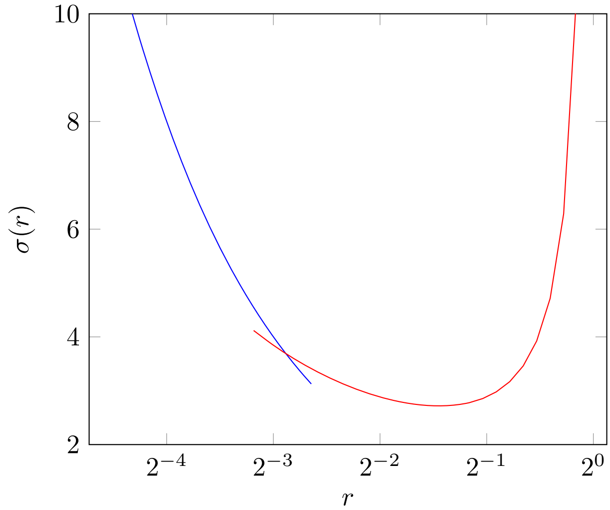


FIGURE 3. The curvature on the widened cusps

Proof. Let us show that $\mathscr{C}_f$ does not contain a primitive unicycle. Since a non-Levy unicycle has spectral radius strictly less than 1, such a (primitive) cycle may not belong to $\mathscr{C}_f$.

Further, all small Thurston maps in $R(f, A, \mathscr{C})$ are nontorus and nonhomeomorphisms, because torus and homeomorphism cycles can only be attached via Levy cycles, because homeomorphisms and torus maps have no attracting periodic points. Theorem 4.10 concludes the proof. □

4.4.3. *Metrics on small spheres.* Consider a cycle of periodic spheres $S \to f(S) \to \cdots \to f^p(S) = S$. Let us denote by $\widehat{S}_i = \widehat{f^i(S)}$ the topological sphere associated with $f^i(S)$ and denote by A_i the marked set of S_i. By Lemma 4.11 the first return map $f^p\colon \widehat{S}_i \circlearrowleft$ is isotopic rel A_i to a rational map. Therefore, let us now assume that each $\widehat{S}_i$ is a marked complex sphere and each $f_i\colon \widehat{S}_i \to \widehat{S}_{i+1}$ is a rational map. Choose next an orbifold structure $\mathrm{ord}_i\colon A_i \to \{1, 2, \dots, \infty\}$ such that $f^p\colon (\widehat{S}_1, \mathrm{ord}_1) \circlearrowleft$ is a partial self-covering but is not a partial covering. We also choose ord_i in such a way that $\mathrm{ord}_i(x) = \infty$ if and only if x is in a periodic critical cycle or x is the image of a boundary curve.

We endow each $(\widehat{S}_i, \mathrm{ord}_i)$ with its natural hyperbolic metric. Then every $f\colon \widehat{S}_i \to \widehat{S}_{i+1}$ is either expanding (if it is not a covering) or an isometry (if it is a covering), and $f^p\colon \widehat{S}_1 \circlearrowleft$ is expanding.

Similarly, we endow each preperiodic sphere $\widehat{S}'$, say marked by A', with a hyperbolic metric such that $f\colon \widehat{S}' \to \widehat{f(S')}$ is either an isometry or an expanding map. The orbisphere structure $\mathrm{ord}'\colon A' \to \{1, 2, 3, \dots, \infty\}$ is chosen so that $\mathrm{ord}_i(x) = \infty$ if and only if x is the image of a boundary curve.

4.4.4. *Plumbing.* Let $\widehat{S}_1$ and $\widehat{S}_2$ be two hyperbolic small spheres with respective cusps at $x_1 \in \widehat{S}_1$ and $x_2 \in \widehat{S}_2$. We now describe an operation, *plumbing*, that joins these small spheres near their cusps. On each of $\widehat{S}_1, \widehat{S}_2$, a small neighbourhood of x is foliated by *horocycles*—curves perpendicular to geodesics starting at x. The plumbing truncates $\widehat{S}_1$ and $\widehat{S}_2$ at x_1 and x_2 along their horocycles of perimeter $\approx 2\pi w$ and joins $\widehat{S}_1, \widehat{S}_2$ along an almost flat cylinder with length $\approx \ell$ such that the resulting sphere still has a negatively curved metric. Since $\widehat{S}_1$ and $\widehat{S}_2$ are covered by punctured discs, it is sufficient to describe the operation between two copies $\mathbb{D}_1^*, \mathbb{D}_2^*$ of the unit disc punctured at 0.

The hyperbolic metric on the unit disc punctured at 0 is written as $\sigma(|z|)|\mathbf{d}z|$ with $\sigma(r) = -1/(r\log r)$. Replace it with the annulus parameterized by $\{\exp(-w\ell/2) \le |z| < 1\}$, and give it a metric $\sigma(|z|)|\mathbf{d}z|$ with

$$\sigma(r) \approx \max\left\{\frac{1}{wr\cos(\epsilon(\log(r) - w\ell/2))}, \frac{-1}{r\log r}\right\};$$

see Figure 3. In that figure, the blue part $1/(wr\cos(\epsilon(\log(r) - w\ell)))$ describes the metric of the one-sheeted hyperboloid of curvature $-\epsilon^2$, as can be readily checked using (5), with its unique minimal closed curve of length $2\pi w$ appearing at radius $r = \exp(-w\ell/2)$ and with length $\approx \ell/2$. The red part $-1/(r\log r)$ is the original metric on the cusp. At $\approx -w\ell/2$ we replace σ by a smooth function that is slightly bigger than $\sigma(-w\ell/2)$; we can do it such that $\log(\sigma)'' \ggg 1$ at $\approx -w\ell/2$. Thus we guarantee that the new function still has a negative curvature by (5).

After the metrics on both cusps have been modified in the above manner, they can be attached along their common boundary curve $\{|z| = \exp(-w\ell/2)\}$, which is geodesic (it corresponds to the core curve of the hyperboloid). The result is a space consisting of two truncated discs with curvature -1 attached by a cylinder of curvature $-\epsilon^2$, perimeter $\approx 2\pi w$, and length $\approx \ell$.

4.4.5. *Global metric.* We now perform the plumbing between the metrized small spheres in $\mathscr{S}$. The following proposition will allow us to endow the annuli of the canonical decomposition with an expanding map.

Proposition 4.12. *There is an assignment*

$$\mathscr{C} \to (1,2) \times (1,2), \qquad \gamma \mapsto (w_\gamma, \ell_\gamma)$$

(where w_γ is the "width" of the annulus corresponding to γ and ℓ_γ is its "length") such that

- *if for a nonperipheral curve $\delta \in f^{-1}(\mathscr{C})$ the map $f\colon \delta \to f(\delta)$ is one-to-one, then $w_{f(\delta)} > w_\delta$;*
- *if for a curve $\gamma \in \mathscr{C}$ there is a unique nonperipheral curve $\delta \in f^{-1}(\mathscr{C})$ isotopic to γ, then $\ell_{f(\delta)} > \ell_\gamma$.*

Proof. We first note that only an ordering of the (w_γ) and (ℓ_γ) is required; once such an ordering is found, they can easily be embedded in the interval $(1,2)$.

If an assignment $\gamma \to w_\gamma$ is forbidden, then there is a sequence $\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0$ of curves in $\mathscr{C}$ such that $w_{\gamma_{i+1}} > w_{\gamma_i}$ holds. This means that $\bigcup_i \partial\gamma_i$ contains a Levy cycle. This contradicts the assumption that $\mathscr{C}_f$ is an anti-Levy multicurve, by Lemma 4.11.

If an assignment $\gamma \to \ell_\gamma$ is forbidden, then there is a sequence $\gamma_0, \gamma_1, \dots, \gamma_n = \gamma_0$ of curves in $\mathscr{C}$ such that $\ell_{\gamma_{i+1}} > \ell_{\gamma_i}$ holds. This means that $\bigcup_i \partial\gamma_i$ is a primitive unicycle, and this contradicts the assumption that $\mathscr{C}_f$ is a Cantor multicurve, again by Lemma 4.11. □

We scale the solutions $(\ell_\gamma), (w_\gamma)$ given by Proposition 4.12 so that $\ell \le \ell_\gamma \le 2\ell$ and $w/2 \le w_\gamma \le w$, for the parameters $\ell \ggg 1/w \ggg 1$ of the construction in §4.4.1.

We consider in turn every small sphere $S \in \mathscr{S}$ containing a curve $\gamma \in \mathscr{C}$ on its boundary. There is then another small sphere $S' \in \mathscr{S}$ also containing γ on its boundary. For $\widehat{S}$ and $\widehat{S'}$, these boundary points appear as cusps in the scaled hyperbolic metrics that were assigned to them in §4.4.6. We truncate the cusps on $\widehat{S}, \widehat{S'}$ along horocycles and attach $\widehat{S}, \widehat{S'}$ through an almost flat hyperboloid as described in §4.4.4. The hyperboloid has curvature $\approx -\epsilon^2$, perimeter $\approx 2\pi w_\gamma$ and length $\approx \ell_\gamma$.

We have, in this manner, constructed a metric sphere $X \simeq (S^2, A)$ by plumbing together truncated small spheres in $\mathscr{S}$. For every $S \in \mathscr{S}$ we denote by S° the image of $\widehat{S}$ in X. We also denote by $\mathscr{A}$ the set of almost flat annuli. We stress that $\mathscr{A}$ is in bijection with $\mathscr{C}$.

