QUASIMARTINGALES ASSOCIATED TO MARKOV PROCESSES

LUCIAN BEZNEA AND IULIAN CÎMPEAN

ABSTRACT. For a fixed right process X we investigate those functions u for which $u(X)$ is a quasimartingale. We prove that $u(X)$ is a quasimartingale if and only if u is the difference of two finite excessive functions. In particular, we show that the quasimartingale nature of u is preserved under killing, time change, or Bochner subordination. The study relies on an analytic reformulation of the quasimartingale property for $u(X)$ in terms of a certain variation of u with respect to the transition function of the process. We provide sufficient conditions under which $u(X)$ is a quasimartingale, and finally, we extend to the case of semi-Dirichlet forms a semimartingale characterization of such functionals for symmetric Markov processes, due to Fukushima.

1. INTRODUCTION

Let us consider a (right) Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x)$ with state space E. In the celebrated paper [ÇiJaPrSh 80] the authors prove that a real-valued function u on E has the property that $u(X)$ is a semimartingale for each $\mathbb{P}^x$ if and only if there exists a sequence of finely open sets $(E_n)_{n \geq 1}$ such that $\bigcup_n E_n = E$, the exit times T_n of E_n tend to infinity a.s., and u is the difference of two 1-excessive functions on each E_n. This characterization was later approached by Fukushima in [Fu 99] from a Dirichlet forms theory perspective. More precisely, he showed that if X is associated with a symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ and $u \in \mathcal{F}$, then $\widetilde{u}(X)$ is a semimartingale if and only if there exist a nest $(F_n)_{n \geq 1}$ and constants $(c_n)_{n \geq 1}$ such that for each $n \geq 1$,

$$(1.1) \qquad |\mathcal{E}(u, v)| \leq c_n \|v\|_\infty \ \text{ for all } v \in \mathcal{F}_{b, F_n},$$

where $\widetilde{u}$ denotes a quasi-continuous version of u. The ideea of Fukushima in order to prove the sufficiency of inequality (1.1) was to assume first that $\mathcal{E}$ is a regular Dirichlet form so that, by Riesz representation, one has $\mathcal{E}(u, v) = \nu(v)$ for some Radon measure ν on E. The next step was to show that ν is a smooth measure, which means that the continuous additive functional of zero energy from Fukushima decomposition is of bounded variation, hence $\widetilde{u}(X)$ is a semimartingale. The extension to quasi-regular symmetric Dirichlet forms was achieved via the so called "transfer method". This result was then used by the author in order to develop a

deep stochastic counterpart of BV functions in both finite and infinite dimensions; beside the above-mentioned paper, we refer the reader also to [Fu 00] and the references therein. As a matter of fact, the approach using Dirichlet forms dates back to the work of Bass and Hsu in [BaHs 90] where they showed that the reflected Brownian motion in a Lipschitz domain is a semimartingale, a result which was later extended to (strong) Caccioppoli sets in [CFW93], where the authors investigate the quasimartingale structure of the process. It is worth mentioning that in [CFW93] the authors consider quasimartingales only on finite intervals and not on the entire positive semi-axis, as we do (see Definition 2.1). Although it might seem a small difference, it is in fact the key point which makes our hole study achievable and, to the best of our knowledge, new.

The aim of this paper is twofold: first, we investigate those real-valued functions u on E for which $u(X)$ is a quasimartingale, and second, we study those functions u for which $u(X)$ is a semimartingale by looking at their local quasimartingale structure. We briefly present below the structure and the main results of the paper:

In section 2 we show that the quasimartingale property of $u(X)$ may be reformulated in terms of the variation

$$V(u) := \sup_{\tau} \left\{ \sum_{i=1}^{n} P_{t_{i-1}} |u - P_{t_i - t_{i-1}} u| + P_{t_n} |u| \right\}$$

of u with respect to the semigroup $(P_t)_{t \geq 0}$ of the process, which allows us to perform the study from a purely analytic point of view. The central results are Theorem 2.6, mainly saying that $\{x \in E : u(X) \text{ is a quasimartingale with respect to } \mathbb{P}^x\} = \{V(u) < \infty\}$, and Corollary 2.7, according to which $u(X)$ is a quasimartingale (which by convention means for all $\mathbb{P}^x, x \in E$) if and only if u may be decomposed as the difference of two finite excessive functions. In particular, if the process is irreducible and $(e^{-\alpha t} u(X_t))_{t \geq 0}$ is a $\mathbb{P}^{x_0}$-quasimartingale for one $x_0 \in E$, then it is a $\mathbb{P}^x$-quasimartingale for all $x \in E$. A Riesz type decomposition and some remarks on the space of differences of excessive functions are discussed in the end of the section.

In section 3 we show that the quasimartingale property of functions is preserved under killing, time change, and Bochner subordination. In addition, we show that for a multiplicative functional M with permanent points E_M, $(e^{-\alpha t} M_t u(X_t))_t$ is a quasimartingale if and only if $(e^{-\alpha t} u|_{E_M}(X^M))_t$ is a quasimartingale, where X^M stands for the killed process by M; see Corollary 3.3. Also, in Proposition 3.5 we show that if $(e^{-\alpha t} u(X_t))_t$ is a quasimartingale, then so is the process $(e^{-\alpha \tau_t} u(Y_t))_t$, where τ is the inverse of an additive functional of X and Y denotes the corresponding time change process.

In section 4 we provide tractable conditions for u such that $(e^{-\alpha t} u(X_t))_t$ is a quasimartingale. We distinguish two ways of considering such conditions, which we treat separately: the first one involves the resolvent $\mathcal{U} = (U_\alpha)_\alpha$ of the process, while the second approach is performed in an $L^p(\mu)$-context, where μ is a σ-finite subinvariant measure. Briefly the key point is to search for an estimate of the type $U_\alpha(|P_t u - u|) \lesssim t$ for the first approach and of the type $\mu(|P_t u - u| f) \lesssim t \|f\|_\infty$ in the L^p-context, but we refer the reader to Propositions 4.1 and 4.2 for the precise statemens. See also Proposition 4.5 for a condition in terms of the dual generator on L^p-spaces.

In the last section we look at quasimartingale and semimartingale functionals from the Dirichlet form theory point of view. More precisely, if $(\mathcal{E}, \mathcal{F})$ is a (non-symmetric) Dirichlet form, then for an element $u \in \mathcal{F}$, an inequality of the type

$$(1.2) \qquad\qquad |\mathcal{E}(u, v)| \leq c\|v\|_\infty \ \text{ for all } v \in \mathcal{F}_b$$

ensures that $(e^{-\alpha t}\widetilde{u}(X))_t$ is a quasimartingale; see Theorem 5.2. As a matter of fact, we show that this is true under a more general situation, when $\|v\|_\infty$ in (1.2) is replaced by $\|v\|_\infty + \|v\|_{L^2(\mu)}$; cf. Theorem 5.1. Then, in Theorem 5.3 we extend the semimartingale characterization due to Fukushima mentioned in the beginning of the Introduction, to nonsymmetric Dirichlet forms. Furthermore, in Corollary 5.4 we consider the situation when u is not necessarily in $\mathcal{F}$ (e.g., $u \in \mathcal{F}_{\mathrm{loc}}$), under the additional hypothesis that the form has the local property. At this point we would like to emphasize that in contrast with previous work, in order to prove the sufficiency of conditions (1.1) or (1.2) we do not use Fukushima decomposition or Revuz correspondence. Instead, we employ heavily the results of the previous sections, and in fact, this approach enables us to extend Theorem 5.3 to semi-Dirichlet forms without further conditions; we do this in Theorem 5.5.

The paper ends with a few remarks concerning situations when it is sufficient to check inequalities (1.1) or (1.2) for v belonging to a proper subspace of $\mathcal{F}$, like cores or special standard cores.

2. Quasimartingales of Markov processes

Before considering Markov processes, let us recall some classic facts about quasi-martingales defined on a general probability space.

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space satisfying the usual hypotheses. An $\mathcal{F}_t$-adapted, right-continuous integrable process $(Z_t)_{t \geq 0}$ is called $\mathbb{P}$-*quasimartingale* if

$$\mathrm{Var}^{\mathbb{P}}(Z) := \sup_\tau \mathbb{E}\left\{ \sum_{i=1}^n |\mathbb{E}[Z_{t_i} - Z_{t_{i-1}} | \mathcal{F}_{t_{i-1}}]| + |Z_{t_n}| \right\} < \infty,$$

where the supremum is taken over all partitions $\tau : 0 = t_0 \leq t_1 \leq \cdots \leq t_n < \infty$.

A classic result is Rao's theorem, according to which any quasimartingale has a unique decomposition as a sum between a local martingale and a predictable process with paths of locally integrable variation. In fact, the following characterization inspired our work (see, e.g., [Pr 05, page 117]).

Theorem 2.2 (Rao). *Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be as in Definition 2.1. A real-valued process is a $\mathbb{P}$-quasimartingale if and only if it is the difference of two positive right-continuous $\mathcal{F}_t$-adapted supermartingales.*

Conversely, one can show that any semimartingale with bounded jumps is locally a quasimartingale.

Hereinafter, we consider a right Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x)$ with state space a Lusin topological space E endowed with the Borel σ-algebra $\mathcal{B}$, transition function $(P_t)_{t \geq 0}$, and resolvent $\mathcal{U} = (U_\alpha)_{\alpha > 0}$. If X has lifetime ξ and cemetery point Δ, we make the convention $u(\Delta) = 0$ for all functions $u : E \to [-\infty, +\infty]$.

The aim of this section is to study those functions $u : E \to \mathbb{R}$ for which $u(X)$ is a $\mathbb{P}^x$-quasimartingale for all $x \in E$.

Definition 2.3. Let $\alpha \geq 0$. A real-valued $\mathcal{B}$-measurable function u is called an α-*quasimartingale function* for X if $(e^{-\alpha t}u(X_t))_{t\geq 0}$ is a $\mathbb{P}^x$-quasimartingale for all $x \in E$. When $\alpha = 0$, we shall drop the index from notations.

Remark 2.4. If u is a quasimartingale function, then $\sup_t P_t|u|(x) = \sup_t \mathbb{E}^x|u(X_t)|$ $\leq \mathrm{Var}^{\mathbb{P}^x}(u(X)) < \infty$, $x \in E$. Also, by the $\mathbb{P}^x$-a.s. right continuity of the trajectories $t \mapsto u(X_t)$, u must be finely continuous; see [BlGe 68, Theorem 4.8].

Notations. For a real-valued function u and a partition τ of $\mathbb{R}+$, $\tau : 0 = t_0 \leq t_1 \leq \ldots \leq t_n < \infty$, we set

$$V_\tau^{(P_t)}(u) := \sum_{i=1}^n P_{t_{i-1}}|u - P_{t_i - t_{i-1}}u| + P_{t_n}|u|,$$

$$V^{(P_t)}(u) := \sup_\tau V_\tau^{(P_t)}(u),$$

where the supremum is taken over all finite partitions of $\mathbb{R}_+$. If there is no risk of confusion, we shall write $V_\tau(u)$ and $V(u)$ instead of $V_\tau^{(P_t)}(u)$ (resp. $V^{(P_t)}(u)$). Also, for $\alpha > 0$, we set $V_\tau^\alpha(u) := V_\tau^{(P_t^\alpha)}(u)$ and $V^\alpha(u) := V^{(P_t^\alpha)}(u)$, where $P_t^\alpha := e^{-\alpha t}P_t, \alpha > 0$.

Recall that for $\alpha \geq 0$, a $\mathcal{B}$-measurable function $f : E \to [0, \infty]$ is called α-*supermedian* if $P_t^\alpha f \leq f$, $t \geq 0$. If f is α-supermedian and $\lim_{t\to 0} P_t^\alpha f = f$, then it is called α-*excessive*. The convex cone of all α-supermedian (resp. α-excessive) functions is denoted by $S(\mathcal{U}^\alpha)$ (resp. $E(\mathcal{U}^\alpha)$). If $\alpha = 0$, we shall drop the index α from notations.

A set $A \in \mathcal{B}$ is called *absorbing* if $R_\alpha^{E\setminus A}1 = 0$ on A, where $R_\alpha^A 1 := \inf\{s \in E(\mathcal{U}^\alpha) : s \geq 1_A\}$. We recall that if A is absorbing, then it is finely open, $U_\alpha 1_{E\setminus A} = 0$ on A, and the restriction of X to A is again a right process; see, e.g., [Sha 88] or [BeRö 11]. Standard examples of absorbing sets are $[v = 0]$ and $[v < \infty]$ for some $v \in E(\mathcal{U}^\alpha)$ and $\alpha \geq 0$.

Definition 2.5. A sequence $(\tau_n)_{n\geq 1}$ of finite partitions of $\mathbb{R}_+$ is called *admissible* if it is increasing, $\bigcup_{k\geq 1} \tau_k$ is dense in $\mathbb{R}_+$, and if $r \in \bigcup_{k\geq 1} \tau_k$, then $r + \tau_n \subset \bigcup_{k\geq 1} \tau_k$ for all $n \geq 1$.

The next theorem and its first corollary are the main results of this section.

Theorem 2.6. *Let u be a real-valued $\mathcal{B}$-measurable function such that $P_t|u| < \infty$ for all t. Then the following assertions hold.*

  i) $\mathrm{Var}^{\mathbb{P}^x}(u(X)) = V(u)(x)$, $x \in E$.
  ii) *If $u_1, u_2 \in S(\mathcal{U})$ such that $u = u_1 - u_2$ on the set $[u_1 + u_2 < \infty]$, then $V(u) \leq u_1 + u_2$ on $[u_1 + u_2 < \infty]$.*
  iii) *If u is finely continuous, then there exist $u_1, u_2 \in E(\mathcal{U})$ such that $[V(u) < \infty] = [u_1 + u_2 < \infty]$ and $u = u_1 - u_2$ on $[V(u) < \infty]$. In this case, the set $[V(u) < \infty]$ is absorbing and $[V(u) < \infty] = [\sup_n V_{\tau_n}(u) < \infty] = [\lim_n V_{\tau_n}(u) < \infty]$ for any admissible sequence of partitions $(\tau_n)_n$.*

One of the fundamental connections between potential theory and Markov processes is the relation between excessive functions and (right-continuous) supermartingales. More precisely, it is well known that for a nonnegative real-valued measurable function u we have that $u(X)$ is a supermartingale if and only if u is

excessive; see, e.g., [LG 06, Proposition 13.7.1 and Theorem 14.7.1]. The following essential consequence of Theorem 2.6 (and its proof) states that this connection may be extended between the space of differences of excessive function on the one hand, and quasimartingales on the other hand, while at the same time revealing a Hahn–Jordan type decomposition.