Suppose $B \in \mathscr{A}$ is an annulus connecting small spheres S_1° and S_2°. Let B_1 be the subannulus of B consisting of points in B that are closer to S_1° than S_2°. Since B_1 is constructed by enlarging the metric in $\widehat{S}_1 \setminus S_1^\circ$ we can view $B \hookrightarrow \widehat{S}_1 \setminus S_1^\circ$; we will refer to this map as *natural*. By construction, we have the following.

Lemma 4.13. *The natural map $B \to \widehat{S}_1 \setminus S_1^\circ$ is contracting.* □

4.4.6. *Dynamics at plumbings.* For a periodic cycle $f\colon \widehat{S}_i \to \widehat{S}_{i+1}$ as in §4.4.3 consider a point $x \in \widehat{S}_i$ that is a cusp with respect to the hyperbolic metric. We perform the plumbing described in §4.4.4 and §4.4.5, and explain how the dynamics are to be adjusted.

Suppose that $x \in \widehat{S}_1$ is periodic of period q. Let $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ be the unit disc punctured at 0. Since x is a cusp, the universal cover $\mathbb{D} \to (\widehat{S}_1, \mathrm{ord}_1)$ factors as $\mathbb{D} \to \mathbb{D}^* \xrightarrow{\pi_x} (\widehat{S}_1, \mathrm{ord}_1)$ with π_x extended to 0 by $\pi_x(0) = x$. Denote by $H_r^* \subset \mathbb{D}^*$ the circle, i.e., horocycle, centered at 0 with Euclidean radius r. For sufficiently small r the image $H_r := \pi_x(H_r^*)$ is a small simple closed curve around $x \in \widehat{S}_1$.

Let $d > 1$ be the local degree of f^q at x, and let $U \subset \widehat{S}_1$ be the Fatou component containing x. Choose a Böttcher coördinate $\phi\colon U \to \mathbb{D}$ conjugating $f^q\colon U \circlearrowleft$ to $z \to z^d\colon \mathbb{D} \circlearrowleft$. Denote by $E'_r \subset \mathbb{D}$ the circle centered at 0 with Euclidean radius r. Then $E_r := \phi^{-1}(E'_r)$ is an *equipotential* of U. By construction, $f^q(E_r) = E_{r^d}$. We propagate these equipotentials along the orbit of x in such a manner that we have $f(E_r) = E_{r^{\deg_x(f)}}$, where the latter is an equipotential around $f(x)$.

Since $\phi \circ \pi_x$ is conformal at 0 there is a $\tau = |\pi'_x(0)| > 0$ such that H_r is approximately $E_{\tau r}$: for small r the horocycle H_r lies in the $O(r^2)$-neighbourhood of $E_{\tau r}$ and, moreover, the hyperbolic length of $E_{\tau r}$ is $-1/\log(r) + O(-r/\log r)$. (We recall that $-1/\log(r)$ is the hyperbolic length of H_r.)

We now set up some notation. The Fatou component around $f(x)$ is denoted by U'. The hyperboloids used in the plumbing at $x, f(x)$ are, respectively, denoted by B and B'. The neighbourhood of x in the original $\widehat{S}_1$ is foliated by horocycles, and so are B, B' and the neighbourhood of $f(x)$ in the original $\widehat{S}_2$. The plumbings at $x, f(x)$ are done, respectively, on horocycles of radii R_0, R'_0. We now use d for the local degree of f at x.

We modify the map $f\colon U \to U'$ into a map from the truncation of U at H_{R_0} into $U' \cup B'$ as follows. Choose $R_+ \in (0,1)$ such that $R_+^d \in (R_0^d, R'_0)$; this is possible by Proposition 4.12. Let $(I_r)_{r \in [R_0, R_+]}$ be a family of disjoint curves in U interpolating horocycles to equipotentials: $I_{R_0} = H_{R_0}$ and $I_{R_+} = E_{\tau R_+}$ such that $\bigcup_{r \in [R_0, R_+]} I_r$ is a cylinder and such that every I_r is very close to H_r and to $E_{\tau r}$. Similarly, let $(I'_r)_{r\in[R_0^d,R_+^d]}$ be a family of disjoint curves in U' interpolating horocycles to equipotentials: $I'_{R_0^d} = H_{R_0^d}$ and $I'_{R_+^d} = E_{\tau R_+^d}$. For $r > R_+$, we map $E_{\tau r} \subset U$ to $E_{\tau r^d}$ in U', and in the case $r^d < R'_0$ we project it naturally to B'; the latter map is expanding by Lemma 4.13. For $r \in [R_0, R_+)$ we map $I_r \subset U$ to I'_{r^d} in U' and project it naturally to B'; the obtained map is expanding by Lemma 4.13. Observe that the new map sends the horocycle H_{R_0} in U to a horocycle in B'. For $r < R_0$ we shall specify the map on B in §4.4.8.

We now propagate the adjusted dynamics to all preperiodic preimages of x that are cusps with respect to the hyperbolic metric. We perform the same operation at all cusps.

4.4.7. *Dynamics at small spheres.* Recall that X consists of truncated small spheres and of almost flat cylinders connecting truncated spheres.

Consider first a small sphere $S \in \mathscr{S}$ and its f-image S'. We have a rational map $f_S := f\colon \widehat{S} \to \widehat{S}'$. For all points in S° with its f_S-image in S'° we set F to be f_S. The remaining points are bounded by $f_S^{-1}(\partial S'^\circ)$. We now extend F to S°.

Consider a curve $\gamma \in f_S^{-1}(\partial S'^\circ)$. Then either γ is nonessential rel A or $\gamma \in \mathscr{C}$ rel A. In the first case γ bounds a peripheral disc U containing at most one point in A. By construction (see §4.4.5) there is a very long almost flat annulus $B \in \mathscr{A}$ attached to $F(\gamma)$. Since $f_S\restriction_U$ is expanding, we may extend F to U (see Lemma 4.13) in such a manner that $F\restriction_U$ is expanding. If there is an $a \in A\cap U$, then we require that $F(a) = f(a)$ and that F maps a neighbourhood of a analytically (i.e., locally conformal except at a, where the map need not be an isomorphism) to a neighbourhood of $f(a)$. In the second case, $\gamma \in \mathscr{C}$ rel A, and we are then in the situation of a plumbing; the modification of the dynamics was described in §4.4.6.

4.4.8. *Dynamics at annuli.* So far F is defined on small spheres; let us assume that $F\restriction_S = f\restriction_S$ for every $S \in \mathscr{S}$. We now extend F to $X \simeq (S^2, A)$ in an expanding manner so that $F \simeq f$.

Consider an annulus $B \in \mathscr{A}$. Suppose that f maps B to a sequence of annuli and spheres $B_1, S_1, B_2, S_2, \dots, B_t$ with $B_i \in \mathscr{A}$ and $S_i \in \mathscr{S}$. Consider two cases.

Suppose first $t = 1$. Then $\ell_B - \ell_{B_1} \ggg 1$ because the values $\ell_B > \ell_{B_1}$ from Proposition 4.12 are rescaled so that $\ell_B, \ell_{B_1} \ggg 1$. Also, either $w_B > w_{B_1}$ or $w_B > w_{B_1}/2$ in case $f\restriction_B$ has degree greater than 1. Therefore, we can map B to B_1 minus a small (i.e., of scale $\lll \ell$) neighbourhood of ∂B_1 (which is already in the image of small spheres) in an expanding manner so that the obtained map F is isotopic to f rel ∂B. Indeed, identify B and $B_1 \setminus$ (small neighbourhood of ∂B_1) with $\mathbb{S}^1 \times [0,1]$, recalling that B, B_1 are almost flat. Then set F to be $(x,y) \to (dx + my, y)$, where $d \geq 1$ is the degree of $f\restriction_B$ and $m \geq 0$ is the twisting parameter. Since m, d are independent of $\ell \ggg 1 \ggg w$, the map $F\restriction_B$ is expanding.

Suppose next $t > 1$. Subdivide B into $B'_1, S'_1, B'_2, \dots, S'_{t-1}, B'_t$ so that each S_i is an annulus of length $\approx w$ and each B'_j is an annulus of length $\approx \ell/t$. Again, since $\ell \ggg 1 \ggg w$ we can define $F\restriction_B \simeq f\restriction_B$ in such a manner that F expands S'_i and B'_j into S_i and B_j, respectively.

4.4.9. *Perturbation of the metric.* We have constructed a metric space (X, μ) and a map $F\colon X \circlearrowleft$ which weakly ($\geq$) expands the metric and such that an iterate of F is expanding.

Lemma 4.14. *There is a small perturbation μ' of μ such that $F\colon X \circlearrowleft$ expands μ'.*

Proof. Let $F^p\colon X \circlearrowleft$ be an expanding iteration of F. By construction, μ is a smooth Riemannian metric such that F is conformal (rel μ) in a small neighbourhood of A.

Denote by A^∞ the set of periodic critical cycles of $F\restriction_A\circlearrowleft$. Recall that A^∞ is the set of points at infinite distance from $X \setminus A^\infty$ for μ. We also recall that cone points of μ belong to $A \setminus A^\infty$.

For $i \le p-1$ consider the pulled-back metric $\mu_i := (F^{-i})^*\mu$. Then μ_i is a Riemannian metric with cones in $f^{-i}(A\setminus A^\infty)$ and singularities in $f^{-i}(A^\infty)$. Moreover, F weakly expands μ_i.

Write $\mu_i(z)$ as a conformal metric $\sigma_i(z)|\mathbf{d}z|$ for $z \in X$ written in complex charts. For a sufficiently large $K > 1$ the inequality $\sigma_i(z) > K$ holds only in a small neighbourhood of $f^{-i}(A)$. Let $A^p \supset A^\infty$ be the set of periodic points in A. For sufficiently large K and for z close to $f^{-i}(A) \setminus A^p$, we define $\bar\sigma_i(z) \approx \min\{\sigma_i(z), K\}$ so that F still weakly expands the truncated metric $\bar\mu_i(z) = \bar\sigma_i(z)|\mathbf{d}z|$. We leave σ_i unchanged away from the neighbourhood of A^p.

We claim that for a sufficiently small $\varepsilon > 0$ the quadratic form

$$\mu' := \mu + (\bar\mu_1 + \cdots + \bar\mu_{p-1})\varepsilon$$

is positive definite (i.e., μ' is a metric) and that F expands μ'. Indeed, away from A^p all $\bar\mu_i$ are finite metrics. Therefore, if ε is sufficiently small, then μ' is positive definite away from A^p; so μ' is a metric. Since F is conformal in a small neighbourhood of A^p, all $\bar\mu_i$ and μ are conformal metrics in a common charts. Hence μ' is positive-definite as a sum of conformal metrics.

Since F^p is expanding, F expands at least one of $\mu, \bar\mu_1, \dots, \bar\mu_{p-1}$. Therefore, F expands μ'. □

4.5. **Isotopy of expanding maps.** Let $f, g\colon (S^2, A) \circlearrowleft$ be two expanding maps. Denote by $\mathcal{F}(f)$ and $\mathcal{F}(g)$ the Fatou sets of f and g, respectively. We may partially order the maps f, g by declaring that g is "smaller than" f if $A \cap \mathcal{F}(g) \subset A \cap \mathcal{F}(f)$. In this partial order, maximal elements are Böttcher expanding maps, and we will show that every map is obtained from a Böttcher expanding map by collapsing Fatou components.

Lemma 4.15. *Let $f, g\colon (S^2, A) \circlearrowleft$ be two expanding maps with $A\cap\mathcal{F}(f) = A\cap\mathcal{F}(g)$. Then f and g are conjugate by $h \simeq \mathbb{1}$ if and only if $f \simeq g$.*

Moreover, if $\#A \ge 3$, then h is unique; see [5, §C].

Proof. We show that if f, g are isotopic, then they are conjugate by $h \simeq \mathbb{1}$. This is an application of the pullback argument.

Choose $h_0, h_1 \simeq \mathbb{1}$ such that $h_1 f = g h_0$. We adjust h_0 so that it respects Böttcher coordinates around periodic points in $A \cap \mathcal{F}(f)$. Thus h_0 is equal to h_1 in a small neighbourhood of $A \cap \mathcal{F}(f)$.

Inductively, let h_n be the lift of h_{n-1}, i.e., $h_n f = g h_{n-1}$. By construction, all h_n coincide in a small neighbourhood of $A \cap \mathcal{F}(f)$.

Since f is expanding away from $A\cap\mathcal{F}(f)$, the sequence h_n tends to a continuous map $h_\infty\colon (S^2, A) \circlearrowleft$ satisfying $h_\infty f = g h_\infty$.

Observe now that we also have $h_n^{-1} g = f h_{n-1}^{-1}$. Since g is expanding away from $A \cap \mathcal{F}(g)$, the sequence h_n^{-1} tends to a continuous map $h'_\infty\colon (S^2, A) \circlearrowleft$ satisfying $h'_\infty g = f h'_\infty$. Clearly, $h'_\infty h_\infty = \mathbb{1}$; i.e., h_∞ is a homeomorphism. □

For a Thurston map $f\colon (S^2, A) \circlearrowleft$, a *Levy arc* is a nontrivial path, with (possibly equal) starting and ending points in A, that is isotopic rel A to one of its iterated lifts. Let A' be a forward-invariant subset of A. We say that A' is *homotopically isolated* if there is no Levy arc connecting two points in A'.

Lemma 4.16. *Suppose that $f\colon (S^2, A) \circlearrowleft$ is a Böttcher expanding map, that $A' \subset A \cap \mathcal{F}(f)$ is forward invariant, and that $\mathcal{F}'$ is the set of points in $\mathcal{F}(f)$ attracted by A'. Then A' is homotopically isolated if and only if the following properties hold:*

(1) *if O is a connected component of $\mathcal{F}'$, then $\overline{O}$ is a closed topological disc and, moreover, $A \cap \partial O = \emptyset$;*
(2) *if O_1, O_2 are different connected components of $\mathcal{F}'$, then $\overline{O}_1 \cap \overline{O}_2 = \emptyset$.*

Proof. Suppose first that A' is not homotopically isolated. Let ℓ be a Levy arc connecting points $a, b \in A'$. Then ℓ can be realized as an inner ray R_1 followed by an inner ray R_2. If $a \neq b$, then the closures of the Fatou components centered at a and b intersect. If $a = b$ but $R_1 \neq R_2$, then the closure of the Fatou component centered at a is not a closed disc, since it is pinched at $a = b$. If $R_1 = R_2$, then the landing point of R_1 belongs to A.

Conversely, let us assume that A' is homotopically isolated. We first verify that $A \cap \partial O = \emptyset$. Indeed, if $a \in A \cap \partial O$, then the internal ray R of O landing at a is preperiodic. For n large enough, the ray $f^n(R)$ is a periodic ray of $f^n(O)$ connecting its center, which is a point in A', to $f^n(a) \in A$. Therefore, a loop starting at the center of $f^n(O)$, then following $f^n(R)$, then circling $f^n(a)$, and then following $f^n(R)$ back to the center of $f^n(O)$ is a Levy arc.

If the conclusion of the lemma does not hold, then either there is a periodic component O of $\mathcal{F}'$ which is not a disk, and then there are two different inner rays R_1, R_2 of O that land together; or there are two periodic connected components O_1, O_2 of $\mathcal{F}'$ and respective inner rays $R_1 \subset O_1$ and $R_2 \subset O_2$ that land together.

If R_1, R_2 are inner rays of O that land together, then we have $f^n(R_1) \neq f^n(R_2)$ for all $n \geq 0$. Indeed, otherwise the common landing point of R_1, R_2 would be precritical, contradicting $A \cap \partial O = \emptyset$. Furthermore, for all n sufficiently large $f^n(R_1) \cup f^n(R_2)$ is a closed curve, nonnull-homotopic rel A. Indeed, if $f^n(R_1) \cup f^n(R_2)$ were trivial for some n, then $f^m(R_1) \cup f^m(R_2)$ would be trivial for all $m \in \{0, 1, \dots, n\}$. Then let D_m be the open disc bounded by $f^m(R_1) \cup f^m(R_2)$ and not intersecting A. We see that $f^m\colon D_0 \to D_m$ has degree one. Denote by ϕ_m the angle in D_m between $f^m(R_1)$ and $f^m(R_2)$ measured at the center $f^m(a)$ of $f^m(O)$. Then $\phi_m = \deg_a(f^m)\phi_0$. Since $\phi_0 > 0$ because $R_1 \neq R_2$, and $\deg_a(f^m) \to \infty$ as $m \to \infty$ because O is a Fatou component, we see that $f^n\colon D_0 \to D_n$ has degree greater than one for all sufficiently large n.

In all cases, we obtain for some $n > m \geq 0$ an arc $f^n(R_1) \cup f^n(R_2)$ that is isotopic to $f^m(R_1) \cup f^m(R_2)$ A, so $f^n(R_1) \cup f^n(R_2)$ is a Levy arc. $\square$

Suppose that $\sim$ is a closed equivalence relation on S^2 whose equivalence classes are connected and filled-in (namely, with connected complement) compact subsets of S^2 and suppose that not all points of S^2 are equivalent. In this case Moore's theorem [15] states that the quotient space $S^2/{\sim}$ is homeomorphic to S^2.

Corollary 4.17. *Suppose that $f\colon (S^2, A) \circlearrowleft$ is a Böttcher expanding map and suppose that $A' \subset A \cap \mathcal{F}(f)$ is a forward invariant homotopically isolated subset* *of A. Let $\mathcal{F}'$ be the set of points in $\mathcal{F}(f)$ attracted by A'. Then the equivalence relation $\sim$ on S^2 specified by*

$$x \sim y \iff \begin{cases} x = y \quad \text{or} \\ x, y \text{ are in the closure of the same connected component of } \mathcal{F}' \end{cases}$$

is an f-invariant equivalence relation satisfying Moore's theorem. View $((S^2, A)/{\sim}) \simeq (S^2, A)$. The induced map $f/{\sim}\colon ((S^2, A)/{\sim}) \circlearrowleft$ is topologically expanding and is isotopic rel A to f.

Proof. It is clear that $f/{\sim}$ is topologically expanding. If we view $((S^2, A)/{\sim}) \simeq (S^2, A)$, then f and $f/{\sim}$ have isomorphic bisets; therefore $f \simeq f/{\sim}$. $\square$

Proposition 4.18. *Let $f, g\colon (S^2, A) \circlearrowleft$ be two expanding maps such that $f \simeq g$ and $A \cap \mathcal{F}(g) \subseteq A \cap \mathcal{F}(f)$. Write $A' := A \cap (\mathcal{F}(f) \setminus \mathcal{F}(g))$ and let $\mathcal{F}'$ be the set of points attracted towards A' under iteration of f.*

Then there is a semiconjugacy $\pi\colon (S^2, A) \to (S^2, A)$ from f to g defined by

$$\pi(x) = \pi(y) \iff \begin{cases} x = y \quad \text{or} \\ x, y \text{ are in the closure of the same connected component of } \mathcal{F}'. \end{cases}$$

As in Lemma 4.15, the semiconjugacy π is unique.