Corollary 2.7. *A real-valued $\mathcal{B}$-measurable function u is a quasimartingale function for X if and only if there exist two real-valued functions $u_1, u_2 \in E(\mathcal{U})$ such that $u = u_1 - u_2$; in this case one can take $u_1 := \sup_n V_{\tau_n}(u)$, where $(\tau_n)_{n \geq 1}$ is any fixed sequence of admissible partitions of $\mathbb{R}_+$.*

For the proof of Theorem 2.6 we need the following lemma. Because we found this result only as an exercise (left for the reader) in [Sha 88, Exercise 10.24] or [BlGe 68, Exercise 4.14], we include its complete proof below.

The first hitting time of a set $A \in \mathcal{B}$ by the process X is defined by $T_A := \inf\{t > 0 : X_t \in A\}$. It is well known that T_A is a stopping time; see [BlGe 68] or [Sha 88].

Lemma 2.8. *If u is finely continuous and bounded, then so is $P_s u$ for all $s \geq 0$.*

Proof. Since u is finely continuous, by [BlGe 68, Theorem 4.8], it follows that the mappings $t \mapsto u(X_t)$ are right continuous a.s. Let $s > 0$, and set $f := P_s u$. In order to show that f is finely continuous, it is sufficient to prove that if $\varepsilon > 0$, then x is irregular for $A = f^{-1}([f(x) + \varepsilon, \infty))$ and $B = f^{-1}((-\infty, f(x) - \varepsilon])$. We treat only the first case. Let $(A_n)_n$ be an increasing sequence of closed sets such that $T_{A_n} \searrow T_A$ $\mathbb{P}^x$-a.s. By the zero-one law [BlGe 68, Proposition 5.17], $\mathbb{P}^x(T_A = 0) \in \{0, 1\}$. Assume that x is regular for A, i.e., $T_A = 0$ $\mathbb{P}^x$-a.s. Then by the strong Markov property and the dominated convergence theorem, $\mathbb{E}^x f(X_{T_{A_n}}) = \mathbb{E}^x \{\mathbb{E}^x[u(X_{s+T_{A_n}})|\mathcal{F}_{T_{A_n}}]\} = \mathbb{E}^x u(X_{s+T_{A_n}}) \longrightarrow_n f(x)$. On the other hand, by the definition of T_{A_n} we have that $f(X_{T_{A_n}}) \geq f(x) + \varepsilon$, which contradicts the previous convergence. $\qquad\square$

Proof of Theorem 2.6.

i) By the Markov property, for all $x \in E$,

$$\mathrm{Var}^{\mathbb{P}^x}(u(X)) = \sup_\tau \mathbb{E}^x \left\{ \sum_{i=1}^n |u(X_{t_{i-1}}) - \mathbb{E}^x[u(X_{t_i})|\mathcal{F}_{t_{i-1}}]| + |u(X_{t_n})| \right\}$$

$$= \sup_\tau \left\{ \mathbb{E}^x \left\{ \sum_{i=1}^n |u(X_{t_{i-1}}) - P_{t_i - t_{i-1}} u(X_{t_{i-1}})| + P_{t_n} |u|(x) \right\} \right\}$$

$$= \sup_\tau \left\{ \sum_{i=1}^n \mathbb{E}^x[|u - P_{t_i - t_{i-1}} u|(X_{t_{i-1}})] + P_{t_n} |u|(x) \right\}$$

$$= \sup_\tau \left\{ \sum_{i=1}^n P_{t_{i-1}} |u - P_{t_i - t_{i-1}} u|(x) + P_{t_n} |u|(x) \right\} = V(u)(x).$$

Note that the above expressions make sense because by hypothesis, $P_t |u| < \infty$ for all t.

ii) Since $u_1, u_2 \in S(\mathcal{U})$, we have that $A := [u_1 + u_2 < \infty]$ satisfies $P_t 1_{A^c} = \lim_n P_t 1_{[u_1+u_2>n]} \leq \lim_n \frac{1}{n} P_t(1_{[u_1+u_2>n]}(u_1 + u_2)) \leq \lim_n \frac{u_1+u_2}{n} = 0$ on A

for all $t > 0$. This leads to $1_A P_t f = 1_A P_t(f 1_A)$ for all $\mathcal{B}$-measurable f for which $P_t|f| < \infty$. Indeed,

$$|1_A P_t(f 1_{A^c})| \leq 1_A P_t(|f| 1_{A^c}) = \sup_n 1_A P_t((|f| \wedge n) 1_{A^c}) \leq \sup_n 1_A n P_t 1_{A^c} = 0.$$

By the previous remarks, we get

$$1_A V(u)$$

$$= \sup_\tau \left\{ \sum_{i=1}^n 1_A P_{t_{i-1}} |u - P_{t_i - t_{i-1}} u| + 1_A P_{t_n} |u| \right\}$$

$$= \sup_\tau \left\{ \sum_{i=1}^n 1_A P_{t_{i-1}} |1_A u - 1_A P_{t_i - t_{i-1}}(1_A u)| + 1_A P_{t_n} 1_A |u| \right\}$$

$$\leq \sup_\tau \left\{ \sum_{i=1}^n 1_A P_{t_{i-1}} |1_A u_1 - 1_A P_{t_i - t_{i-1}}(1_A u_1)| + 1_A P_{t_n}(1_A u_1) \right\}$$

$$+ \sup_\tau \left\{ \sum_{i=1}^n 1_A P_{t_{i-1}} |1_A u_2 - 1_A P_{t_i - t_{i-1}}(1_A u_2)| + 1_A P_{t_n}(1_A u_2) \right\}$$

$$= 1_A \sup_\tau \left\{ \sum_{i=1}^n [P_{t_{i-1}} u_1 - P_{t_i} u_1] + P_{t_n} u_1 \right\} + 1_A \sup_\tau \left\{ \sum_{i=1}^n [P_{t_{i-1}} u_2 - P_{t_i} u_2] + P_{t_n} u_2 \right\}$$

$$= 1_A (u_1 + u_2).$$

iii) For each partition $\tau : 0 = t_0 \leq t_1 \leq \cdots \leq t_n < \infty$, we set

$$u_1^\tau := \sum_{i=1}^n P_{t_{i-1}}(u - P_{t_i - t_{i-1}} u)^+ + P_{t_n}(u^+) < \infty,$$

$$u_2^\tau := \sum_{i=1}^n P_{t_{i-1}}(u - P_{t_i - t_{i-1}} u)^- + P_{t_n}(u^-) < \infty.$$

Let $\prec$ denote the ordering of set containment, and suppose that σ and τ are two finite partitions of $\mathbb{R}_+$ such that $\sigma \prec \tau$. We claim that $u_i^\sigma \leq u_i^\tau$, $i = \overline{1,2}$. To see this, let $\sigma : 0 = t_0 \leq t_1 \leq \cdots \leq t_n < \infty$, and note that it is enough to consider τ as a partition obtained from σ by adding an extra point t before t_1, after t_n, or between some t_i and t_{i+1}. In the first case we have $(u - P_{t_1} u)^\pm \leq (u - P_t u)^\pm + (P_t(u - P_{t_1 - t} u))^\pm \leq (u - P_t u)^\pm + P_t(u - P_{t_1 - t} u)^\pm$. If $t \geq t_n$, then $P_{t_n}(u^\pm) \leq P_{t_n}(u - P_{t - t_n} u)^\pm + P_t(u^\pm)$, and if $t_i \leq t \leq t_{i+1}$, then

$$(u - P_{t_{i+1} - t_i} u)^\pm \leq (P_{t - t_i}(u - P_{t_{i+1} - t} u))^\pm + (u - P_{t - t_i} u)^\pm$$

$$\leq P_{t - t_i}(u - P_{t_{i+1} - t} u)^\pm + (u - P_{t - t_i} u)^\pm,$$

hence $P_{t_i}(u - P_{t_{i+1} - t_i} u)^\pm \leq P_t(u - P_{t_{i+1} - t} u)^\pm + P_{t_i}(u - P_{t - t_i} u)^\pm$. Therefore, $u_i^\sigma \leq u_i^\tau$, $i = \overline{1,2}$.

Let now $(\tau_n)_{n \geq 1}$ be an admissible sequence of partitions of $\mathbb{R}_+$, and define

$$u_1 := \sup_n u_1^{\tau_n} = \lim_n u_1^{\tau_n} \quad \text{and} \quad u_2 := \sup_n u_2^{\tau_n} = \lim_n u_2^{\tau_n}.$$

Then $u_1 + u_2 = \sup_n V_{\tau_n}(u) < \infty$ on $[V(u) < \infty]$.

Now, if $r \in \bigcup_{n \geq 1} \tau_n$,

$$P_r u_1 = \sup_n P_r u_1^{\tau_n}$$

$$= \sup_n \left\{ \sum_{i=1}^{n} P_{r+t_{i-1}} (u - P_{t_i - t_{i-1}} u)^+ + P_{r+t_n} (u^+) \right\}$$

$$= \sup_n \left\{ \sum_{i=1}^{n} P_{r+t_{i-1}} (u - P_{r+t_i - (r+t_{i-1})} u)^+ + P_{r+t_n} (u^+) \right\}$$

$$\leq u_1,$$

because the last supremum is taken over a class of partitions included in $\{\tau_n : n \geq 1\}$. Analogously, $P_r u_2 \leq u_2$ for all $r \in \bigcup_{n \geq 1} \tau_n$. Then,

$$u_2 + u = \sup_n \left\{ \sum_{i=1}^{n} P_{t_{i-1}} (u - P_{t_i - t_{i-1}} u)^- + P_{t_n} (u^-) \right.$$

$$\left. + \sum_{i=1}^{n} P_{t_{i-1}} (u - P_{t_i - t_{i-1}} u) + P_{t_n} u \right\}$$

$$= \sup_n \left\{ \sum_{i=1}^{n} P_{t_{i-1}} (u - P_{t_i - t_{i-1}} u)^- + P_{t_n} (u^-) \right\}$$

$$= u_1,$$

and $u = u_1 - u_2$ on $[\sup_n V_{\tau_n}(n) < \infty]$.

Case 1. Assume that u is lower bounded. We claim that $u_1, u_2 \in E(\mathcal{U})$ and $[V(u) < \infty] = [\sup_n V_{\tau_n}(u) < \infty]$. First, note that if $u_1, u_2 \in E(\mathcal{U})$, since $u_1 + u_2 = \sup_n V_{\tau_n}(u)$, $[V(u) < \infty] \subset [\sup_n V_{\tau_n}(u) < \infty]$, and $u = u_1 - u_2$ on $[u_1 + u_2 < \infty]$, by ii) we obtain $[V(u) < \infty] = [\sup_n V_{\tau_n}(u) < \infty]$.

It remains to show that $u_1, u_2 \in E(\mathcal{U})$. By Lemma 2.8 the functions $\varphi_{k,l}^n := \sum_{i=1}^{n} P_{t_{i-1}} [(u - P_{t_i - t_{i-1}} (u \wedge k))^- \wedge l]$ are finely continuous and $u_2 = \sup_n \sup_k \sup_l \varphi_{k,l}^n$ is finely lower semicontinuous. Moreover, if $t \in \mathbb{R}_+$ and $(t_j)_j \subset \bigcup_{n \geq 1} \tau_n$, $t_j \searrow t$, then

$$P_t u_2 = \sup_n \sup_k \sup_l P_t \varphi_{k,l}^n = \sup_n \sup_k \sup_l \lim_j P_{t_j} \varphi_{k,l}^n \leq \liminf_j P_{t_j} u_2 \leq u_2,$$

so u_2 is supermedian, and by [BeBo 04, Corollary 1.3.4], it is excessive. Now, since $u_1 = u_2 + u$ is finely continuous for $t \in \mathbb{R}_+$ and $(t_j)_j$ is as before,

$$P_t u_1 = \sup_k P_t (u_1 \wedge k) = \sup_k \lim_j P_{t_j} (u_1 \wedge k) \leq u_1,$$

and $u_1 \in E(\mathcal{U})$.

Case 2. Let now u be arbitrary. Then $u^+ = u_1 - u_1 \wedge u_2$ and $u^- = u_2 - u_1 \wedge u_2$ are finely continuous and of course, lower bounded. Applying Case 1 to u^+ and u^-, we have that $u = u^+ - u^-$ is the difference of two real-valued excessive functions on $[V(u^+) < \infty] \cap [V(u^-) < \infty]$. Let us show that $[V(u) < \infty] = [\sup_n V_{\tau_n}(u) < \infty] = [V(u^+) < \infty] \cap [V(u^-) < \infty]$, which completes the proof. Arguing as in the proof of ii), one can check that $A = [u_1 + u_2 < \infty] = [\sup_n V_{\tau_n}(u) < \infty]$ satisfies $P_r 1_{A^c} = 0$ on A for all $r \in \bigcup_{n \geq 1} \tau_n$, and further, $V(u^\pm) = \sup_n V_{\tau_n}(u^\pm) \leq u_1 + u_2$

on A. Taking into account the subadditivity of $f \mapsto V(f)$,

$$[V(u) < \infty] \subset [\sup_n V_{\tau_n}(u) < \infty] = A \subset [V(u^+) + V(u^-) < \infty] \subset [V(u) < \infty]. \quad \square$$

We say that the process X is *irreducible* (in the strong sense) if the only nonempty absorbing set is the hole space E. Often in practice, the irreducibility of $\mathcal{U}$ is ensured by the *strong Feller* property (i.e., U_α maps bounded measurable functions into continuous ones) in association with the *topological irreducibility* (i.e., $U_\alpha 1_D > 0$ for all open sets $D \subset E$); cf., e.g., [Ha 10].

Corollary 2.9. *Let u be a real-valued $\mathcal{B}$-measurable finely continuous function and assume that there exists $x_0 \in E$ such that $(e^{-\alpha t} u(X_t))_{t \geq 0}$ is a $\mathbb{P}^{x_0}$-quasimartingale for some $\alpha \geq 0$. The following assertions hold.*

  i) *If $\mathcal{U}$ is irreducible, then $(e^{-\alpha t} u(X_t))_{t \geq 0}$ is a $\mathbb{P}^x$-quasimartingale for all $x \in E$.*

  ii) *If $\mathcal{U}$ is strong Feller and topologically irreducible, then $\mathcal{U}$ is irreducible.*

Proof.

  i) By Proposition 3.1 below we have that $V^\alpha(u)(x_0) = \text{Var}^{\mathbb{P}^{x_0}}((e^{-\alpha t} u(X_t))_{t \geq 0}) < \infty$, hence $A := [V^\alpha(u) < \infty]$ is absorbing (cf. Theorem 2.6, iii)) and nonempty. Since $\mathcal{U}$ is irreducible, it follows that $A = E$.

  ii) Let $B \in \mathcal{B}$ be absorbing, and set $E_0 := [U_1 1_{E \setminus B} = 0] \supset B$. The strong Feller property implies that $E \setminus E_0$ is an open set and $1_{E \setminus B} \geq U_1 1_{E \setminus B} \geq U_1 1_{E \setminus E_0}$ leads to $E_0 = E$. $\qquad \square$

Following [Ge 80], X is called *recurrent* if either $U1_B = 0$ or $U1_B = \infty$ for all $B \in \mathcal{B}$. Getoor showed that $\mathcal{U}$ is recurrent if and only if any excessive function is constant; hence, Corollary 2.7 gives the following quasimartingale characterization of recurrence.

Corollary 2.10. *X is recurrent if and only if every quasimartingale function is constant.*

A Riesz type decomposition. Extending [Me 66] (see also [Sha 88, Chapter VI]), a quasimartingale function f is called *(locally) harmonic* if $f(X)$ is a $\mathbb{P}^x$-(local) martingale for all $x \in E$; it is called a *potential function of class (D)* if for any sequence of stopping times $(T_n)_n \nearrow \infty$, $\mathbb{E}^x[f(X_{T_n})] \to 0$.

Theorem 2.11. *If u is a quasimartingale function for X, then u may be decomposed as $u = h + v$, where h is locally harmonic and v is a potential function of class (D).*

Proof. It follows by Corollary 2.7 and [Sha 88, Theorem (51.10)]. $\qquad \square$

The space of differences of excessive functions. We saw that the space of α-quasimartingale functions of X is in fact the space of differences of real-valued α-excessive functions. We end this section by collecting some useful observations on the dependence on α of the above-mentioned spaces, in the same spirit as [BeLu 16, Remark 2.1].

Recall that $\mathcal{U}$ is called *m-transient* (m is a fixed σ-finite subinvariant measure for $\mathcal{U}$) if there exists $0 < f \in L^1(m)$ such that $Uf < \infty$ m-a.e.

Proposition 2.12. *The following assertions hold.*

    i) *For $\alpha, \beta \geq 0$, if $v \in E(\mathcal{U}_\alpha)$ is real valued such that $U_\beta v < \infty$, then v is a difference of two real-valued β-excessive functions. In particular, $bE(\mathcal{U}_\alpha) - bE(\mathcal{U}_\alpha)$ is independent of $\alpha > 0$.*

    ii) *Let m be a σ-finite subinvariant measure for $\mathcal{U}$. Then the following hold.*

        1) *If $\alpha, \beta \geq 0$, $v \in E(\mathcal{U}_\alpha)$, and $U_\beta v < \infty$ m-a.e., then v is m-a.e. (hence q.e.) the difference of two β-excessive functions. In particular, the L^p-subspaces $L^p(m) \cap E(\mathcal{U}_\alpha) - L^p(m) \cap E(\mathcal{U}_\alpha)$ are independent of $\alpha > 0$ for all $1 \leq p \leq \infty$.*

        2) *If $\mathcal{U}$ is m-transient, then the L^1-subspaces $L^1(m) \cap E(\mathcal{U}_\alpha) - L^1(m) \cap E(\mathcal{U}_\alpha)$ are independent of $\alpha \geq 0$.*

Proof.

    i) Of course, we need to consider only the case $\beta < \alpha$. Let $w := v + (\alpha - \beta)U_\beta v$. Then by hypothesis $w < \infty$, and it is straightforward to check that w is β-excessive. Hence $v = w - (\alpha - \beta)U_\beta v \in E(\mathcal{U}_\beta) - E(\mathcal{U}_\beta)$.

The proof of ii)1) is similar to the one for assertion i).

ii)2). By ii)1), it is sufficient to show that if $0 \leq v \in L^1(m)$, then $Uv < \infty$ m-a.e. But this is true by a characterization of m-transience; see [BeCîRö 18]. $\qquad\square$

3. QUASIMARTINGALE FUNCTIONS OF TRANSFORMED MARKOV PROCESSES

As in section 2, $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x)$ is a right Markov process on E. Before we move on, we would like to remark that although in section 1 we considered only $\mathcal{B}$-measurable functions, the results obtained there remain valid for functions measurable with respect to $\mathcal{B}^u$, the σ-algebra of all universally measurable sets in E.

Quasimartingales under killing. Let $M := (M_t)_{t \geq 0}$ be a right-continuous decreasing multiplicative functional (MF) of X, and let E_M be the set of permanent points for M, $E_M := \{x \in E : \mathbb{P}^x(M_0 = 1) = 1\}$. As in [Sha 88, Proposition 56.5], define the kernels on $p\mathcal{B}^u$ by setting for $f \in p\mathcal{B}^u$, $\alpha \geq 0$, and $t \geq 0$

$$P_M^\alpha f(x) := \begin{cases} \mathbb{E}^x \int_0^\infty e^{-\alpha t} f(X_t) dM_t, & x \in E_M \\ f(x), & x \in E \setminus E_M, \end{cases}$$

$$Q_t f(x) := \mathbb{E}^x \{f(X_t) M_t\},$$

$$W_\alpha f(x) := \mathbb{E}^x \int_0^\infty e^{-\alpha t} M_t f(X_t) dt.$$

It is well known that $(Q_t)_t$ is a sub-Markovian semigroup of kernels on $(E, \mathcal{B}^u)$ whose resolvent is $\mathcal{W} = (W_\alpha)_\alpha \geq 0$.

Proposition 3.1. *Let u be a real-valued $\mathcal{B}^u$-measurable function such that $P_t|u| < \infty$ for all $t \geq 0$. Then for all $x \in E$,*

$$\mathrm{Var}^{\mathbb{P}^x}(Mu(X)) = V^{(Q_t)}u(x).$$

Proof. For $x \in E$,

$$\mathrm{Var}^{\mathbb{P}^x}(Mu(X))$$

$$= \sup_\tau \mathbb{E}^x \left\{ \sum_{i=1}^n |\mathbb{E}^x[M_{t_{i-1}}u(X_{t_{i-1}}) - M_{t_i}u(X_{t_i})|\mathcal{F}_{t_{i-1}}]| + M_{t_n}|u|(X_{t_n}) \right\}$$

$$= \sup_\tau \mathbb{E}^x \left\{ \sum_{i=1}^n |M_{t_{i-1}}u(X_{t_{i-1}}) - M_{t_{i-1}}Q_{t_i - t_{i-1}}u(X_{t_{i-1}})| + M_{t_n}|u|(X_{t_n}) \right\}$$

$$= \sup_\tau \left\{ \sum_{i=1}^n \mathbb{E}^x[M_{t_{i-1}}|u - Q_{t_i - t_{i-1}}u|(X_{t_{i-1}})] + Q_{t_n}|u|(x) \right\}$$

$$= \sup_\tau \left\{ \sum_{i=1}^n Q_{t_{i-1}}|u - Q_{t_i - t_{i-1}}u|(x) + Q_{t_n}|u|(x) \right\}$$

$$= V^{(Q_t)}u(x). \qquad \square$$

Corollary 3.2. *Let u be a real-valued $\mathcal{B}^u$-measurable function. If $\alpha \geq 0$, then u is an α-quasimartingale function if and only if it is the difference of two real-valued α-excessive functions.*

If M is exact, then E_M is finely open, and the restriction $Q_t|_{E_M}$ of $(Q_t)_{t \geq 0}$ to E_M is the transition function of a right Markov process $(X_t^M)_{t \geq 0}$ on E_M; see [Sha 88, Chapter VII].

Corollary 3.3. *Assume that M is perfect. Then the following assertions hold.*

i) *Let f be a real-valued $\mathcal{B}^u$-measurable function such that $U_\alpha|f| < \infty$ for some $\alpha \geq 0$, and set $u := W_\alpha f$. Then u is an α-quasimartingale function for X.*

ii) *Let u be a real-valued $\mathcal{B}^u$-measurable function, such that $Q_t|u| < \infty$ for all $t \geq 0$. Then for all $\alpha \geq 0$*

$$V^{(Q_t^\alpha)}(u) = \begin{cases} V^{(Q_t^\alpha|_{E_M})}(u|_{E_M})(x), & x \in E_M, \\ 0, & x \in E \setminus E_M. \end{cases}$$

In particular, if u is finely continuous, then for all $\alpha \geq 0$, $(e^{-\alpha t}M_t u(X_t))_t$ is a $\mathbb{P}^x$-quasimartingale for all $x \in E$ if and only if $u|_{E_M}$ is an α-quasimartingale function for X^M.

Proof. i) Clearly, it is enough to consider $f \geq 0$. Then, the assertion follows since $u = U_\alpha f - P_M^\alpha U_\alpha f$ and $P_M^\alpha U_\alpha f \in E(\mathcal{U}^\alpha)$; see, e.g., [Sha 88, Proposition 56.5].

ii) The first assertion follows easily since $Q_t f \equiv 0$ on E_M and $M_t \equiv 0$ $\mathbb{P}^x$-a.s. for $x \in E \setminus E_M$, while the second one is entailed by Proposition 3.1. $\qquad \square$

Quasimartingales under time change. Let A be a perfect continuous additive functional of X (AF), and let $F = \mathrm{supp}(A)$ be its fine support. Then the inverse τ_t of A_t, defined by

$$\tau_t(\omega) := \inf\{s : A_s(\omega) > t\},$$

is a stopping time for each $t \geq 0$ and the process $(\tau_t)_{t \geq 0}$ is right continuous. Set $Y_t(\omega) := X_{\tau_t(\omega)}(\omega)$, $\mathcal{G}_t := \mathcal{F}_{\tau_t}$, $t \geq 0$, $\mathcal{G} = \bigcup_{t \geq 0} \mathcal{G}_t$. Then the process $Y = (\Omega, \mathcal{G}, \mathcal{G}_t, Y_t, \mathbb{P}^x)$ is a right process on F and is called the time-changed process

of X with respect to A; see [Sha 88, Chapter VII (more precisely, Theorem 65.9)]. We denote its resolvent by $\widehat{\mathcal{U}}$.

Corollary 3.4. *If u is a quasimartingale function for X, then $u|_F$ is a quasimartingale function for Y. Conversely, if $F = E$, then any quasimartingale function for Y is a quasimartingale function for X.*

Proof. If u is a quasimartingale function for X, then by Corollary 2.7 $u = u_1 - u_2$ with $u_1, u_2 \in E(\mathcal{U})$ and is real valued. But $E(\mathcal{U})|_F \subset E(\widehat{\mathcal{U}})$ (see [Sha 88, 65.12]), so $u|_F$ is a quasimartingale function for Y by the same Corollary 2.7. If $F = E$, the result follows by same arguments, since in this case $E(\mathcal{U}) = E(\widehat{\mathcal{U}})$; cf. [Sha 88, 65.13]. □

The α-quasimartingales are not preserved by time change, since $E(\mathcal{U}^\alpha) \not\subset E(\widehat{\mathcal{U}}^\alpha)$, $\alpha > 0$, in general. However, the following result holds.

Proposition 3.5. *If u is an α-quasimartingale function of X for some $\alpha \geq 0$, then the process $(e^{-\alpha\tau_t}u(Y_t))_{t\geq0}$ is a $\mathbb{P}^x$-quasimartingale with respect to the filtration $(\mathcal{G}_t)_{t\geq0}$ for all $x \in F$.*

Proof. If u is an α-quasimartingale function for X, then by Corollary 2.7, $u = u_1 - u_2$ with $u_1, u_2 \in E(\mathcal{U}^\alpha)$ finite on E. By the Doob stopping theorem we have that

$$\mathbb{E}^x\{e^{-\alpha\tau_t}u_i(X_{\tau_t})\} \leq u_i(x), \ x \in E, \ t \geq 0, \ i = \overline{1, 2}.$$

On the other hand, $(\alpha\tau_t)_{t\geq0}$ is a perfect right-continuous AF of Y, hence $(e^{-\alpha\tau_t})_{t\geq0}$ is an exact and perfect MF of Y; see [Sha 88, 54.11]. Let $(Q_t)_{t\geq0}$ be the transition function of the process Y killed by $(e^{-\alpha\tau_t})_{t\geq0}$. Then

$$Q_t u_i|_F(x) = \mathbb{E}^x\{e^{-\alpha\tau_t}u_i(X_{\tau_t})\} \leq u_i(x), \ x \in F,$$

which means that $u_i|_F$ is (Q_t)-excessive, hence $V^{(Q_t)}(u|_F) < \infty$ (cf. Theorem 2.6, ii)). The result now follows since $\mathrm{Var}^{\mathbb{P}^x}((e^{-\alpha\tau_t}u(X_{\tau_t}))_{t\geq0}) = V^{(Q_t)}(u|_F)(x)$ by Proposition 3.1. □

Quasimartingales under Bochner subordination. Assume that X is transient, and let $\mu := (\mu_t)_{t\geq0}$ be a vaguely continuous convolution semigroup of sub-probability measures on $\mathbb{R}_+$. Define the *subordinate* $(P_t^\mu)_{t\geq0}$ of $(P_t)_{t\geq0}$ by

$$P_t^\mu f := \int_0^\infty P_s f \mu_t(ds) \quad \text{for all } f \in bp\mathcal{B},$$

whose resolvent is denoted by $\mathcal{U}^\mu := (U_\alpha^\mu)_{\alpha\geq0}$. By [Lu 14, Theorem 3.3], $(P_t^\mu)_{t\geq0}$ is the transition function of a right process X^μ on E. Moreover, $E(\mathcal{U}) \subset E(\mathcal{U}^\mu)$; hence, we have the following result.