Proof. It is sufficient to prove this proposition for the case in which f is a Böttcher expanding map. By Lemma 4.16 applied to g we see that A_g is homotopically isolated. Therefore, again by Lemma 4.16 we can collapse $\mathcal{F}'$ to obtain a topologically expanding map $f/\mathcal{F}'$. Since $f/\mathcal{F}' \approx g$, the claim now follows from Lemma 4.15. $\square$

5. Computability of the Levy decomposition

In this section, we give algorithms that prove Corollaries C and D.

Recall that a branched covering $f\colon (S^2, P_f, \mathrm{ord}_f) \circlearrowleft$ is doubly covered by a torus endomorphism if and only if P_f contains exactly four points and $\mathrm{ord}_f(P_f) = \{2\}$. Moreover, in this case $f\colon (S^2, P_f, \mathrm{ord}_f) \circlearrowleft$ is itself an orbifold self-covering and its biset $B(f)$ is right principal. It is easy to see that $G := \pi_1(S^2, P_f, \mathrm{ord}_f)$ is isomorphic to $\mathbb{Z}^2 \rtimes_{-\mathbb{1}} \mathbb{Z}/2$ and that $B(f)$ is of the following form: for a 2×2 integer matrix M with $\det(M) > 1$ and a vector $v \in \mathbb{Z}^2$, denote by $M^v\colon \mathbb{Z}^2 \rtimes_{-\mathbb{1}} \mathbb{Z}/2 \circlearrowleft$ the endomorphism given by a "cross product structure" (see [2, Proposition III.11]):

$$M^v(n, 0) = (Mn, 0) \text{ and } M^v(n, 1) = (Mn + v, 1). \tag{6}$$

Then $B(f)$ is isomorphic to G as a set, with left and right actions given by $g \cdot b \cdot h = M^v(g)bh$ for all $g, b, h \in G$. Moreover, $f\colon (S^2, P_f, \mathrm{ord}_f) \circlearrowleft$ is combinatorially equivalent to the quotient of $z \mapsto Mz + v\colon \mathbb{R}^2/\mathbb{Z}^2 \circlearrowleft$ by the involution $z \mapsto -z$. Indeed, every endomorphism of G is of the form (6).

Algorithm 5.1. GIVEN *a sphere biset* ${}_GB_G$*,*
DECIDE *whether* B *is the biset of a map double covered by a torus endomorphism* AS FOLLOWS:

(1) Compute the action of B on peripheral conjugacy classes in G.
(2) Determine the minimal orbisphere structure $(S^2, \operatorname{ord}_B)$ from the action on peripheral conjugacy classes; see §3.3.
(3) Return `yes` if the Euler characteristic of $(S^2, \operatorname{ord}_B)$ is $= 0$ and ord_B has four marked points, and `no` otherwise.

Algorithm 5.2. GIVEN *a sphere biset* ${}_GB_G$ *of a map double covered by a torus endomorphism,*
COMPUTE *parameters* M, v *for the torus endomorphism* $z \mapsto Mz + v$ AS FOLLOWS:

(1) As in Algorithm 5.1, compute the action of B on peripheral conjugacy classes in G, and determine the quotient map $\pi\colon G \to \overline{G}$ to the minimal orbisphere structure (see §3.3) and the quotient biset ${}_{\overline{G}}\overline{B}_{\overline{G}}$.
(2) Note that $\overline{G}$ is of the form $\mathbb{Z}^2 \rtimes \mathbb{Z}/2$, where $\mathbb{Z}^2$ is generated by all even products of peripheral generators and $\mathbb{Z}/2$ is generated by any chosen generator.
(3) Since the map corresponding to B is a covering, the biset $\overline{B}$ is left-free and right-principal. Choose an arbitrary element $\overline{x} \in \overline{B}$, thus identifying $\overline{B}$ with $\overline{G}$ via $\overline{x}g \leftrightarrow g$.
(4) Let $\{g_0, g_1\}$ be a basis of $\mathbb{Z}^2 \subset \overline{G}$, and choose a peripheral generator h of $\overline{G}$. Write $g_0\overline{x} = \overline{x}g_0^a g_1^b$ and $g_1\overline{x} = \overline{x}g_0^c g_1^d$ for some $a, b, c, d \in \mathbb{Z}$ which form the matrix $M = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$, and write $h\overline{x} = \overline{x}g_0^e g_1^f h$ for some $e, f \in \mathbb{Z}$ forming the vector $v = \left(\begin{smallmatrix} e \\ f \end{smallmatrix}\right)$.

The following algorithm determines whether a biset is {GTor/2}. We shall give, in [5], a much more efficient encoding of nonpostcritical marked periodic points and improve the speed of Algorithm 5.4. The present algorithm relies on the following theorem.

Theorem 5.3 ([20, Main Theorem II])**.** *Let* f *be a Thurston map that is doubly covered by a torus endomorphism. If* f *is Levy-free, then it is* {GTor/2}*.*

Algorithm 5.4. GIVEN *a sphere biset* ${}_GB_G$ *of a map double covered by a torus endomorphism,*
DECIDE *whether* B *is the biset of a* {GTor/2} *map* AS FOLLOWS:

(1) Use Algorithm 5.2 to obtain a 2×2 matrix M expressing the linear part of the endomorphism covering B, and return `no` if M has ± 1 as eigenvalue.
(2) Choose a basis X of B. Using the action of B on peripheral conjugacy classes, determine those (call them A') that correspond to nonpostcritical points.
(3) Make the finite list of all choices $\widehat{A'}$ of periodic points or preperiodic points on the torus that map to each other as the peripheral conjugacy classes map to each other under B_*.
(4) Run the following two steps in parallel. By Theorem 5.3, precisely one of them will terminate.

(5) For an enumeration of all multicurves $\mathscr{C}$, check whether $\mathscr{C}$ is a Levy cycle, and if so return `no`.
(6) For each choice $\widehat{A'}$ of periodic points, compute the biset $\widehat{B(A')}$ of the map $(z \mapsto Mz + v)/\{\pm 1\}$ with $(\frac{1}{2}\mathbb{Z}^2/\mathbb{Z}^2 \cup \widehat{A'})/\{\pm 1\}$ marked, and go through the countably many maps $X \to \widehat{B(A')}$. If one of these maps extends to an isomorphism of bisets, return `yes`.

Algorithm 5.5. GIVEN *a sphere biset* ${}_GB_G$,
DECIDE *whether B is the biset of an expanding map* AS FOLLOWS:

(1) Check, using Algorithm 5.1, whether B is double covered by a torus endomorphism. If not, run the next two steps in parallel. If B is double covered by a torus endomorphism, then run Algorithm 5.2 to obtain a 2×2 matrix M expressing the linear part of the endomorphism, and run Algorithm 5.4 to decide whether ${}_GB_G$ is a geometric biset. If ${}_GB_G$ is not a geometric biset or at least one eigenvalue of M has absolute value less than 1, then return `no`. Otherwise return `yes`.
(2) As in §3.3 pass to a bounded quotient of B. Enumerate all finite subsets of G, and check whether one is the nucleus of (B, X). If so, return `yes`.
(3) Simultaneously, enumerate all multicurves $\mathscr{C}$ on (S^2, A), and check whether any is a Levy obstruction for B. If so, return `no`.

By Theorem A, either step (2) or step (3) will succeed.

The following algorithm computes the Levy decomposition and proves in this manner Corollary D.

Algorithm 5.6. GIVEN *a Thurston map $f\colon (S^2, A) \circlearrowleft$ by its biset,*
COMPUTE *the Levy decomposition of f* AS FOLLOWS:

(0) We are given a G-G-biset $B = B(f)$. Recall that multicurves on (S^2, A) are treated as collections of conjugacy classes in G. Their B-lift is computable by [3, §2.6].
(1) For an enumeration of all multicurves $\mathscr{C}$ on (S^2, A) that never reaches a multicurve before reaching its proper submulticurves, do the following steps.
(2) If the multicurve $\mathscr{C}$ is not invariant or is not Levy, continue in (1) with the next multicurve.
(3) Compute the decomposition of B using the algorithm in [4, Theorem 3.9].
(4) If all return bisets of the decomposition are either of degree 1 or expanding (recognized using Algorithm 5.5) or {GTor/2} (recognized using Algorithm 5.4), then return $\mathscr{C}$.
(5) Proceed with the next multicurve.

6. AMALGAMS

In the previous sections, we considered a single Thurston map—or, equivalently, a sphere biset—and characterized when it is combinatorially equivalent to an expanding map.

In this section, we rather consider a Thurston map that is defined as an "amalgam" of small maps, glued together along a multicurve; we derive a criterion for the amalgam to be expanding. A typical example is a *formal mating*, which is a sphere map admitting an "equator"—a simple closed curve γ isotopic to its lift, which maps back to γ by maximal degree. We first give an algebraic characterization in terms of bisets, and then its geometric translation in terms of internal rays.