Corollary 3.6. *Any quasimartingale function for X is a quasimartingale function for X^μ.*

Example. Recall that a sub-Markovian resolvent of kernels $\mathcal{V} = (V_\alpha)_\alpha$ is said to be S-subordinate to $\mathcal{U}$ if $E(\mathcal{U}) \subset E(\mathcal{V})$; see [HmHm 09] and [Si 99].

By Corollary 2.7, it follows that the class of quasimartingale functions for X is inherited by any right process whose resolvent is S-subordinate to $\mathcal{U}$. We remark that killing, time change, Bochner subordination, and any combination of them, may be regarded as S-subordinations with respect to $\mathcal{U}$; hence, the quasimartingale functions for X are preserved under such transformations. We emphasize that

since the killing, time change, and Bochner subordination transformations do not commute in general, the order of any combination of them is relevant. We illustrate such a situation by looking at (Bochner) subordinate killed and killed subordinate Brownian motion. We follow [SoVo 03]; see also [HmJa 14, Example 7].

Let $(X_t)_{t\geq 0}$ be a d-dimensional Brownian motion on $\mathbb{R}^d$, and let $(\xi_t)_{t\geq 0}$ be an α-stable subordinator starting at 0, $\alpha \in (0,1)$. Let $Y_t = X_{\xi_t}$ be the right process whose transition function is the subordinate $(P_t^\mu)_{t\geq 0}$ of $(P_t)_{t\geq 0}$ by means of the convolution semigroup μ induced by $(\xi_t)_{t\geq 0}$. The generator of Y is $-(-\Delta)^\alpha$, the fractional power of the negative Laplacian. Let now $D \subset \mathbb{R}^d$ be a domain, and denote by Y^D the killed upon leaving D, which is a right process obtained by killing Y with the exact MF $M_t = 1_{[0,T_{D^c})}(t)$, $t \geq 0$, where $T_{D^c}(\omega) := \inf\{t > 0 \mid Y_t(\omega) \in D^c\}$.

Changing the order of transformations, let Z be the right process obtained by first killing X upon leaving D and then subordinating the killed Brownian motion by means of μ. The generator of Z is $-(-\Delta|_D)^\alpha$. As remarked in [HmJa 14], Z is S-subordinate to Y^D, hence:

Corollary 3.7. *Any quasimartingale function for Y^D is a quasimartingale function for Z.*

4. Criteria for quasimartingale functions

In this section we present some sufficient conditions for a function to be an α-quasimartingale function. In the first part we develop the study from the resolvent point of view, while in the last part we place ourselves in an L^p-context (C_0-semigroups and infinitesimal generators) with respect to a subinvariant measure.

A resolvent approach. Again, we deal with a fixed right Markov process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}^x)$ on $(E, \mathcal{B})$, with transition function $(P_t)_{t\geq 0}$ and resolvent $\mathcal{U} = (U_\alpha)_\alpha$.

The main result of this subsection is the following.

Proposition 4.1. *Let u be a real-valued $\mathcal{B}$-measurable finely continuous function.*

i) *Assume there exist $\alpha \geq 0$ and $c \in p\mathcal{B}$ such that*

$$U_\alpha(|u| + c) < \infty, \quad \limsup_{t\to\infty} P_t^\alpha |u| < \infty, \quad |P_t u - u| \leq ct, t \geq 0,$$

and the functions $t \mapsto P_t(|u| + c)(x)$ are Riemann integrable. Then u is an α-quasimartingale function.

ii) *Assume there exist $\alpha \geq 0$, $c \in p\mathcal{B}$ such that*

$$|P_t u - u| \leq ct, t \geq 0, \quad \sup_{t \in \mathbb{R}_+} P_t^\alpha(|u| + c) =: b < \infty.$$

Then u is a β-quasimartingale function for all $\beta > \alpha$.

iii) *Assume there exists $x_0 \in E$ such that for some $\alpha \geq 0$*

$$U_\alpha(|u|)(x_0) < \infty, \quad U_\alpha(|P_t u - u|)(x_0) \leq \text{const} \cdot t, \ t \geq 0.$$

Then $[V^\beta(u) < \infty] \neq \emptyset$ and if $\mathcal{U}$ is irreducible (e.g., strong Feller and topologically irreducible), then u is a β-quasimartingale function for all $\beta > \alpha$.

Proof. Assume that the conditions in i) are satisfied, and let

$$\tau_n := \left\{ \frac{k}{2^n} \ : \ 0 \leq k \leq n \cdot 2^n \right\}, \ n \geq 1.$$

Clearly, $(\tau_n)_{n \geq 1}$ is an admissible sequence of partitions of $\mathbb{R}_+$ (see Definition 2.5); hence, by Theorem 2.6 iii), we have to check that $\lim_n V^\alpha_{\tau_n}(u) < \infty$. First, note that by hypotheses

$$|P^\alpha_t u - u| \leq |P_t u|(1 - e^{-\alpha t}) + ct \leq (\text{const}|P_t u| + c)t \leq \text{const}(|u| + c)t$$

for all t small enough. Therefore,

$$\lim_n V^\alpha_{\tau_n}(u) = \lim_n \left\{ \sum_{k=1}^{n2^n - 1} P^\alpha_{\frac{k-1}{2^n}} |P^\alpha_{\frac{1}{2^n}} u - u| + P^\alpha_{n2^n} |u| \right\}$$

$$\leq \limsup_n \left\{ \sum_{k=1}^{n2^n - 1} P^\alpha_{\frac{k-1}{2^n}} |P^\alpha_{\frac{1}{2^n}} u - u| \right\} + \limsup_n P^\alpha_{n \cdot 2^n} |u|.$$

By hypothesis, $\limsup_n P^\alpha_{n \cdot 2^n} |u| < \infty$. As for the other term, we have

$$\limsup_n \left\{ \sum_{k=1}^{n2^n - 1} P^\alpha_{\frac{k-1}{2^n}} |P^\alpha_{\frac{1}{2^n}} u - u| \right\} \leq \text{const} \cdot \limsup_n \left\{ \frac{1}{2^n} \sum_{k=1}^{n2^n - 1} P^\alpha_{\frac{k-1}{2^n}} (|u| + c) \right\}$$

$$= \text{const} \cdot \int_0^\infty e^{-\alpha t} P_t(|u| + c) dt$$

$$= \text{const} \cdot U_\alpha(|u| + c) < \infty.$$

i) Let $\beta > 0$. Similar to the above computations and noticing that $\lim_{t \to \infty} P^\beta_t |u| = 0$,

$$\lim_n V^\beta_{\tau_n}(u) \leq \text{const} \cdot \limsup_n \sum_{k=1}^{n2^n - 1} P^\beta_{\frac{k-1}{2^n}} (|u| + c) \frac{1}{2^n}$$

$$\leq \text{const} \cdot b \limsup_n \sum_{k=1}^{n2^n - 1} e^{-(\beta - \alpha)\frac{k-1}{2^n}} \frac{1}{2^n}$$

$$= \text{const} \cdot b \int_0^\infty e^{-(\beta - \alpha)t} dt < \infty.$$

ii) Let $\beta > \alpha$. Once we show that $[V^\beta(u) < \infty] \neq \emptyset$, the second assertion follows by Corollary 2.9. Taking into account Theorem 2.6 iii), we will

show that $U_\alpha(\lim_n V_{\tau_n}^\beta(u))(x_0) < \infty$. Notice first that $\delta_{x_0} \circ U_\alpha$ is an α-sub-invariant measure for $(P_t)_t$, i.e., $U_\alpha(P_t^\alpha f)(x_0) \leq U_\alpha f(x_0)$ for all $f \in p\mathcal{B}$. Employing this property and using the hypotheses

$$U_\alpha(\lim_n V_{\tau_n}^\beta(u))(x_0) = \lim_n U_\alpha(V_{\tau_n}^\beta(u))(x_0)$$

$$= \lim_n \left\{ \sum_{k=1}^{n \cdot 2^n - 1} U_\alpha(P_{\frac{k-1}{2^n}}^\beta |u - P_{\frac{1}{2^n}}^\beta u|)(x_0) + U_\alpha(P_n(|u|))(x_0) \right\}$$

$$\leq \limsup_n \sum_{k=1}^{n2^n-1} e^{-(\beta-\alpha)\frac{k-1}{2^n}} U_\alpha |u - P_{\frac{1}{2^n}}^\beta u|(x_0)$$

$$+ \limsup_n e^{-(\beta-\alpha)n} U_\alpha(|u|)(x_0)$$

$$\leq \text{const} \cdot \int_0^\infty e^{-(\beta-\alpha)t} dt < \infty. \qquad \square$$

An L^p-approach. Throughout this subsection we assume that μ is a σ-finite sub-invariant measure for $(P_t)_{t\geq 0}$. Hence $(P_t)_{t\geq 0}$ and $\mathcal{U}$ extend to a strongly continuous semigroup (resp. resolvent) family of contractions on $L^p(\mu)$, $1 \leq p < \infty$. The corresponding generators $(\mathsf{L}_p, D(\mathsf{L}_p) \subset L^p(\mu))$ are defined by

$$D(\mathsf{L}_p) = \{U_\alpha f | f \in L^p(\mu)\},$$
$$\mathsf{L}_p(U_\alpha f) = \alpha U_\alpha f - f \quad \text{for all } f \in L^p(\mu), 1 \leq p < \infty,$$

with the remark that this definition is independent of $\alpha > 0$.

The corresponding notations for the dual structure are $\widehat{P}_t$ and $(\widehat{\mathsf{L}}_p, D(\widehat{\mathsf{L}}_p))$, and we note that the adjoint of L_p is $\widehat{\mathsf{L}}_{p^*}$; $\frac{1}{p} + \frac{1}{p^*} = 1$.

The main results of section 2, namely Theorem 2.6 and its Corollary 2.7, can be reformulated in the $L^p(\mu)$ context. Although the proofs follow the same main ideas, they become simpler due to the strong continuity of $(P_t)_{t\geq 0}$ on $L^p(\mu)$. Because we are mainly interested in the situation when $V(u) < \infty$ on E except for some negligible set but also for simplicity, we present below the L^p-version of Corollary 2.7 only.

Proposition 4.2. *The following assertions are equivalent for a $\mathcal{B}$-measurable function $u \in \bigcup_{1 \leq p < \infty} L^p(\mu)$.*

   i) *$u(X)$ is $\mathbb{P}^x$-quasimartingale for μ-a.e., $x \in E$.*
   ii) *$V(u) < \infty$ μ-a.e.*
   iii) *For an admissible sequence of partitions of $(\tau_n)_{n\geq 1}$ of $\mathbb{R}_+$, $\sup_n V_{\tau_n}(u) < \infty$ μ-a.e.*
   iv) *There exist $u_1, u_2 \in E(\mathcal{U})$ finite m-a.e. such that $u = u_1 - u_2$ μ-a.e.*

Proof. We prove only iii) $\Rightarrow$ iv), just to point out the benefit of the strong continuity of $(P_t)_{t\geq 0}$ on $L^p(\mu)$.

As in the proof of Theorem 2.6 iii), if we define $\widetilde{u}_1$ and $\widetilde{u}_2$ μ-a.e. by

$$\widetilde{u}_1 = \sup_n u_1^{\tau_n} = \sup_n \left\{ \sum_{i=1}^{n} P_{t_{i-1}}(u - P_{t_i - t_{i-1}}u)^+ + P_{t_n}(u^+) \right\},$$

$$\widetilde{u}_2 = \sup_n u_2^{\tau_n} = \sup_n \left\{ \sum_{i=1}^{n} P_{t_{i-1}}(u - P_{t_i - t_{i-1}}u)^- + P_{t_n}(u^-) \right\},$$

then $\widetilde{u}_i$ are finite m-a.e., and one can show that $P_r\widetilde{u}_i \leq \widetilde{u}_i$ for all $r \in \bigcup_{n \geq 1} \tau_n$, $i = \overline{1, 2}$, and $u = \widetilde{u}_1 - \widetilde{u}_2$ μ-a.e.

If $t \in [0, \infty)$ and $\bigcup_{n \geq 1} \tau_n \supset (t_k)_k \searrow t$, then for $i = \overline{1, 2}$ and μ-a.e.,

$$P_t\widetilde{u}_i = \sup_n P_t u_i^{\tau_n} = \sup_n \lim_k P_{t_k} u_i^{\tau_n} \leq \lim_k P_{t_k} \widetilde{u}_i \leq \widetilde{u}_i,$$

with the remark that the second holds μ-a.e. because $u_i^{\tau_n} \in L^p(\mu)$ and $(P_t)_{t \geq 0}$ is strongly continuous. Then, cf. e.g., [BeCiRö 18], Proposition 2.4, there exist two $\mathcal{B}$-measurable functions $u_1, u_2 \in E(\mathcal{U})$ such that $\widetilde{u}_i = u_i$ μ-a.e., and finally, $u = u_1 - u_2$ μ-a.e.   □

Remark 4.3.

    i) We point out that for the proof of Proposition 4.2 we did not really use the fact that μ is subinvariant, but just that $(P_t)_{t \geq 0}$ is strongly continuous on $L^p(\mu)$. In particular, Proposition 4.2 remains true for $u \in L^\infty(\mu)$ if we regard $(P_t)_{t \geq 0}$ as a strongly continuous semigroup on $L^1(\widehat{U}_1 f \cdot \mu)$ for some $0 < f \in L^1(\mu)$.

    ii) If u is a $\mathcal{B}$-measurable finely continuous function from $\bigcup_{1 \leq p \leq \infty} L^p(\mu)$ satisfying any of the equivalent assertions in Proposition 4.2, then the decomposition $u = u_1 - u_2$ with $u_1, u_2 \in E(\mathcal{U})$ holds q.e.

Now, we focus our attention on a class of α-quasimartingale functions which arises as a natural extension of $D(\mathsf{L}_p)$. First of all, it is clear that any function $u \in D(\mathsf{L}_p)$, $1 \leq p < \infty$, has a representation $u = U_\alpha f = U_\alpha(f^+) - U_\alpha(f^-)$ with $U_\alpha(f^\pm) \in E(\mathcal{U}^\alpha) \cap L^p(\mu)$. In particular, u has an α-quasimartingale version for all $\alpha > 0$. Moreover, $\|P_t u - u\|_p = \left\| \int_0^t P_s \mathsf{L}_p u \, ds \right\|_p \leq t \|\mathsf{L}_p u\|_p$. Conversely, if $1 < p < \infty$, $u \in L^p(\mu)$, and $\|P_t u - u\|_p \leq \text{const} \cdot t$, $t \geq 0$, then due to the reflexivity of L^p we have that the family $\{\frac{P_t u - u}{t}\}_{t \geq 0}$ is weakly relatively compact, and by duality one can easily check that any weakly convergent subsequence $(\frac{P_{t_n} u - u}{t_n})_{t_n \to 0}$ has the same limit. Therefore $\frac{P_t u - u}{t}$ is weakly convergent to a limit from $L^p(\mu)$ as t tends to 0, and by [Sa 99, Lemma 32.3], it is strongly convergent and $u \in D(\mathsf{L}_p)$. But this is no longer the case if $p = 1$, and in general, $\|P_t u - u\|_1 \leq \text{const} \cdot t$ does not imply $u \in D(\mathsf{L}_1)$. However, this last condition on $L^1(\mu)$ is still sufficient to guarantee that u is an α-quasimartingale function. In fact, the following general characterization holds.

Proposition 4.4. *Let* $1 \leq p < \infty$ *and suppose* $\mathcal{A} \subset \{u \in L_+^{p^*}(\mu) : \|u\|_{p^*} \leq 1\}$, $\widehat{P}_s\mathcal{A} \subset \mathcal{A}$ *for all* $s \geq 0$, *and* $E = \bigcup_{f \in \mathcal{A}} \text{supp}(f)$ μ-a.e. *Then the following assertions are equivalent for* $u \in L^p(\mu)$.

    i) $\sup_{f \in \mathcal{A}} \int_E |P_t u - u| f \, d\mu \leq \text{const} \cdot t$ *for all* $t \geq 0$.

    ii) *For every* $\alpha > 0$, *there exist* $u_1, u_2 \in E(\mathcal{U}^\alpha)$ *which satisfies* i), $\sup_{f \in \mathcal{A}} \int_E (u_1 + u_2) f \, d\mu < \infty$, *and* $u = u_1 - u_2$ μ-a.e.

Proof. Since ii) $\Rightarrow$ i) is clear, let us prove the other implication. Assume that u satisfies i). Then taking $\widehat{P}_s f$ instead of f in condition i), we get for all $s, t \geq 0$

$$\int_E P_s^\alpha |P_t^\alpha u - u| f d\mu \leq \int_E P_s^\alpha |P_t u - u| f d\mu + \int_E P_s^\alpha |P_t u - P_t^\alpha u| f d\mu$$
$$\leq \left[\text{const} \cdot t + (1 - e^{-\alpha t}) \|u\|_p \|f\|_{p^*}\right] e^{-\alpha s}$$
$$\leq \text{const} \cdot t e^{-\alpha s}.$$

Let now $\tau_n := \left\{ \dfrac{k}{2^n} : 0 \leq k \leq n 2^n \right\}$, $n \geq 1$. Then, for $\alpha > 0$

$$\int_E \sup_n V_{\tau_n}^\alpha(u) f d\mu = \lim_n \sum_{k=1}^{n 2^n} \int_E P_{\frac{k-1}{2^n}}^\alpha |P_{\frac{1}{2^n}}^\alpha u - u| f d\mu$$
$$\leq \text{const} \cdot \lim_n \sum_{k=1}^{n 2^n} e^{-\alpha \frac{k-1}{2^n}} \frac{1}{2^n}$$
$$= \text{const} \cdot \int_0^\infty e^{-\alpha t} dt < \infty$$

for all $f \in \mathcal{A}$. Hence $\sup_n V_{\tau_n}^\alpha(u) < \infty$ μ-a.e., and by Proposition 4.2 we have that $u = u_1 - u_2$ μ-a.e. with $u_1, u_2 \in E(\mathcal{U}^\alpha)$. Moreover, inspecting the way u_1 and u_2 have been constructed, we have that $u_1 + u_2 = \sup_n V_{\tau_n}^\alpha(u)$ μ-a.e., hence $\sup_{f \in \mathcal{A}} \int_E (u_1 + u_2) f d\mu < \infty$. Moreover, for $r \in \bigcup_{n \geq 1} \tau_n$ and $i = \overline{1, 2}$,

$$u_i = \lim_n \left\{ \sum_{k=1}^{r 2^n} P_{\frac{k-1}{2^n}}^\alpha (u - P_{\frac{1}{2^n}}^\alpha u)^\pm + P_r^\alpha (u - P_{\frac{1}{2^n}}^\alpha u)^\pm \right.$$
$$\left. + \sum_{i=1}^{n 2^n} P_r^\alpha P_{\frac{i-1}{2^n}}^\alpha (u - P_{\frac{1}{2^n}}^\alpha u)^\pm + P_r^\alpha P_{n-r}^\alpha (u^\pm) \right\}$$
$$= \lim_n \left\{ \sum_{k=1}^{r 2^n} P_{\frac{k-1}{2^n}}^\alpha (u - P_{\frac{1}{2^n}}^\alpha u)^\pm + P_r^\alpha (u - P_{\frac{1}{2^n}}^\alpha u)^\pm \right\} + P_r^\alpha u^i.$$

Therefore,

$$\int_E |u_i - P_r^\alpha u_i| f d\mu \leq \lim_n \sum_{k=1}^{r 2^n} \int_E P_{\frac{k-1}{2^n}}^\alpha |u - P_{\frac{1}{2^n}}^\alpha u| f d\mu$$
$$\leq \text{const} \cdot \int_0^r e^{-\alpha t} dt$$
$$= \text{const} \cdot r$$

for all $f \in \mathcal{A}$, $i = \overline{1, 2}$, $r \in \bigcup_{n \geq 1} \tau_n$, where the above constant is independent of $f \in \mathcal{A}$, $i = \overline{1, 2}$, and $r \in \bigcup_{n \geq 1} \tau_n$.

We claim that $\int_E (u_i - P_t^\alpha u_i) f d\mu \leq \text{const} \cdot t$ for all $t \geq 0$, $i = \overline{1, 2}$, and $f \in \mathcal{A}$. Since the desired inequality holds for all $r \in \bigcup_{n \geq 1} \tau_n$ and $0 \leq u_i - P_r^\alpha u_i \leq u_i$, by dominated convergence it is sufficient to show that for each $f \in \mathcal{A}$, $P_{r_k}^\alpha u_i$ converges $f \cdot \mu$-a.e. on a subsequence to $P_t^\alpha u_i$, whenever $\bigcup_{n \geq 1} \tau_n \ni r_k \searrow_k t \geq 0$. To see this, let $\nu := \widehat{U}_\alpha f \cdot \mu$ and note that $u_i \in L^1(\nu)$. Since ν is a subinvariant measure for

$(P_t^\alpha)_{t\geq 0}$, we have that $(P_t^\alpha)_{t\geq 0}$ is strongly continuous on $L^1(\nu)$; hence, if $\bigcup_{n\geq 1}\tau_n \ni r_k \searrow t \geq 0$, it follows that on a subsequence, $(P_{r_k}^\alpha u_i)_{k\geq 1}$ converges ν-a.e. to $P_t^\alpha u_i$. Since $f\cdot\mu \ll \nu$, we obtain that the above convergence holds $f\cdot\mu$-a.e. So,

$$\int_E |u_i - P_t^\alpha u_i| f d\mu \leq \text{const}\cdot t,$$

and finally

$$\int_E |u_i - P_t u_i| f d\mu \leq \text{const}\cdot t + (1 - e^{-\alpha t})\int_E u_i f d\mu \leq \text{const}\cdot t$$

for all $t\geq 0$, $i = \overline{1,2}$, and independently on $f\in\mathcal{A}$. $\qquad\square$

We can interpert condition i) from Proposition 4.4 in terms of the adjoint generator as follows.

Proposition 4.5. *Let $p\in(1,\infty)$ and $q\in[1,\infty]$. The following assertions are equivalent for $u\in L^p(\mu)$.*

- i) $|\mu(u\,\widehat{\mathsf{L}}_{p^*}v)| \leq \text{const}\cdot(\|v\|_\infty + \|v\|_q)$ *for all $v\in D(\widehat{\mathsf{L}}_{p^*})$.*
- ii) u *satisfies i) from Proposition 4.4 for all $\mathcal{A} = \{v\in L^{p^*}(\mu) : \|v\|_\infty + \|v\|_q \leq 1\}$.*

Proof. i) $\Rightarrow$ ii). Let $f\in L^\infty(\mu)\cap L^q(\mu)\cap L^{p^*}(\mu)$. For $t\geq 0$ let $w := \frac{1}{t}\text{sgn}(P_t u - u)f \in L^{p^*}(\mu)$ and $v := \int_0^t \widehat{P}_s w ds \in D(\widehat{\mathsf{L}}_{p^*})$. Then $\widehat{\mathsf{L}}_{p^*}v = \widehat{P}_t w - w$, $\|v\|_\infty + \|v\|_q \leq 2(\|f\|_\infty + \|f\|_q)$, and

$$\frac{1}{t}\int_E |P_t u - u| f d\mu = \int_E (P_t u - u) w d\mu = \int_E u(\widehat{P}_t w - w) d\mu$$

$$= \int_E u\,\widehat{\mathsf{L}}_{p^*}v d\mu \leq 2\cdot\text{const}\cdot(\|f\|_\infty + \|f\|_q).$$

Therefore, $\int_E |P_t u - u| f d\mu \leq \text{const}\cdot t$ for all $t\geq 0$ and $f\in\mathcal{A}$.

ii) $\Rightarrow$ i). If $\int_E |P_t u - u| f d\mu \leq \text{const}\cdot t(\|f\|_\infty + \|f\|_q)$, then by replacing f with $\text{sgn}(P_t u - u)f$, we get

$$\frac{1}{t}\int_E u(\widehat{P}_t f - f) d\mu \leq \text{const}\cdot(\|f\|_\infty + \|f\|_q).$$

Now, if $f\in D(\widehat{\mathsf{L}}_{p^*})$, then assertion i) follows by letting t tend to 0. $\qquad\square$

Example: Adding jumps to a Markov process. Assume that X is a standard process and N is a Markov kernel on E. As before, μ is a σ-finite subinvariant measure for $\mathcal{U}$. We assume further that $\mu\circ\mathsf{N} \leq \mu$. It is well known that there exists a second Markov process Y on E whose infinitesimal generator is given by $\mathsf{Q} := \mathsf{L} - 1 + \mathsf{N}$; $D(\mathsf{L}) = D(\mathsf{Q})$; cf. [Ba 79] or [BeSt 94, Theorem 1.8]; see also [Op 16] for more general perturbations with kernels for generators of Markov processes. Let $\mathcal{V} = (V_\alpha)_\alpha$ denote the resolvent of Y. Then $V_\alpha = U_{\alpha+1} + U_{\alpha+1}\mathsf{N}V_\alpha$ and

$$\mu(V_1 f) = \mu\left(\sum_{n=0}^\infty U_2(\mathsf{N}U_2)^n f\right) \leq \sum_{n=1}^\infty \frac{1}{2^n}\mu(f) = \mu(f)$$

for all $f\in L^1_+(\mu)$, which means that μ is $\mathcal{V}$-subinvariant. Therefore, we can extend Q on $L^p(\mu)$, $1\leq p < \infty$, by $\mathsf{Q}_p := \mathsf{L}_p - 1 + \mathsf{N}_p$, $D(\mathsf{Q}_p) = D(\mathsf{L}_p)$, where L_p and N_p are the corresponding $L^p(\mu)$-extensions of L and N.

Let $(S_t)_{t\geq 0}$ be the transition function of Y. Since μ is $(S_t)_{t\geq 0}$-subinvariant, we have that $(\widetilde{S_t})_{t\geq 0}$ extends to a C_0-semigroup of contractions on $L^p(\mu)$, $1\leq p<\infty$, for which we keep the same notation.

Clearly, since $E(\mathcal{V}_\alpha)\subset E(\mathcal{U}_{\alpha+1})$, we get by Corollary 2.7 that any α-quasimartingale function for Y is an $(\alpha+1)$-quasimartingale function for X. Also, as remarked in [BeLu 16, Proposition 4.5], the spaces of differences of bounded functions from $E(\mathcal{U}_{\alpha+1})$ and from $E(\mathcal{V}_\alpha)$, respectively, are the same. Next, we show that the class of quasimartingale functions which are produced by the estimate in Proposition 4.4 i) (and in Proposition 4.5 ii)) are the same for both X and Y.

Corollary 4.6. *Let $1\leq p<\infty$, $u\in L^p(\mu)$, and let $\mathcal{A}$ be a bounded subset in L^{p^*}. Then* i) *from Proposition 4.4 is satisfied with respect to $(P_t)_{t\geq 0}$ if and only if it is satisfied with respect to $(S_t)_{t\geq 0}$.*

Proof. The result follows easily since Q_p is a bounded perturbation of L_p, and by, e.g., [EnNa 99, Corollary 1.11], there exists a constant c such that $\|P_t-S_t\|_p\leq t\cdot c$, $t\geq 0$. $\qquad\square$

5. Applications to Dirichlet forms

Let E be a Hausdorff topological space with Borel σ-algebra $\mathcal{B}$, let μ be a σ-finite measure on $\mathcal{B}$, and let $\mathcal{E}$ be a bilinear form on $L^2(\mu)$ with dense domain $\mathcal{F}$; $\mathcal{E}_\alpha(\cdot,\cdot)=\mathcal{E}(\cdot,\cdot)+\alpha(\cdot,\cdot)$, $\alpha>0$.