6.1. **Sphere trees of bisets.** We briefly recall from [4, Definition 3.7] the notion of *sphere tree of biset*: firstly, we are given a tree $\mathfrak{X}$ of groups, namely a tree with a group attached to every vertex and edge, and inclusions $G_e \to G_{e^-}$ and isomorphisms $G_e \leftrightarrow G_{\overline{e}}$ from an edge e, respectively, to its source e^- and its reverse $\overline{e}$. Secondly, we are given analogously a tree $\mathfrak{B}$ of bisets and two graph morphisms $\lambda, \rho\colon \mathfrak{B} \to \mathfrak{X}$ such that ρ is a graph covering and λ is monotonous (preimages of connected sets are connected).

The graph of groups $\mathfrak{X}$ has a *fundamental group* $\pi_1(\mathfrak{X}, *)$ at each vertex $* \in \mathfrak{X}$; this is the group of expressions of the form $(g_0, e_0, g_1, \dots, e_{n-1}, g_n)$ with $(e_0, \dots, e_{n-1})$ a closed path in $\mathfrak{X}$ based at $*$ and $g_i \in G_{e_i^-}$, subject to natural relations coming from the edge group inclusions. Likewise, the graph of bisets $\mathfrak{B}$ has a *fundamental biset*, which is an ordinary biset for the fundamental group. Just as sphere bisets (up to isomorphism) capture Thurston maps (up to isotopy), sphere trees of bisets capture Thurston maps with an invariant multicurve.

Consider a sphere group G and a sphere G-G-biset B. A *Levy cycle* in B is a periodic sequence of conjugacy classes $g_0^G, \dots, g_{m-1}^G, g_m^G = g_0^G$ such that each g_i^G is a B-lift of g_{i+1}^G; namely, there are biset elements $b_0, \dots, b_{m-1} \in B$ such that $g_i b_i = b_i g_{i+1}$ holds for all $i = 0, \dots, m-1$. More succinctly, in the product biset $B^{\otimes m}$ we have a commutation relation $g_0 b = b g_0$.

Lemma 6.1. *Let $f\colon (S^2, A) \circlearrowleft$ be a Thurston map not doubly covered by a torus endomorphism map. Then f admits a Levy cycle if and only if $B(f)$ admits one.*

Proof. If $(g_0^G, \dots, g_{m-1}^G)$ is a Levy cycle in $B(f)$, then $B(f)$ is not contracting, so f is not expanding by Theorem A and thus contains a Levy cycle again by Theorem A.

Conversely, let $(\gamma_0, \dots, \gamma_{m-1})$ be a Levy cycle for f, and write each γ_i as a conjugacy class g_i^G. Since each γ_{i+1} has an f-lift isotopic to γ_i, there are biset elements $b_0, \dots, b_{m-1}$ such that $g_i^{\pm G} b_i \ni b_i g_{i+1}$. Up to replacing some g_i by their inverses, we may assume $g_i^G b_i \ni b_i g_{i+1}$ except possibly $g_{m-1}^G b_{m-1} \ni b_{m-1} g_0^{-1}$. In that case, increase m to $2m$ and set $g_{m+i} = g_i^{-1}$ for $i = 0, \dots, m-1$ so as to have $g_i^G b_i \ni b_i g_{i+1}$ for all i, namely $g_i^{h_i} b_i = b_i g_{i+1}$ for some elements $h_i \in G$. Finally, set $c_i \coloneqq h_i b_i$ to obtain $g_i c_i = c_i g_{i+1}$ for all i. Thus $(g_0^G, \dots, g_{m-1}^G)$ is a Levy cycle in B. □

The following definition captures the notion of algebraic Levy cycles for graphs of bisets.

Definition 6.2. Let $\mathfrak{B}$ be a sphere tree of bisets. A *periodic pinching cycle* for $\mathfrak{B}$ is

(1) a sequence of m closed paths $\gamma_j \coloneqq (v_{0,j}, e_{1,j}, v_{1,j}, \dots, e_{n,j}, v_{n,j} = v_{0,j})$ in the tree $\mathfrak{B}$, for $j = 0, \dots, m-1$, such that $\rho(\gamma_{j+1}) = \lambda(\gamma_j)$, indices read modulo m;
(2) a sequence of $m \times n$ biset elements $b_{i,j} \in B_{e_{i,j}}$ and group elements $g_{i,j} \in G_{\rho(v_{i,j})}$, for $i = 0, \dots, n-1$ and $j = 0, \dots, m-1$, satisfying
$$g_{i,j+1} b^-_{i+1,j} = b^+_{i,j} g_{i,j} \text{ for all } i, j,$$
indices being read cyclically. △

Consider a periodic pinching cycle. Note that elements $g_{0,j}\rho(e_{i,j})g_{1,j}\cdots\rho(e_{n,j})$, for $j=0,\dots,m-1$, define elements of the fundamental group of $\mathfrak{X}$ based at $\rho(v_{0,j})$, and that their conjugacy classes again produce a Levy cycle for the fundamental biset of $\mathfrak{B}$.

We shall always assume that periodic pinching cycles are nontrivial: $m,n>0$ and the elements $g_{0,j}\rho(e_{i,j})g_{1,j}\cdots\rho(e_{n,j})$ are reduced in the fundamental group of $\mathfrak{X}$.

Recall also that, in a tree of bisets $\mathfrak{B}$, vertices of $\mathfrak{B}$ are classified as *essential* and *inessential*; every vertex $v\in\mathfrak{X}$ has a unique λ-preimage $\iota(v)\in\mathfrak{B}$ that is essential. Consider a vertex $v\in\mathfrak{X}$, and assume that $(\rho\circ\iota)^m(v)=v$ for some $m>0$. The corresponding *return biset* is $B_{\iota(v)}\otimes\cdots\otimes B_{\iota(\rho\circ\iota)^{m-1}(v)}$, and it is a G_v-G_v-biset. We denote by $R(\mathfrak{B})$ the set of all return bisets of $\mathfrak{B}$.

Let $\mathfrak{X}$ be a tree of sphere groups with fundamental group $G=\pi_1(\mathfrak{X},*)$. Recall that the edge groups G_e in $\mathfrak{X}$ embed as cyclic subgroups of G. Choose a generator $t_e\in G_e$ for every edge $e\in\mathfrak{X}$, and consider the collection of their conjugacy classes $\mathscr{C}=\{t_e^G\mid e\in E(\mathfrak{X})\}$. We call $\mathscr{C}$ the *edge multicurve* of $\mathfrak{X}$.

Given a sphere biset B, recall that its *portrait* is the induced map $B_*\colon A\circlearrowleft$ on the set of peripheral conjugacy classes. A portrait is *hyperbolic* if every periodic cycle of B_* contains a critical peripheral class; i.e., if B is the biset of a rational map f, then all critical points of f are in the Fatou set.

Theorem 6.3. *Let $\mathfrak{B}$ be a sphere tree of bisets, and let $B\coloneqq\pi_1(\mathfrak{B})$ denote its fundamental biset. Assume that the portrait of B is hyperbolic. Then B is sphere contracting if and only if the following all hold:*

1. *all return bisets in $R(\mathfrak{B})$ are contracting;*
2. *the edge multicurve of $\mathfrak{B}$ contains no Levy cycle;*
3. *there is no nontrivial periodic pinching cycle for $\mathfrak{B}$.*

Proof. Each of the conditions is clearly necessary: if a return biset of $\mathfrak{B}$ is not contracting, then its image in B is still not contracting; if an edge multicurve is a Levy cycle, then it is a Levy cycle for B; and, by definition, a periodic pinching cycle has an iterated lift that is isotopic to itself, so every periodic pinching cycle generates a Levy obstruction.

Conversely, assume that every return biset in $\mathfrak{B}$ is contracting, that the edge multicurve $\mathscr{C}$ of $\mathfrak{B}$ is Levy-free, and that B is not contracting. Then by Theorem A there is a Levy cycle in B. Write $G=\pi_1(\mathfrak{X},*)$, and let $\{\ell_0^G,\dots,\ell_{m-1}^G\}$ denote this Levy cycle. The conjugacy classes ℓ_j^G are not reduced to conjugacy classes in vertex or edge groups, because return bisets are contracting and the edge multicurve is Levy-free, so every ℓ_j^G admits a representative ℓ_j of the form $g_{0,j}f_{1,j}g_{1,j}\cdots f_{n(j),j}\in\pi_1(\mathfrak{X},w_j)$; here $f_{1,j}\cdots f_{n(j),j}$ is a loop in $\mathfrak{X}$ based at w_j, and $g_{i,j}\in G_{f_{i,j}^+}$. Furthermore, if we require each $n(j)$ to be minimal, then this expression of a representative is unique up to cyclic permutation.

Since $\{\ell_0^G,\dots,\ell_{m-1}^G\}$ is a Levy cycle, there are $b_0,\dots,b_{m-1}\in B$ with $\ell_jb_j=b_j\ell_{j+1}$ for all j. Furthermore, since the tree of bisets $\mathfrak{B}$ is left fibrant, every $b_j\in B$ may be written as $b_j=h_jc_j$ for some $c_j\in B_{v_{0,j+1}}$ the vertex biset of a vertex $v_{0,j}\in\mathfrak{B}$ with $\rho(v_{0,j})=w_j$, and some element $h_j\in\pi_1(\mathfrak{X},w_j,w_{j+1})$ in the path groupoid of $\mathfrak{X}$. We get

$$\ell_j^{h_j}c_j=c_j\ell_{j+1}\text{ for all }j=0,\dots,m-1.$$

Now, again because $\mathfrak{B}$ is left fibrant, each path ℓ_j lifts by ρ to a unique path $\gamma_j := (v_{0,j}, e_{1,j}, v_{1,j}, \dots, e_{n(j),j}, v_{n(j),j} = v_{0,j})$, and the above equation gives $\lambda(\gamma_{j+1}) = \ell_j^{h_j}$. In particular, the length of $\ell_j^{h_j}$ is at most the length of ℓ_{j+1}; it follows that all $\ell_j^{h_j}$ are cyclically reduced and all have the same length n.