Recall that $(\mathcal{E},\mathcal{F})$ is called a *coercive closed* form if the following hold.

i) $(\mathcal{E},\mathcal{F})$ is positive definite and closed on $L^2(\mu)$.

ii) $\mathcal{E}$ satisfies the (weak) sector condition, i.e., there exists a constant k such that

$$|\mathcal{E}_1(u,v)|\leq k\mathcal{E}_1(u,u)^{\frac{1}{2}}\mathcal{E}_1(v,v)^{\frac{1}{2}}\quad\text{for all }u,v\in\mathcal{F}.$$

The coercive closed form $(\mathcal{E},\mathcal{F})$ is called a *Dirichlet* form if $u^+\wedge 1\in\mathcal{F}$ and both

iii) $\mathcal{E}(u+u^+\wedge 1,u-u^+\wedge 1)\geq 0$ and

iv) $\mathcal{E}(u-u^+\wedge 1,u+u^+\wedge 1)\geq 0$

hold for all $u\in\mathcal{E}$. If only iii) is satisfied, then $(\mathcal{E},\mathcal{F})$ is called a *semi-Dirichlet* form.

A bilinear form $(\mathcal{E},\mathcal{F})$ on $L^2(\mu)$ is called a *lower-bounded* (semi-) Dirichlet form if there exists $\alpha>0$ such that $(\mathcal{E}_\alpha,\mathcal{F})$ is a (semi-) Dirichlet form. If $(\mathcal{E},\mathcal{F})$ is a coercive closed form, let $(P_t)_{t\geq 0}$ be the C_0-semigroup of contractions on $L^2(m)$ associated to $\mathcal{E}$, whose dual is denoted by $(\widehat{P}_t)_{t\geq 0}$. Recall that condition iii) (resp. iv)) is equivalent with the sub-Markov property for $(P_t)_{t\geq 0}$ (resp. $(\widehat{P}_t)_{t\geq 0}$); see [MaRö 92, I.4.4].

Adopting the notations from [Fu 99] for a closed set $F\subset E$, we set

$$\mathcal{F}_F=\{v\in\mathcal{F}\,:\,v=0\ m\text{-a.e. on }E\setminus F\},$$

$$\mathcal{F}_{b,F}=\{v\in\mathcal{F}_F\,:\,v\in L^\infty(\mu)\}.$$

An increasing sequence of closed sets $(F_n)_{n\geq 1}$ is called an $\mathcal{E}$-*nest* if $\bigcup_{n=1}^\infty\mathcal{F}_{F_n}$ is $\mathcal{E}_1$-dense in $\mathcal{F}$. An element $f\in\mathcal{F}$ is called $\mathcal{E}$-*quasi-continuous* if there exists a nest $(F_n)_{n\geq 1}$ such that $f|_{F_n}$ is continuous for each $n\geq 1$.

A (semi-) Dirichlet form $(\mathcal{E},\mathcal{F})$ on $L^2(\mu)$ is called *quasi-regular* if there exist an $\mathcal{E}$-nest consisting of compact sets, an $\mathcal{E}_1$-dense subset of $\mathcal{F}$ whose elements admit $\mathcal{E}$-quasi-continuous versions, and a countable family of $\mathcal{E}$-quasi-continuous elements from $\mathcal{F}$ which separates the points of $\bigcup_{n=1}^\infty E_n$ for a certain $\mathcal{E}$-nest $(E_n)_{n\geq 1}$. It is

well known that the quasi-regularity property is a necessary and sufficient condition for a semi-Dirichlet form to be (properly) associated to a μ-tight special standard process X (i.e., the semigroup $(P_t)_{t\geq 0}$ of $(\mathcal{E}, \mathcal{F})$ is generated by the transition function of X); see [MaOvRö 95] or [MaRö 92] for details. On the other hand, it was shown in [BeBoRö 06] that for any semi-Dirichlet form $(\mathcal{E}, \mathcal{F})$ on a Lusin measurable space E, one can always find a larger space E_1 such that $E_1 \setminus E$ has measure zero and $(\mathcal{E}, \mathcal{F})$ regarded on E_1 becomes quasi-regular.

Hereinafter, all of the forms are assumed to be quasi-regular; in particular every element $u \in \mathcal{F}$ admits a quasi-continuous version denoted by $\widetilde{u}$.

In the rest of this article we often appeal to the following well-known decompositions for the elements of the domain $\mathcal{F}$:

Ortogonal decomposition via hitting distribution. For a nearly Borel set $A \subset E$ and a quasi-continuous function $u \in \mathcal{F}$ we define the α-order hitting distribution $R_\alpha^{A^c} u(x) := \mathbb{E}^x[e^{-\alpha T_{A^c}} u(X_{T_{A^c}})]$, $\alpha > 0$. Then $R_\alpha^{A^c} u \in \mathcal{F}$ is quasi-continuous, $u - R_\alpha^{A^c} u \in \mathcal{F}_A$, and $\mathcal{E}_\alpha(R_\alpha^{A^c} u, v) = 0$ for all $v \in \mathcal{F}_A$. When $\mathcal{E}$ is a Dirichlet form, $\widehat{R}_\alpha^{A^c} u$ may be defined analogously, replacing X with the dual process $\widehat{X}$. When $\mathcal{E}$ is merely a semi-Dirichlet form, the existence of the dual process is more delicate, and for simplicity we prefer to define $\widehat{R}_\alpha^{A^c} u$ as the unique element from $\mathcal{F}$ such that $u - \widehat{R}_\alpha^{A^c} u \in \mathcal{F}_A$ and $\mathcal{E}_\alpha(v, \widehat{R}_\alpha^{A^c} u) = 0$ for all $v \in \mathcal{F}_A$; see e.g., [Os 13], Section 3.5.

Fukushima's decomposition. (see [MaRö 92, Chapter VI, Theorem 2.5], or [FuOsTa 11]) For each $u \in \mathcal{F}$ there exist a martingale additive functional of finite energy $(M_t)_{t\geq 0}$ (MAF) and a continuous additive functional $(N_t)_{t\geq 0}$ of zero energy such that $\widetilde{u}(X) - \widetilde{u}(X_0) = M + N$; we denote by $|N|_t$ the variation of N on $[0, t]$.

For the rest of this section our aim is to explore conditions for an element $u \in \mathcal{F}$ (or more generally in $\mathcal{F}_{\text{loc}}$) ensuring that $\widetilde{u}(X)$ is a $\mathbb{P}^x$-semimartingale (α-quasimartingale) q.e. $x \in E$; in this case we shall say shortly that $\widetilde{u}(X)$ *is a semi(α-quasi)martingale.*

The case of (non-symmetric) Dirichlet forms. Going back to Proposition 4.4 and Corollary 4.5, we note that the sub-Markov property of the dual semigroup was quite helpful, and for this reason we shall first deal with Dirichlet forms. However, in the end of this section we shall see that the results can be extended to semi-Dirichlet forms in their full generality. It is worth mentioning that all of the forthcoming criteria for quasimartingale functions can be directly transferred to lower bounded (semi-) Dirichlet forms whose semigroups are associated to right processes, but for simplicity we deal only with (semi-) Dirichlet forms.

Theorem 5.1. *The following assertions are equivalent for an element $u \in \mathcal{F}$.*

   i) *$|\mathcal{E}(u,v)| \leq \text{const} \cdot (\|v\|_\infty + \|v\|_2)$ for all $v \in \mathcal{F}_b$.*
   ii) *For one (hence all) $\alpha > 0$, there exist $u_1, u_2 \in E(\mathcal{U}^\alpha)$ such that*
       ii)2) $\sup_{\|f\|_\infty + \|f\|_2 \leq 1} \int_E (u_1 + u_2) f d\mu < \infty.$
       ii)2) $u = u_1 - u_2 \ \mu\text{-a.e.}$
       ii)2) $\sup_{\|f\|_\infty + \|f\|_2 \leq 1} \int_E |P_t u_i - u_i| f d\mu \leq \text{const} \cdot t, \ i = 1, 2.$

iii) *For one (hence all) $\alpha > 0$, $\widetilde{u}(X)$ is an α-quasimartingale and*

$$\sup_{\|f\|_\infty + \|f\|_2 \leq 1} \mathbb{E}^{f \cdot \mu}[|N|_t] \leq \text{const} \cdot t$$

for sufficiently small $t \geq 0$.

In particular, if u satisfies i), then there exists a smooth measure ν such that $\mathcal{E}(u, v) = \nu(\widetilde{v})$ for all $v \in \mathcal{F}_b$.

Proof. i) $\Rightarrow$ ii). Let $t > 0$, $f \in L^\infty(\mu) \cap L^2(\mu)$, $w := \text{sgn}(P_t u - u)f$, and set $v := \int_0^t \widehat{P}_s w \, ds$. Then

$$\int_E |P_t u - u| f \, d\mu = \int_E (P_t u - u) w \, d\mu = \mathcal{E}(u, v) \leq \text{const} \cdot t(\|f\|_\infty + \|f\|_2).$$

By Proposition 4.4 (take $\mathcal{A} := \{f \in L^2(\mu) : \|f\|_\infty + \|f\|_2 \leq 1\}$) we obtain ii).

ii) $\Rightarrow$ iii). Since $\widetilde{u}$ is quasi-continuous, we have $\widetilde{u} = u_1 - u_2$ q.e.; hence, the first assertion is clear (by Corollary 3.2 for example).

Since $(e^{-\alpha t} u_i(X_t))_{t \geq 0}$, $i = \overline{1, 2}$, are right continuous supermartingales, by [Sha 88, Section VI], there exist (uniquely) local martingales M^i, $M_0^i = 0$, and predictable right-continuous increasing and nonnegative processes A^i such that $e^{-\alpha t} u_i(X_t) - u_i(X_0) = M_t^i - A_t^i$, $t \geq 0$.

If $(T_n^i)_n$ are stopping times increasing a.s. to infinity such that the stopped processes $(M_{t \wedge T_n^i}^i)_{t \geq 0}$ are uniformly integrable martingales, we get

$$\mathbb{E}^x[A_{t \wedge T_n^i}^i] = -\mathbb{E}^x[e^{-\alpha(t \wedge T_n)} u_i(X_{t \wedge T_n})] + u_i(x) \leq u_i(x) - e^{-\alpha t} P_t u_i(x), \ x \in E.$$

Therefore, $\mathbb{E}^x[A_t^i] \leq u_i(x) - e^{-\alpha t} P_t u_i(x)$ and if $f \in L^\infty(\mu) \cap L^2(\mu)$

$$\mathbb{E}^{f \cdot \mu}[A_t^i] \leq \mu((u_i - e^{-\alpha t} P_t u_i)f) \leq \mu(|u_i - P_t u_i|f) + (1 - e^{-\alpha t})\mu(u_i \widehat{P}_t f)$$
$$\leq \text{const} \cdot t(\|f\|_\infty + \|f\|_2).$$

Also, $e^{-\alpha t} \widetilde{u}(X_t) - \widetilde{u}(X_0) = \overline{M_t} + \overline{A_t}$, where $\overline{M} := M^1 - M^2$ is a local martingale and $\overline{A} := A^2 - A^1$ is a predictable right-continuous process of bounded variation.

On the other hand, since $\widetilde{u}(X)$ is an α-quasimartingale, it follows that N, the CAF from Fukushima decomposition, is a continuous semimartingale; hence, it is the sum of a local martingale and a continuous process with bounded variation (see, e.g., [Pr 05, page 131]). But N has zero energy, so the quadratic variation of its martingale part is zero; hence, N is of bounded variation. Then, integrating by parts,

$$e^{-\alpha t} \widetilde{u}(X_t) - \widetilde{u}(X_0) = \int_0^t e^{-\alpha s} dM_s + e^{-\alpha t} N_t - \alpha \int_0^t M_s e^{-\alpha s} ds - \widetilde{u}(X_0)(1 - e^{-\alpha t}).$$

By the uniqueness of the canonical decomposition of $(e^{-\alpha t} \widetilde{u}(X_t))_{t \geq 0}$, we get that

$$\overline{A_t} = e^{-\alpha t} N_t - \alpha \int_0^t M_s e^{-\alpha s} ds - \widetilde{u}(X_0)(1 - e^{-\alpha t}).$$

Therefore,

$$N_t = e^{\alpha t} \overline{A_t} + \alpha e^{\alpha t} \int_0^t M_s e^{-\alpha s} ds + \widetilde{u}(X_0) e^{\alpha t}(1 - e^{-\alpha t})$$

and

$$|N|_t \leq e^{\alpha t}(A_t^1 + A_t^2) + \alpha e^{\alpha t} \int_0^t |M_s| e^{-\alpha s} ds + |\widetilde{u}(X_0)| e^{\alpha t}(1 - e^{-\alpha t}).$$

But by the previously obtained estimates for $\mathbb{E}^{f \cdot \mu}[A_t^i]$, we get

$$\begin{aligned}
\mathbb{E}^{f \cdot \mu}[|N|_t] &\leq \text{const} \cdot t e^{\alpha t}(\|f\|_\infty + \|f\|_2) \\
&\quad + e^{\alpha t}(1 - e^{-\alpha t})\mu(f|u|) + e^{\alpha t}(1 - e^{-\alpha t})t\mathbb{E}^{f \cdot \mu}[|M_t|] \\
&\leq \text{const} \cdot t(\|f\|_\infty + \|f\|_2)
\end{aligned}$$

for conveniently small t, since $\mathbb{E}^{f \cdot \mu}[|M_t|] \leq \|f\|_2 \mathbb{E}^\mu[M_t^2]$ and M is of finite energy (i.e., $\lim_{t \to 0} \frac{1}{t}\mathbb{E}^\mu[M_t^2] < \infty$).

iii) $\Rightarrow$ i). By Revuz correspondence (see [MaRö 92, Theorem 2.4]), if $v = \widehat{U}_\alpha f$ for some $\alpha > 0$ and $f \in L^2(\mu) \cap L^\infty(\mu)$,

$$\mathcal{E}(u,v) = \lim_{t \to 0} \frac{1}{t} \int_E (P_t u - u)v \, d\mu = \lim_{t \to 0} \frac{1}{t}\mathbb{E}^{v \cdot \mu}[N_t] = \nu(\widetilde{v}),$$

where ν is the signed Revuz measure associated to N. By an approximation argument, $|\mathcal{E}(u,v)| = |\nu(\widetilde{v})| \leq \text{const }(\|v\|_\infty + \|v\|_2)$ for all $v \in \mathcal{F}_b$. $\qquad \square$

Further versions of Theorem 5.1 can be taken into account. For example, the following result extends Theorem 6.2 from [Fu 99] to the nonsymmetric case, and it can be proved in the same manner as Theorem 5.1, so we omit it.

Theorem 5.2. *The following assertions are equivalent for $u \in \mathcal{F}$.*

    i) $|\mathcal{E}(u,v)| \leq \text{const} \cdot \|v\|_\infty$ *for all $v \in \mathcal{F}_b$.*
    ii) *For each $\alpha > 0$, $\widetilde{u}(X)$ is an α-quasimartingale and $\mathbb{E}^\mu[|N|_t] \leq \text{const} \cdot t$ for small t.*
    iii) *There exists a smooth signed measure (the Revuz measure of N) ν such that ν is finite and $\mathcal{E}(u,v) = \nu(\widetilde{v})$ for all $v \in \mathcal{F}$.*

Now, we turn our attention to the situation when any of the equivalent assertions of Theorem 5.2 holds only locally. The following result extends Theorem 6.1 from [Fu 99] to the nonsymmetric case.

Theorem 5.3. *The following assertions are equivalent for $u \in \mathcal{F}$.*

    i) $\widetilde{u}(X)$ *is a semimartingale.*
    ii) *There exists a nest $(F_n)_{n \geq 1}$ and constants c_n such that*

$$|\mathcal{E}(u,v)| \leq c_n\|v\|_\infty \quad \text{for all } v \in \mathcal{F}_{b,F_n}.$$

Proof. i) $\Rightarrow$ ii). As in the poof of ii) $\Rightarrow$ iii) in Theorem 5.1, if $\widetilde{u}(X)$ is a semimartingale, then N (the CAF in Fukushima decomposition) is a continuous semimartingale of zero energy, hence it is of bounded variation. By [MaRö 92, Theorem 2.4], N is in Revuz correspondence with a signed smooth measure ν, with an attached nest of compacts $(F_n)_{n \geq 1}$ such that $\nu(F_n) < \infty$. Then just as in the proof of Theorem 5.4.2. in [FuOsTa 11], one obtains that $\mathcal{E}(u,v) = \nu(\widetilde{v})$ for all $v \in \mathcal{F}$.

ii) $\Rightarrow$ i). Without loss we can assume that $\mu(F_n) < \infty$. Also, since $(F_n)_n$ is a nest, we have that $\lim_n T_{F_n^c} \geq \xi$ a.s. Due to a result of Meyer (see, e.g., [Pr 05, Theorem 6]) it is sufficient to show that $(\widetilde{u}(X_t)1_{[0,T_{F_n^c})}(t))_{t \geq 0}$ is a semimartingale for each n (such an argument was also employed in [ÇiJaPrSh 80], after Theorem 4.6). On the other hand, $(e^{-t}R_1^{F_n^c}\widetilde{u}(X_t)1_{[0,T_{F_n^c})}(t))_{t \geq 0}$ is a difference of two right-continuous supermartingales, so we only have to check that $(\widetilde{u} - R_1^{F_n^c}\widetilde{u})(X)$ is a semimartingale.

But, if $v \in \mathcal{F}_b$,

$$|\mathcal{E}_1(\widetilde{u} - R_1^{F_n^c}\widetilde{u}, v)| = |\mathcal{E}_1(\widetilde{u} - R_1^{F_n^c}\widetilde{u}, v - \widehat{R}_1^{F_n^c}v)| = |\mathcal{E}_1(\widetilde{u}, v - \widehat{R}_1^{F_n^c}v)|$$

$$\leq (c_n + \int_{F_n} |u| d\mu) \|v - \widehat{R}_1^{F_n^c}v\|_\infty$$

$$\leq 2(c_n + \int_{F_n} |u| d\mu) \|v\|_\infty,$$

and by Theorem 5.1 it follows that $(\widetilde{u} - R_1^{F_n^c}\widetilde{u})(X)$ is a semimartingale. $\qquad\square$

Recall that $(\mathcal{E}, \mathcal{F})$ is said to be *local* if for all pairs of elements $u, v \in \mathcal{F}$ with disjoint compact supports, it holds that $\mathcal{E}(u, v) = 0$. By [MaRö 92, Chapter V, Theorem 1.5], $(\mathcal{E}, \mathcal{F})$ is local if and only if the associated process is a diffusion.

When $\mathcal{E}$ is local, Theorem 5.3 remains true if u is assumed to be only locally in $\mathcal{F}$. Actually, the following even more general statement holds.

Corollary 5.4. *Assume that $(\mathcal{E}, \mathcal{F})$ is local. Let u be a real-valued $\mathcal{B}$-measurable finely continuous function, and let $(v_k)_k \subset \mathcal{F}$ such that $v_k \longrightarrow_{k \to \infty} u$ pointwise except a μ-polar set and boundedly on each F_n. Further, suppose that there exist constants c_n such that*

$$|\mathcal{E}(v_k, v)| \leq c_n \|v\|_\infty \ \ for \ all \ v \in \mathcal{F}_{b, F_n}.$$

Then $u(X)$ is a semimartingale.

Proof. As we already mentioned in the proof of Theorem 5.3 ii) $\Rightarrow$ i), it is sufficient to show that $(u - R_1^{F_n^c}u)(X)$ is a semimartingale. Also, we saw that $|\mathcal{E}_1(\widetilde{v}_k - R_1^{F_n^c}\widetilde{v}_k, v)| \leq \text{const} \cdot \|v\|_\infty$ for all $v \in \mathcal{F}_b$, where the constant in the right-hand side may depend on n. Now, by Theorem 5.1 we have that by setting $\widetilde{v}_k^n := \widetilde{v}_k - R_1^{F_n^c}\widetilde{v}_k$,

$$\mu(\widetilde{v}_k^n(\widehat{P}_t^1 f - f)) \leq \text{const} \cdot t(\|f\|_\infty + \|f\|_2).$$

But by hypothesis, $R_1^{F_n^c}\widetilde{v}_k(x) = \mathbb{E}^x[e^{-T_{F_n^c}}\widetilde{v}_k(X_{T_{F_n^c}})]$ converges μ-a.s. and boundedly to $R_1^{F_n^c}u$ as k tends to infinity. So by setting $u_n := u - R_1^{F_n^c}u$ and by dominated convergence,

$$\mu((P_t^1 u_n - u_n)f) = \mu(u_n(\widehat{P}_t^1 f - f)) = \lim_k \mu(\widetilde{v}_k^n(\widehat{P}_t^1 f - f))$$

$$\leq \text{const} \cdot t(\|f\|_\infty + \|f\|_2)$$

for all $f \in L^1(\mu)$. Therefore, by Proposition 4.4 we get that $u_n(X)$ is a semimartingale. $\qquad\square$

5.1. **Extensions to semi-Dirichlet forms.** We reiterate that for the previous results of this section, where we considered only Dirichlet forms, we used the fact that the adjoint semigroup $(\widehat{P}_t)_{t \geq 0}$ was sub-Markovian; e.g., in order to have the estimate $\|\int_0^t \widehat{P}_s f \, ds\|_\infty \leq t\|f\|_\infty$. In this subsection we show that, as a matter of fact, the sub-Markov property of the adjoint semigroup is not crucial, and most of the previous results remain valid for semi-Dirichlet forms. More precisely, although in order to extend Theorems 5.1 and 5.2 i) $\Rightarrow$ ii) to semi-Dirichlet forms, essentially with the same proofs, it is sufficient to assume the existence of a strictly positive bounded co-excessive function, Theorem 5.3 ii) $\Rightarrow$ i) remains true without any

further assumptions, due to a standard localization procedure. Finally, before we present the announced extensions, we emphasize once again that the case of lower bounded semi-Dirichlet forms follows easily, by working with $\mathcal{E}_\alpha$ instead of $\mathcal{E}$, for instance.

Hereinafter, we keep the same context and notations as before, but we assume that $(\mathcal{E}, \mathcal{F})$ is merely a (quasi-regular) semi-Dirichlet form on $L^2(E, \mu)$; i.e., we drop condition iv) from the beginning of this section.

Before we present the announced extension, in order to fix the notations, let us recall the following localization procedure. Let G be a finely open set, and consider the bilinear form

$$\mathcal{E}^G(u, v) := \mathcal{E}(u, v) \quad \text{for all } u, v \in D(\mathcal{E}^G) := \mathcal{F}_G .$$

Then by [BeBo 04, Theorem 7.6.11] (see also [Os 13, Theorem 3.5.7]), $(\mathcal{E}^G, \mathcal{F}_G)$ is a (quasi-regular) semi-Dirichlet form whose associated process is X^G with state space $G \cup \{\Delta\}$, obtained by killing X upon leaving G:

$$X_t^G := \begin{cases} X_t & \text{if } 0 \le t < T_{G^c}, \\ \Delta & \text{if } t \ge T_{G^c}. \end{cases}$$

The associated semigroup and resolvent are denoted by $(P_t^G)_{t \ge 0}$ and $(U_\alpha^G)_\alpha$.

Theorem 5.5. *Let $u \in \mathcal{F}$, and assume there exist a nest $(F_n)_{n \ge 1}$ and constants $(c_n)_{n \ge 1}$ such that*

$$\mathcal{E}(u, v) \le c_n \|v\|_\infty \quad \text{for all } v \in \mathcal{F}_{b, F_n}.$$

Then $\widetilde{u}(X)$ is a semimartingale.

Proof. Let us fix a quasi-continuous element $0 < f_0 \in \mathcal{F}$ and a sequence of positive constants $\alpha_k \nearrow_k \infty$. By [MaRö 92, Theorem 2.13] we have that $\alpha_k \widehat{U}_{\alpha_k} f_0 \xrightarrow[k]{\mathcal{E}_1^{1/2}} f_0$, hence by [MaOvRö 95, Proposition 2.18(i)], there exists a nest $(F_n')_{n \ge 1}$ such that (by passing to a subsequence if necessary) $\lim_k \alpha_k \widehat{U}_{\alpha_k} f_0 = f_0$ uniformly on each F_n'. Consequently, replacing F_n with $F_n \cap F_n'$, we may assume that $(F_n)_{n \ge 1}$ is a nest such that $\sup_k \|1_{F_n} \alpha_k \widehat{U}_{\alpha_k} f_0\|_\infty < \infty$. Also, without loss of generality we suppose that $\mu(F_n) < \infty$ for all $n \ge 1$.

Now, let us consider the fine interiors $G_n := \overset{\circ}{F_n}{}^f$ and the localized semi-Dirichlet forms $(\mathcal{E}^{G_n}, \mathcal{F}_{G_n})$. As before, the idea is to localize u by setting

$$u_n := \widetilde{u} - R_1^{G_n^c} \widetilde{u} \in \mathcal{F}_{G_n} \quad \text{for all } n \ge 1,$$

so that by setting $c_k^n := c_n + \alpha_k \|u_n\|_{L^1(G_n, \mu)} + \|u\|_{L^1(G_n, \mu)}$, for all $v \in \mathcal{F}_{G_n}$,

$$|\mathcal{E}_{\alpha_k+1}^{G_n}(u_n, v)| = |\mathcal{E}_{\alpha_k+1}(u_n, v)| = |\mathcal{E}_1(u, v) + \alpha_k(u_n, v)_2| \le c_k^n \|v\|_\infty.$$

On the one hand, we claim that $(u_n(X_t)1_{[0, T_{G_n^c})}(t))_{t \ge 0}$ is a $\mathbb{P}^x$-semimartingale q.e. $x \in G_n$. To see this, let us introduce for all $\alpha > 0$ and $n \ge 1$,

$$v_k^n := \alpha_k \widehat{U}_{\alpha_k}^{G_n}(f_0|_{G_n}) \in \mathcal{F}_{G_n},$$

and note that v_k^n is $\mathcal{E}_{\alpha_k}^{G_n}$-co-excessive (i.e., $\widehat{P}_s^{G_n, \alpha_k} v_k^n \le v_k^n$) and $v_k^n \xrightarrow[k]{\mathcal{E}_1^{G_n}} f_0|_{G_n} > 0$. Furthermore, by the way we chose the nest $(F_n)_{n \ge 1}$

$$d_n := \sup_k \|v_k^n\|_\infty \le \sup_k \|1_{F_n} \alpha_k \widehat{U}_{\alpha_k} f_0\|_\infty < \infty.$$

It follows that for all $r, t > 0$,

$$\int_{G_n} P_r^{G_n, \alpha_k+1} |P_t^{G_n, \alpha_k+1} u_n - u_n| v_k^n \, d\mu$$

$$= |\mathcal{E}_{\alpha_k+1}^{G_n}(u_n, \int_0^t \widehat{P}_s^{G_n, \alpha_k+1}(\mathrm{sgn}(P_t^{G_n, \alpha_k+1} u_n - u_n) \widehat{P}_r^{G_n, \alpha_k+1} v_k^n \, ds))|$$

$$\leq c_k^n \| \int_0^t \widehat{P}_s^{G_n, \alpha_k+1}(\mathrm{sgn}(P_t^{G_n, \alpha_k+1} u_n - u_n) \widehat{P}_r^{G_n, \alpha_k+1} v_k^n \, ds) \|_\infty$$

$$\leq c_k^n \int_0^t \|\widehat{P}_{s+r}^{G_n, \alpha_k+1} v_k^n\|_\infty \, ds \leq c_k^n \int_0^t e^{-(r+s)} \|v_k^n\|_\infty \, ds$$

$$\leq c_k^n d_n e^{-r} t.$$

Let now $\tau_l := \{\frac{i}{2^l} : 0 \leq i \leq l2^l\}$, $l \geq 1$. As in the proof of Proposition 4.4 i) $\Rightarrow$ ii), we get

$$\int_{G_n} \sup_l V_{\tau_l}^{(P_t^{G_n, \alpha_k+1})}(u_n) \, v_k^n \, d\mu < \infty,$$

hence, by, e.g., Proposition 4.2, the process $(e^{-(\alpha_k+1)t} u_n(X_t^{G_n}))_{t \geq 0}$, and more importantly, the process $u_n(X^{G_n})$, are $\mathbb{P}^x$-semimartingales for $v_k^n \cdot \mu$-a.e. $x \in G_n$ for all $k > 0$. This means that $(u_n(X_t) 1_{[0, T_{G_n^c})}(t))_{t \geq 0}$ is a $\mathbb{P}^x$-semimartingale q.e. $x \in G_n$.