We may now redefine ℓ_j as the appropriate cyclic permutation of itself so that $\ell_j c_j = c_j \ell_{j+1}$ holds for all $j = 0, \dots, m-2$, and we have $\ell_{m-1}^{h_{m-1}} c_{m-1} = c_{m-1}\ell_0$, where $\ell_{m-1}^{h_{m-1}}$ is a cyclic permutation of ℓ_{m-1}. At worst replacing m by mn and letting ℓ_{km+j} be the appropriate cyclic permutation of ℓ_j for all $j = 0, \dots, m-1$ and all $k = 0, \dots, n-1$, we may ensure that $\ell_j c_j = c_j \ell_{j+1}$ holds for all j. Set $c_{0,j} := c_j$ and choose $c_{i,j} \in B_{f_{i,j}}$ so that $g_{i,j+1} c^-_{i+1,j} = c^+_{i,j} g_{i,j}$ holds. We have constructed a periodic pinching cycle. $\square$

Furthermore, it is decidable whether $\mathfrak{B}$ admits a periodic pinching cycle: for example, Algorithm 5.5 tells us whether the fundamental biset B is expanding; in that case, there is no periodic pinching cycle, while if not, then a periodic pinching cycle may be found by enumerating all mn-tuples of biset and group elements as in Definition 6.2.

6.2. **Trees of correspondences.** The algebraic construction above is closely related to the topological construction of an "amalgam" $\mathfrak{F}$ of maps. We shall not stress too precisely the conditions that must be satisfied by $\mathfrak{F}$, but rather give an intuitive connection to the previous subsection: on the one hand, such a formalism is well developed in [18]; on the other hand, the algebraic picture is the one that we use in practice.

We may start with the following data: first, one is given a finite tree $\mathfrak{T}$ expressing a decomposition of a marked sphere (S^2, A). Let there be a topological sphere S_v for every vertex $v \in \mathfrak{T}$, and a cylinder (written S_e) for every edge $e \in \mathfrak{T}$. There is a finite set $A_v \subset S_v$ of marked points assigned to each vertex $v \in \mathfrak{T}$. If whenever e touches v one removes a small disk around a certain marked point from S_v and attaches its boundary to a boundary of the cylinder S_e, after gluing one obtains a marked sphere (S^2, A) so that A is $\bigcup_v A_v \setminus \{\text{removed points}\}$.

Second, one is given a tree of correspondences: a tree $\mathfrak{F}$ also expressing a decomposition of a marked sphere and two graph morphisms $\lambda, \rho \colon \mathfrak{F} \to \mathfrak{T}$. To every vertex and edge $z \in \mathfrak{F}$ one is given a "topological correspondence" between the spaces $\lambda(z)$ and $\rho(z)$. More precisely, for each vertex $v \in \mathfrak{F}$ one is given a marked sphere (S_v, A_v), a covering map $S_v \setminus A_v \to S_{\rho(v)} \setminus A_{\rho(v)}$, and an inclusion $S_v \to S_{\lambda(v)}$ (note that $\lambda(v)$ need not be a vertex). Similarly, for every edge $e \in \mathfrak{F}$ one is given a cylinder S_e together with a covering map $S_e \to S_{\rho(e)}$ and an inclusion $S_e \to S_{\lambda(e)}$. The marked set A is assumed to be forward invariant and contains all critical values of all correspondences F_z.

Typical examples to consider are matings (as we saw in the introduction), for which the trees $\mathfrak{T}$ and $\mathfrak{F}$ have a single edge. The correspondence at each vertex $v_\pm$ is the polynomial $p_\pm$, and the correspondence at the edge is $z \mapsto z^d$ if the cylinder is modelled on $\mathbb{C}^*$.

We denote by $R(\mathfrak{F})$ the *small maps* of $\mathfrak{F}$, namely the return maps to vertex spheres obtained by composing the correspondences along cycles. Again in the example of matings, the small maps are $p_\pm$.

By the "van Kampen theorem" for bisets (see [3] and [4, Theorem C]) we may freely move between the languages of trees of correspondences $\mathfrak{F}$, sphere trees of bisets $\mathfrak{B}$, sphere bisets with invariant algebraic multicurve $(B, \mathscr{C})$ represented as conjugacy classes in the fundamental group, and Thurston maps with invariant multicurve $f\colon (S^2, A, \mathscr{C}) \circlearrowleft$. We call f the *limit* of $\mathfrak{F}$.

Let $\mathfrak{F}$ be a tree of correspondences with Böttcher expanding return maps. Let $\mathscr{C}$ denote the invariant multicurve associated with the edges of $\mathfrak{F}$, namely $\mathscr{C}$ is the set of core curves of cylinders represented by edges of $\mathfrak{T}$. We assume that $\mathscr{C}$ is Levy-free. Let $\mathscr{C}_0$ denote the union of primitive unicycles in $\mathscr{C}$. Consider $\gamma \in \mathscr{C}_0$ and denote by S_e the cylinder with core curve γ, for $e \in \mathfrak{T}$. Since γ is contained in a primitive unicycle, there is a unique $f \in \mathfrak{F}$ with $\lambda(f) = e$. We call the core curve of $S_{\rho(f)}$ the *image* of γ. In this manner, there is a well-defined (up to isotopy) first return map $f_\gamma\colon \gamma \to \gamma$; up to isotopy we assume that f_γ is conjugate to $z \to z^d\colon S^1 \circlearrowleft$, with $d > 1$ because $\mathscr{C}$ is Levy-free.

The curve γ is on the boundary of two small periodic spheres, call them S_1 and S_2. By assumption, the first return maps on S_1 and S_2 are Böttcher expanding. There are periodic Fatou components $F_1 \subset S_1$ and $F_2 \subset S_2$ such that γ is viewed as the circle at infinity of F_1 and F_2. Then points in γ parameterize internal rays of F_1 and F_2, and periodic internal rays are parameterized by periodic points of $f_\gamma\colon \gamma \circlearrowleft$, namely by rationals of the form $m/(d^n - 1)$ for some $m, n \in \mathbb{N}$.

Definition 6.4. Let $\mathfrak{F}$ be a tree of correspondences with expanding return maps. Let $\mathscr{C}$ denote the invariant multicurve associated with the edges of $\mathfrak{T}$. Let $\mathscr{C}_0$ denote the union of the primitive unicycles in $\mathscr{C}$.

A *periodic pinching cycle* for $\mathfrak{F}$ is a sequence $z_1, \dots, z_n$ of periodic points on $\mathscr{C}_0$ and a sequence of internal rays $I_1^\pm, \dots, I_n^\pm$ in the Fatou components of small maps in $\mathfrak{F}$ touching $\mathscr{C}_0$ such that, indices read modulo n,

- I_i^+ and I_{i+1}^- are both parameterized by z_i, and lie in neighbouring spheres;
- I_i^+ and I_i^- both land at the same point and in the same sphere. △

As mentioned above, topological periodic pinching cycles are the form that Levy cycles take in trees of correspondences with expanding return maps: given a Levy cycle, we may put it in minimal position with respect to the multicurve $\mathscr{C}$ associated with the edges of the tree, and thus decompose the Levy cycle into periodic arcs, with each arc contained in a small sphere and connecting two boundary circles.

If we choose basepoints on the small spheres and boundary circles, and paths from the boundary circle basepoint to the neighbouring sphere basepoints, we may translate these arcs into loops in fundamental groups of small spheres.

Even though it is not necessary for our argument, let us explain more precisely how to construct an algebraic periodic pinching cycle out of a topological one. For simplicity assume that all small spheres and cylinders are fixed. Choose basepoints $*_t$ on all small spheres and curves S_t in $\mathfrak{T}$, identifying the group G_t with $\pi_1(S_t, *_t)$ and the biset B_t with homotopy classes of paths from $*_t$ to an f-preimage of $*_t$. Choose for each edge $e \in \mathfrak{T}$ a path ℓ_e from $*_e$ to $*_{e^-}$.

Consider a periodic pinching cycle for $\mathfrak{F}$, and assume again for simplicity that all rays $I_i^\pm$ are fixed. To every fixed point z_i, say $z_i \in S_{t(i)}$, there corresponds a biset element $b_i \in B_{t(i)}$: choose a path γ_i in $S_{t(i)}$ from $*_{t(i)}$ to z_i, and set $b_i := \gamma_i \# f^{-1}(\gamma_i^{-1})$. Since f is expanding, the infinite concatenation of lifts b_i^∞ is a path from $*_{t(i)}$ to z_i. Note that here we are using the identification of the circle

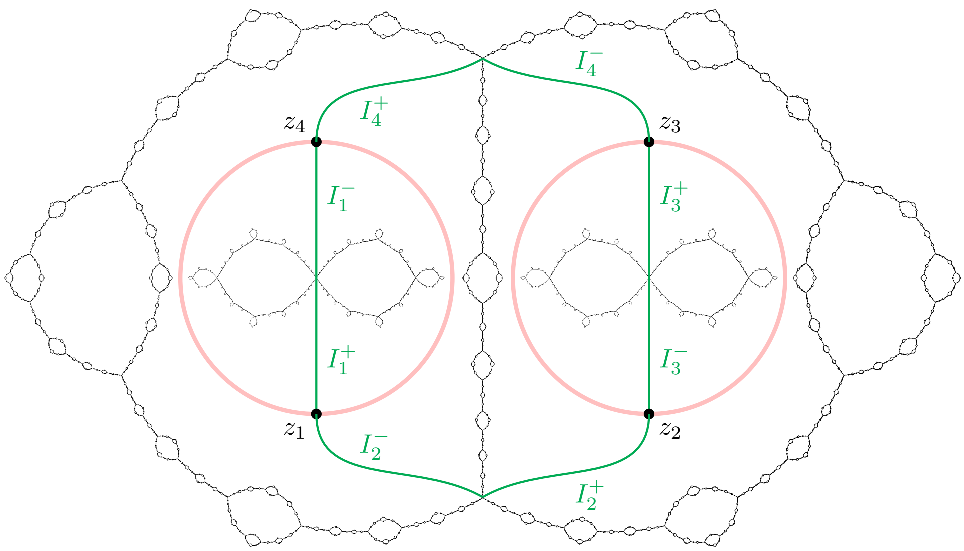


FIGURE 4. A periodic pinching cycle. There is a central fixed sphere mapping under $z^3\frac{2z-1}{2-z}$, and two spheres attached on the Fatou components of 0 and 1 mapping under $3z^2-2z^2$. The periodic pinching cycle is in green, and the edges of the tree of spheres are in red.

$S_{t(i)}$ with the Julia set of z^d for some $d>1$ and with the Julia set $\mathcal{J}(B_{t(i)})$ of the biset $B_{t(i)}$; recall from §3.2 that it consists of equivalence classes of bounded (here constant b_i^∞) infinite sequences in $B_{t(i)}$. Let the rays $I_i^\pm$ belong to sphere $S_{v(i)}$, and set $g_i := \ell_{t(i-1)}^{-1}\#b_{i-1}^\infty\#I_i^-\#(I_i^+)^{-1}\#(b_i^\infty)^{-1}\#\ell_{t(i)} \in G_{v(i)}$. Then these data $b_i, g_i, v(i), t(i)$ determine an algebraic periodic pinching cycle with $m=1$. In general, the periodic pinching cycle for $\mathfrak{F}$ will be periodic but not fixed, and m will be >1.

Given a Thurston map $f\colon (S^2,A)\circlearrowleft$, recall that its *portrait* is the induced map $f\colon A\circlearrowleft$ with its local degree. A portrait is *hyperbolic* if all its cycles contain a point of degree >1.

Theorem 6.5. *Let $\mathfrak{F}$ be a tree of maps with hyperbolic portraits. Then its limit $f\colon (S^2,A)\circlearrowleft$ is isotopic to an expanding map if and only if all of the following hold:*

(1) *all small maps of $\mathfrak{F}$ are isotopic to expanding maps;*
(2) *the invariant multicurve associated with the edges of $\mathfrak{T}$ is Levy-free;*
(3) *there is no nontrivial periodic pinching cycle for $\mathfrak{F}$.*

Proof. This is a direct translation of Theorem 6.3. It it instructive to give a geometric proof of the only nontrivial implication, namely that if f admits a Levy cycle L, then it admits a periodic pinching cycle.

Put L in minimal position with respect to $\mathscr{C}$. By Proposition 2.4(2), a Levy cycle may only intersect a primitive unicycle. Choose a curve $\ell\in L$, and let $z_1,\dots,z_n$ denote, in cyclic order along ℓ, the intersections of ℓ with $\mathscr{C}$. Assuming that all small maps are expanding, the pieces of ℓ between points z_i and z_{i+1} belong to

Fatou components and their boundaries, and may be assumed to be internal rays. In this manner we have obtained a periodic pinching cycle. □

6.3. **Higher-degree matings.** We are ready to prove Theorem E. Note that, in the case of matings, periodic pinching cycles of periodic angles are precisely the periodic pinching cycles defined above for amalgams.

6.3.1. *Polynomials.* Let f be a complex polynomial of degree $d \geq 2$. The *filled-in Julia set* $\mathcal{K}(f)$ of f is

$$\mathcal{K}(f) = \{z \in \mathbb{C} \mid f^n(z) \not\to \infty \text{ as } n \to \infty\}.$$

Assume that $\mathcal{K}(f)$ is connected, and let ϕ be the inverse of the Böttcher coördinate associated with the Fatou component of ∞, so we have $\phi\colon \widehat{\mathbb{C}} \setminus \mathcal{K}(f) \to \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ satisfying $\phi(f(z)) = \phi(z)^d$ and $\phi(\infty) = \infty$ and $\phi'(\infty) = 1$. For $\theta \in \mathbb{R}/\mathbb{Z}$, the associated *external ray* $R_f(\theta)$ is defined as $\{\phi^{-1}(re^{2i\pi\theta}) \mid r > 1\}$.

We have $\mathcal{J}(f) = \partial\mathcal{K}(f)$. Assume now that f is postcritically finite; in particular, $\mathcal{J}(f)$ is locally connected. Then the landing point $\pi(\theta) := \lim_{r\to 1^+} \phi_f^{-1}(re^{2i\pi\theta})$ of the ray $R_f(\theta)$ exists for all θ and defines a continuous map $\pi\colon \mathbb{R}/\mathbb{Z} \to \mathcal{J}(f)$.

On the other hand, consider a basepoint $* \in \mathbb{C} \setminus A$ very close to ∞ so that its preimages $*_0, \dots, *_{d-1}$ are all also very close to ∞. Let $t \in \pi_1(\mathbb{C}, *)$ denote a short counterclockwise loop around ∞, and choose for all $i = 0, \dots, d-1$ a path ℓ_i from $*$ to $*_i$ that remains in the neighbourhood of ∞, and in such a manner that the paths $\ell_i \# f^{-1}(t)$ and ℓ_{i+1} are homotopic for all $i = 0, \dots, d-2$ and that $\ell_{d-1} \# f^{-1}(t)$ is homotopic to $t \# \ell_0$. Here by "$s \# f^{-1}(t)$" we denote the concatenation of a path s with the unique f-lift of t that starts where s ends.

The following proposition illustrates the link between Julia sets (see also §4.2) and bisets in the concrete case of polynomials.

Proposition 6.6. *The set $X := \{\ell_0, \dots, \ell_{d-1}\}$ is a basis of $B(f)$. Let $\rho\colon \{0, \dots, d-1\}^\infty \to \mathbb{R}/\mathbb{Z}$ be the base-d encoding map $x_1x_2\cdots \mapsto \sum x_i d^{-i}$. Then the following diagram commutes:*

$$\begin{tikzcd}
X^\infty \arrow[r, equal] \arrow[dd, "/\sim"'] & \{0, \dots, d-1\}^\infty \arrow[d, "\rho"] \\
& \mathbb{R}/\mathbb{Z} \arrow[d, "\pi"] \\
\mathcal{J}(B(f)) \arrow[r, leftrightarrow] & \mathcal{J}(f)
\end{tikzcd}$$

where $\sim$ is the asymptotic equivalence relation defined in §3.2.

Proof. Consider $x_1x_2\cdots \in X^\infty$ with each $x_i = \ell_{m_i}$ for some $m_i \in \{0, \dots, d-1\}$. Then the path $x_1 \# f^{-1}(x_2) \# f^{-2}(x_3) \cdots$ is a well-defined path in $\mathbb{C} \setminus \mathcal{K}(f)$, which has a limit because f is expanding and has the same limit as $R_f(\theta)$ for $\theta = \rho(m_1m_2\cdots)$ because with respect to the hyperbolic metric of $\mathbb{C} \setminus K(f)$ there is a $\delta > 0$ such that $x_1 \# f^{-1}(x_2) \# f^{-2}(x_3) \cdots$ is in the δ-neighbourhood of $R_f(\theta)$. □

Proof of Theorem E. (1) $\Rightarrow$ (2): assume that $p_+ \uplus p_- \colon \mathbb{S} \circlearrowleft$ is combinatorially equivalent to an expanding map $h\colon S^2 \circlearrowleft$. Denote by Σ the quotient of $\mathbb{S}$ in which all external rays are shrunk to points.

Let $\mathcal{J}_\pm$ denote the Julia set of $p_\pm$, respectively, and denote their common image in Σ by $\mathcal{J}$. We have a well-defined map $p_+ \amalg p_- \colon \mathcal{J} \circlearrowleft$, and we shall see that it is conjugate to $h\colon \mathcal{J}(h) \circlearrowleft$. Let $\pi_\pm(\theta)$ denote the landing point of the external ray with angle θ on $\mathcal{J}_\pm$.