On the other hand, as in the proof of Theorem 5.3 ii) $\Rightarrow$ i), the process $(R_1^{G_n^c} \widetilde{u}(X_t) 1_{[0, T_{G_n^c})})_{t \geq 0}$ is a semimartingale, hence

$$\widetilde{u}(X_t) 1_{[0, T_{G_n^c})}(t) = u_n(X_t) 1_{[0, T_{G_n^c})}(t) + R_1^{G_n^c} \widetilde{u}(X_t) 1_{[0, T_{G_n^c})}, \quad t \geq 0$$

is a semimartingale. By the result of Meyer already used in Theorem 5.3, it is sufficient to show that $\lim_n T_{G_n^c} \geq \xi$ a.s. But this property is true for $(F_n)_{n \geq 1}$ and it is inherited by $(G_n)_{n \geq 1}$ because for any $f \in E(\mathcal{U}^1)$, $R_1^{F_n^c} f = R_1^{G_n^c} f$. $\square$

The case when u is merely locally in $\mathcal{F}$ can be treated just like Corollary 5.4, so we state this observation as a corollary, but we omit the proof.

Corollary 5.6. *The statement of Corollary* 5.4 *remains valid if* $(\mathcal{E}, \mathcal{F})$ *is a semi-Dirichlet form.*

5.2. **Final remarks.** For practical reasons, it is useful to know whether it is sufficient to check inequalities (1.2) and (1.1) only for $v \in \mathcal{F}_0$, where $\mathcal{F}_0$ is a certain proper subspace of $\mathcal{F}$. We point out below some ideas of choosing $\mathcal{F}_0$.

a) Assume that $(\mathcal{E}, \mathcal{F})$ is a (nonsymmetric) Dirichlet form, and take $\mathcal{F}_0 := D(\widehat{\mathsf{L}}) \cap L^\infty(\mu)$. If inequality (1.2) is verified for all $v \in \mathcal{F}_0$, then $\widetilde{u}(X)$ is an α-quasimartingale for all $\alpha > 0$. This is true by Proposition 4.5.

b) Assume that $(\mathcal{E}, \mathcal{F})$ is a semi-Dirichlet form. We consider the following extension of condition $(\mathcal{L})$ from [Fu 99]: a subspace $\mathcal{F}_0 \subset \mathcal{F}_b$ satisfies *condition* $(\mathcal{S})$ if $\mathcal{F}_0$ is $\mathcal{E}_1^{1/2}$-dense in $\mathcal{F}$ and there exists a bounded continuous function $\phi : \mathbb{R} \mapsto \mathbb{R}$ such that

- $\phi(\mathcal{F}_0) \subset \mathcal{F}_0$;
- $\phi(t) = t$ if $t \in [-1, 1]$;
- if $(v_n)_{n \geq 1} \subset \mathcal{F}_0$ is $\mathcal{E}_1^{1/2}$-convergent, then $(\phi(v_n))_n$ is $\mathcal{E}_1^{1/2}$-bounded.

As a candidate for a space satisfying condition $(\mathcal{S})$, one should have in mind a *core* in the sense of [FuOsTa 11] (for regular Dirichlet forms), or the space of

cylindrical functions in the infinite-dimensional situation (see, e.g., [MaRö 92]), while ϕ could be a smooth unit contraction.

In the same spirit as [Fu 99, Lemma 6.1], we have the following result.

Lemma 5.7. *Let $u \in \mathcal{F}$ and $\mathcal{F}_0$ satisfy condition $(\mathcal{S})$. If inequality (1.2) holds for all $v \in \mathcal{F}_0$, then it holds for all $v \in \mathcal{F}_b$.*

Proof. Let $v \in \mathcal{F}$ and assume that $\|v\|_\infty \le 1$. If $(v_n)_{n \ge 1} \subset \mathcal{F}_0$ is $\mathcal{E}_1^{1/2}$-convergent to v, then from the boundedness condition on $(\phi(v_n))_n$ and by the Banach–Saks theorem, there exists a subsequence $(\phi(v_{n_k}))_{n_k}$ whose Cesaro means $\frac{i=1}{k} \sum_1^k \phi(v_{n_i})$ converges to $\phi(v) = v$ with respect to $\mathcal{E}_1^{1/2}$. Therefore $|\mathcal{E}(u, v)| = \lim_k |\mathcal{E}(u, \frac{1}{k} \sum_{i=1}^k \phi(v_{n_i}))| \le c\|\phi\|_\infty$. □

Regarding inequality (1.1), we have the following.

Lemma 5.8. *Let $u \in \mathcal{F}$ and $\mathcal{F}_0$ satisfy condition $(\mathcal{S})$ such that inequality (1.1) holds for $\mathcal{F}_{b,F_n}$ replaced by $\mathcal{F}_{b,F_n} \cap \mathcal{F}_0$. In addition, assume that $\mathcal{F}_0$ is an algebra and that for each $n \ge 1$, there exists $\psi_n \in \mathcal{F}_0 \cap \bigcup_k \mathcal{F}_{b,F_k}$ such that $\psi_n = 1$ on F_n. Then inequality (1.1) holds for all $n \ge 1$ and $v \in \mathcal{F}_{b,F_n}$, with possible different constants c_n.*

Proof. Fix $n \ge 1$, $v \in \mathcal{F}_{F_n}$ such that $\|v\|_\infty \le 1$, and let $k(n) \ge 1$ such that $\psi_n \in \mathcal{F}_{b,F_{k(n)}}$. Take $(v_m)_m \subset \mathcal{F}_0$ which is $\mathcal{E}_1^{1/2}$-convergent to v. Then

$$\mathcal{E}_1(\phi(v_m)\psi_n, \phi(v_m)\psi_n) \le \mathcal{E}_1(\phi(v_m), \phi(v_m))^{1/2} \|\psi_n\|_\infty^{1/2} + \mathcal{E}_1(\psi_n, \psi_n)^{1/2} \|\phi(v_m)\|_\infty^{1/2},$$

which means that $(\phi(v_m)\psi_n)_m$ is $\mathcal{E}_1^{1/2}$-bounded and employing once again the Banach–Saks theorem just like we did in the proof of the previous lemma, we get

$$|\mathcal{E}(u, v)| \le c_{k(n)} \|\phi\psi_n\|_\infty,$$

where the right-hand term does not depend on v (in fact it is the new constant replacing c_n). □

Candidates for $\mathcal{F}_0$ satisfying the assumption of Lemma 5.8 are the *special standard cores* in the sense of [FuOsTa 11]; see also [Fu 99, page 27].

References

[Ba 79]   R. F. Bass, *Adding and subtracting jumps from Markov processes*, Trans. Amer. Math. Soc. **255** (1979), 363–376, DOI 10.2307/1998181. MR542886

[BaHs 90]   R. F. Bass and P. Hsu, *The semimartingale structure of reflecting Brownian motion*, Proc. Amer. Math. Soc. **108** (1990), no. 4, 1007–1010, DOI 10.2307/2047960. MR1007487

[BeBo 04]   L. Beznea and N. Boboc, *Potential theory and right processes*, Mathematics and its Applications, vol. 572, Kluwer Academic Publishers, Dordrecht, 2004. MR2153655

[BeBoRö 06]   L. Beznea, N. Boboc, and M. Röckner, *Quasi-regular Dirichlet forms and L^p-resolvents on measurable spaces*, Potential Anal. **25** (2006), no. 3, 269–282, DOI 10.1007/s11118-006-9016-2. MR2255348

[BeCîRö 18]   Beznea, L., Cîmpean, I., and Röckner, M., Irreducible recurrence, ergodicity, and extremality of invariant measures for resolvents, Stoch. Proc. and their Appl., **128** (2018), 1405–1437

[BeLu 16]   L. Beznea and O. Lupaşcu, *Measure-valued discrete branching Markov processes*, Trans. Amer. Math. Soc. **368** (2016), no. 7, 5153–5176, DOI 10.1090/tran/6514. MR3456175

[BeRö 11]   L. Beznea and M. Röckner, *From resolvents to càdlàg processes through compact excessive functions and applications to singular SDE on Hilbert spaces*, Bull. Sci. Math. **135** (2011), no. 6-7, 844–870, DOI 10.1016/j.bulsci.2011.07.002. MR2838104

[BeSt 94]   L. Beznea and L. Stoica, *From diffusions to processes with jumps*, Probability theory and mathematical statistics (Vilnius, 1993), TEV, Vilnius, 1994, pp. 53–74. MR1649571

[BlGe 68]   R. M. Blumenthal and R. K. Getoor, *Markov processes and potential theory*, Pure and Applied Mathematics, Vol. 29, Academic Press, New York-London, 1968. MR0264757

[CFW93]   Z. Q. Chen, P. J. Fitzsimmons, and R. J. Williams, *Reflecting Brownian motions: quasimartingales and strong Caccioppoli sets*, Potential Anal. **2** (1993), no. 3, 219–243, DOI 10.1007/BF01048506. MR1245240

[ÇiJaPrSh 80] E. Çinlar, J. Jacod, P. Protter, and M. J. Sharpe, *Semimartingales and Markov processes*, Z. Wahrsch. Verw. Gebiete **54** (1980), no. 2, 161–219, DOI 10.1007/BF00531446. MR597337

[EnNa 99]   K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. MR1721989

[Fu 99]   M. Fukushima, *On semi-martingale characterizations of functionals of symmetric Markov processes*, Electron. J. Probab. **4** (1999), no. 18, 32, DOI 10.1214/EJP.v4-55. MR1741537

[Fu 00]   M. Fukushima, *BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space*, J. Funct. Anal. **174** (2000), no. 1, 227–249, DOI 10.1006/jfan.2000.3576. MR1761369

[FuOsTa 11] M. Fukushima, Y. Oshima, and M. Takeda, *Dirichlet forms and symmetric Markov processes*, Second revised and extended edition, De Gruyter Studies in Mathematics, vol. 19, Walter de Gruyter & Co., Berlin, 2011. MR2778606

[LG 06]   Le Gall, J. F., Intégration, probabilités et processus aléatoires, *Ecole Normale Supérieure de Paris*, September 2006.

[Ge 80]   R. K. Getoor, *Transience and recurrence of Markov processes*, Seminar on Probability, XIV (Paris, 1978/1979), Lecture Notes in Math., vol. 784, Springer, Berlin, 1980, pp. 397–409. MR580144

[Ha 10]   Hairer, M., Convergence of Markov Processes, Lecture Notes, University of Warwick, http://www.hairer.org/notes/Convergence.pdf, 2010.

[HmHm 09]   F. Hmissi and M. Hmissi, *On subordination of resolvents and application to right processes*, Stochastics **81** (2009), no. 3-4, 345–353, DOI 10.1080/17442500903080298. MR2549492

[HmJa 14]   M. Hmissi and K. Janssen, *On $\mathcal{S}$-subordination and applications to entrance laws*, Rev. Roumaine Math. Pures Appl. **59** (2014), no. 1, 105–121. MR3296839

[Lu 14]   O. Lupaşcu, *Subordination in the sense of Bochner of L^p-semigroups and associated Markov processes*, Acta Math. Sin. (Engl. Ser.) **30** (2014), no. 2, 187–196, DOI 10.1007/s10114-014-2751-1. MR3150232

[MaOvRö 95] Z. M. Ma, L. Overbeck, and M. Röckner, *Markov processes associated with semi-Dirichlet forms*, Osaka J. Math. **32** (1995), no. 1, 97–119. MR1323103

[MaRö 92]   Z. M. Ma and M. Röckner, *Introduction to the theory of (nonsymmetric) Dirichlet forms*, Universitext, Springer-Verlag, Berlin, 1992. MR1214375

[Me 66]   P.-A. Meyer, *Probability and potentials*, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto, Ont.-London, 1966. MR0205288

[Op 16]   A.-G. Oprina, *Perturbation with kernels of Markovian resolvents*, Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci. **17** (2016), no. 4, 300–306. MR3582628

[Os 13]   Y. Oshima, *Semi-Dirichlet forms and Markov processes*, De Gruyter Studies in Mathematics, vol. 48, Walter de Gruyter & Co., Berlin, 2013. MR3060116

[Pr 05]   P. E. Protter, *Stochastic integration and differential equations*, Stochastic Modelling and Applied Probability, vol. 21, Springer-Verlag, Berlin, 2005. Second edition. Version 2.1; Corrected third printing. MR2273672

[Sa 99]   K.-i. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original; Revised by the author. MR1739520

[Sha 88]   M. Sharpe, *General theory of Markov processes*, Pure and Applied Mathematics, vol. 133, Academic Press, Inc., Boston, MA, 1988. MR958914

[Si 99]   T. Simon, *Subordination in the wide sense for Lévy processes*, Probab. Theory Related Fields **115** (1999), no. 4, 445–477, DOI 10.1007/s004400050245. MR1728917

[SoVo 03]   R. Song and Z. Vondraček, *Potential theory of subordinate killed Brownian motion in a domain*, Probab. Theory Related Fields **125** (2003), no. 4, 578–592, DOI 10.1007/s00440-002-0251-1. MR1974415

SIMION STOILOW INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, RESEARCH UNIT No. 2, P.O. Box 1-764, RO-014700 BUCHAREST, ROMANIA—AND—UNIVERSITY OF BUCHAREST, FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, AND CENTRE FRANCOPHONE EN MATHÉMATIQUE DE BUCAREST, ROMANIA.

Email address: `lucian.beznea@imar.ro`

SIMION STOILOW INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, RESEARCH UNIT No. 2, P.O. Box 1-764, RO-014700 BUCHAREST, ROMANIA

Email address: `iulian.cimpean@imar.ro`