Fix a basepoint of $*$ at infinity, and choose a set X of paths from $*$ to all its $(p_+ \uplus p_-)$-preimages on the circle at infinity; the cardinality of X is the common degree of p_+ and p_-. The bisets $B(p_+)$ and $B(p_-)$ may be chosen to have the same basis X consisting of these paths. Note that the basis X is the standard one for p_+ but is reversed for p_-. Let their respective nuclei be $N_\pm$. Denoting by $\sim_\pm$ the corresponding asymptotic equivalence relations we have, according to §3.2, conjugacies

$$X^\infty/{\sim_+} \cong \mathcal{J}_+ \text{ and } X^\infty/{\sim_-} \cong \mathcal{J}_-.$$

The bisets $B(h)$ and $B(p_+ \uplus p_-)$ are isomorphic, and since h is expanding the nucleus of $B(h)$ is contained in $(N_+ \cup N_-)^\ell$ for some $\ell \in \mathbb{N}$. It follows that the equivalence relation $\sim_h$ associated with the nucleus of $N(h)$ is generated, as an equivalence relation, by $\sim_+ \cup \sim_-$. By Proposition 6.6 we therefore have

$$\begin{aligned}\mathcal{J}(h) \cong X^\infty/{\sim_h} &\cong \frac{(X^\infty/{\sim_+}) \sqcup (X^\infty/{\sim_-})}{[w]_{\sim_+} = [w]_{\sim_-} \text{ for all } w \in X^\infty}\\ &\cong \frac{\mathcal{J}_+ \sqcup \mathcal{J}_-}{\pi_+(\theta) = \pi_-(-\theta) \text{ for all } \theta \in S^1} \cong \mathcal{J} \subseteq \Sigma,\end{aligned}$$

conjugacies between the dynamics of h, $p_+ \uplus p_-$, and $p_+ \amalg p_-$.

We then extend this conjugacy between the Julia sets of h and $p_+ \amalg p_-$ to Fatou components, which are all discs. The critical portraits of $p_+ \uplus p_-$ and of $p_+ \amalg p_-$ coincide, so their periodic Fatou components are in natural bijection. Since every Fatou component is ultimately periodic, we extend the bijection by pulling back by $p_+ \uplus p_-$ and $p_+ \amalg p_-$, respectively. The bijection between the Julia sets restricts to bijections between boundaries of Fatou components, which are ultimately periodic embedded circles in the Julia sets; this uniquely extends the bijection between Julia sets to a conjugacy $(S^2, h) \to (\Sigma, p_+ \amalg p_-)$.

(2) ⇒ (3) is clear, because a pinching cycle is made of external rays, so it shrinks to a node in X, and therefore X is not a topological sphere.

(3) ⇒ (1) is Theorem 6.5. □

We remark that the criterion due to Mary Rees and Tan Lei gives strong constraints on pinching cycles of periodic angles in degree 2. First, the associated external rays must land at dividing fixed points. Second, in Definition 6.4 it may be assumed that $n = 2$, namely each curve in a pinching cycle intersects the equator in exactly two points. This is not true anymore in higher degree; here is an example in degree 3.

Example 6.7. Consider the polynomials $q_\pm = \frac{1}{2}z^3 \pm \frac{3}{2}z$. The polynomial q_+ has two fixed critical points at $\pm i$, and q_- exchanges its two critical points at ± 1.

Let p_+ be the tuning of q_+ in which the local map z^2 is replaced by the Basilica map $z^2 - 1$ on the immediate basins of $\pm i$, and let p_- be the tuning of q_- in which the return map $z^2 \circ z^2$ on the immediate basin of 1 is replaced by $(1-z)^2 + 1 \circ z^2$. Then $p_\pm$ are polynomials of degree 3, with 4 finite postcritical points forming 2 periodic 2-cycles. The supporting rays for p_+ are $\{\{1/8, 11/24\}, \{5/8, 23/24\}\}$, and those for p_- are $\{\{1/8, 19/24\}, \{5/8, 7/24\}\}$; the maps are $\approx z^3 \pm 2.12132z$.

The only periodic external rays landing together for q_+ are at angles 0 and 1/2, while the only periodic external rays landing together for q_- are at angles 1/4 and

3/4. It follows that the only pairs of external rays landing together for p_+ and p_- are

$$\begin{aligned} R_{p_+}(1/8),&R_{p_+}(3/8) & R_{p_-}(1/8),&R_{p_-}(7/8) \\ R_{p_+}(0),&R_{p_+}(1/2) & R_{p_-}(1/4),&R_{p_-}(3/4) \\ R_{p_+}(5/8),&R_{p_+}(7/8) & R_{p_-}(3/8),&R_{p_-}(5/8). \end{aligned}$$

It then follows that the sequence of rays $R_{p_+}(1/8)$, $R_{p_+}(3/8)$, $R_{p_-}(3/8)$, $R_{p_-}(5/8)$, $R_{p_+}(5/8)$, $R_{p_+}(7/8)$, $R_{p_-}(7/8)$, and $R_{p_-}(1/8)$ is a periodic pinching cycle, so $p_+ \uplus p_-$ is not equivalent to an expanding map. On the other hand, there does not exist any periodic pinching cycle with $n = 2$.

References

[1] Laurent Bartholdi, André G. Henriques, and Volodymyr V. Nekrashevych, *Automata, groups, limit spaces, and tilings*, J. Algebra **305** (2006), no. 2, 629–663, DOI 10.1016/j.jalgebra.2005.10.022. MR2266846

[2] Laurent Bartholdi and Dzmitry Dudko, *Algorithmic aspects of branched coverings* (English, with English and French summaries), Ann. Fac. Sci. Toulouse Math. (6) **26** (2017), no. 5, 1219–1296, DOI 10.5802/afst.1566. MR3746628

[3] Laurent Bartholdi and Dzmitry Dudko, *Algorithmic aspects of branched coverings I/V. Van Kampen's theorem for bisets*, Groups Geom. Dyn. **12** (2018), no. 1, 121–172, DOI 10.4171/GGD/441. MR3781419

[4] Laurent Bartholdi and Dzmitry Dudko, *Algorithmic aspects of branched coverings II/V. Sphere bisets and their decompositions* (2016), submitted, available at `arXiv:math/1603.04059`.

[5] Laurent Bartholdi and Dzmitry Dudko, *Algorithmic aspects of branched coverings III/V. Erasing maps, orbispaces, and the Birman exact sequence* (2018), submitted, available at `arXiv:math/1802.03045`.

[6] Mario Bonk and Daniel Meyer, *Expanding Thurston maps*, Mathematical Surveys and Monographs, vol. 225, American Mathematical Society, Providence, RI, 2017. MR3727134

[7] Xavier Buff, Adam L. Epstein, Sarah Koch, Daniel Meyer, Kevin Pilgrim, Mary Rees, and Tan Lei, *Questions about polynomial matings* (English, with English and French summaries), Ann. Fac. Sci. Toulouse Math. (6) **21** (2012), no. 5, 1149–1176, DOI 10.5802/afst.1365. MR3088270

[8] Adrien Douady and John H. Hubbard, *A proof of Thurston's topological characterization of rational functions*, Acta Math. **171** (1993), no. 2, 263–297, DOI 10.1007/BF02392534. MR1251582

[9] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012. MR2850125

[10] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, 1978. MR507725

[11] Peter Haïssinsky and Kevin M. Pilgrim, *Coarse expanding conformal dynamics* (English, with English and French summaries), Astérisque **325** (2009), viii+139 pp. (2010). MR2662902

[12] Peter Haïssinsky and Kevin M. Pilgrim, *An algebraic characterization of expanding Thurston maps*, J. Mod. Dyn. **6** (2012), no. 4, 451–476. MR3008406

[13] W. J. Harvey, *Boundary structure of the modular group*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), Ann. of Math. Stud., vol. 97, Princeton Univ. Press, Princeton, N.J., 1981, pp. 245–251. MR624817

[14] Atsushi Kameyama, *The Thurston equivalence for postcritically finite branched coverings*, Osaka J. Math. **38** (2001), no. 3, 565–610. MR1860841

[15] R. L. Moore, *Concerning upper semi-continuous collections of continua*, Trans. Amer. Math. Soc. **27** (1925), no. 4, 416–428, DOI 10.2307/1989234. MR1501320

[16] Volodymyr Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs, vol. 117, American Mathematical Society, Providence, RI, 2005. MR2162164

[17] Kevin M. Pilgrim, *Canonical Thurston obstructions*, Adv. Math. **158** (2001), no. 2, 154–168, DOI 10.1006/aima.2000.1971. MR1822682

[18] Kevin M. Pilgrim, *Combinations of complex dynamical systems*, Lecture Notes in Mathematics, vol. 1827, Springer-Verlag, Berlin, 2003. MR2020454

[19] Nikita Selinger, *Thurston's pullback map on the augmented Teichmüller space and applications*, Invent. Math. **189** (2012), no. 1, 111–142, DOI 10.1007/s00222-011-0362-3. MR2929084

[20] Nikita Selinger and Michael Yampolsky, *Constructive geometrization of Thurston maps and decidability of Thurston equivalence*, Arnold Math. J. **1** (2015), no. 4, 361–402, DOI 10.1007/s40598-015-0024-4. MR3434502

[21] Mitsuhiro Shishikura and Tan Lei, *A family of cubic rational maps and matings of cubic polynomials*, Experiment. Math. **9** (2000), no. 1, 29–53. MR1758798

[22] Lei Tan, *Matings of quadratic polynomials*, Ergodic Theory Dynam. Systems **12** (1992), no. 3, 589–620, DOI 10.1017/S0143385700006957. MR1182664

[23] Ben Scott Wittner, *On the bifurcation loci of rational maps of degree two*, Thesis (Ph.D.)–Cornell University, ProQuest LLC, Ann Arbor, MI, 1988. MR2636558

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