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VARIETIES WITH $\mathbb{P}$-UNITS

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ABSTRACT. We study the class of compact Kähler manifolds with trivial canonical bundle and the property that the cohomology of the trivial line bundle is generated by one element. If the square of the generator is zero, we get the class of strict Calabi–Yau manifolds. If the generator is of degree 2, we get the class of compact hyperkähler manifolds. We provide some examples and structure results for the cases where the generator is of higher nilpotency index and degree. In particular, we show that varieties of this type are closely related to higher-dimensional Enriques varieties.

1. INTRODUCTION

In this paper we will study a certain class of compact Kähler manifolds with trivial canonical bundle which contains all strict Calabi–Yau varieties as well as all hyperkähler manifolds. For the bigger class of manifolds with trivial first Chern class $c_1(X) = 0 \in \mathrm{H}^2(X, \mathbb{R})$ there exists the following nice structure theorem, known as the *Beauville–Bogomolov decomposition*; see [Bea83]. Namely, each such manifold X admits an étale covering $X' \to X$ which decomposes as

$$X' = T \times \prod_i Y_i \times \prod_j Z_j,$$

where T is a complex torus, the Y_i are hyperkähler, and the Z_j are simply connected strict Calabi–Yau varieties of dimension at least 3.

Given a variety X, the graded algebra $\mathrm{H}^*(\mathcal{O}_X) := \bigoplus_{i=0}^{\dim X} \mathrm{H}^i(X, \mathcal{O}_X)[-i]$ is considered an important invariant; see, in particular, Abuaf [Abu15] who calls $\mathrm{H}^*(\mathcal{O}_X)$ the *homological unit* of X and conjectures that it is stable under derived equivalences. In this paper, we want to study varieties which have trivial canonical bundle and the property that the algebra $\mathrm{H}^*(\mathcal{O}_X)$ is generated by one element.

The main motivations are the following two observations. Let X be a compact Kähler manifold.

Observation 1.1. X is a strict Calabi–Yau manifold if and only if the canonical bundle ω_X is trivial and $\mathrm{H}^*(\mathcal{O}_X) \cong \mathbb{C}[x]/x^2$ with $\deg x = \dim X$. These conditions can be summarized in terms of objects of the bounded derived category $\mathrm{D}(X) := \mathrm{D}^b(\mathrm{Coh}(X))$ of coherent sheaves. Namely, X is a strict Calabi–Yau manifold if and only if $\mathcal{O}_X \in \mathrm{D}(X)$ is a *spherical object* in the sense of Seidel and Thomas [ST01].

The above is a very simple reformulation of the standard definition of a strict Calabi–Yau manifold as a compact Kähler manifold with trivial canonical bundle

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such that $\mathrm{H}^i(\mathcal{O}_X) = 0$ for $i \neq 0, \mathsf{dim}\, X$. The second observation is probably less well known.

Observation 1.2. X is a hyperkähler manifold of dimension $\mathsf{dim}\, X = 2n$ if and only if ω_X is trivial and $\mathrm{H}^*(\mathcal{O}_X) \cong \mathbb{C}[x]/x^{n+1}$ with $\mathsf{deg}\, x = 2$. This is equivalent to the condition that $\mathcal{O}_X \in \mathrm{D}(X)$ is a $\mathbb{P}^n$-*object* in the sense of Huybrechts and Thomas [HT06].

Indeed, the structure sheaf of a hyperkähler manifold is one of the well-known examples of a $\mathbb{P}^n$-object; see [HT06, Ex. 1.3(ii)]. The fact that $\mathrm{H}^*(\mathcal{O}_X)$ also characterizes the compact hyperkähler manifolds follows from [HNW11, Prop. A.1].

Inspired by this, we study the class of compact Kähler manifolds X with the property that $\mathcal{O}_X \in \mathrm{D}(X)$ is what we call a $\mathbb{P}^n[k]$-*object*; see Definition 2.4. Concretely, this means:

(C1) the canonical bundle ω_X is trivial;
(C2) there is an isomorphism of $\mathbb{C}$-algebras $\mathrm{H}^*(\mathcal{O}_X) \cong \mathbb{C}[x]/x^{n+1}$ with $\mathsf{deg}\, x = k$.

By Serre duality, such a manifold is of dimension $\mathsf{dim}\, X = \mathsf{deg}(x^n) = n \cdot k$. For $n = 1$, we get exactly the strict Calabi–Yau manifolds, while for $k = 2$, we get the hyperkähler manifolds.

In this paper we will study the case of higher n and k. We construct examples and prove some structure results. If $\mathcal{O}_X$ is a $\mathbb{P}^n[k]$-object with $k > 2$, the manifold X is automatically projective; see Lemma 3.10. Hence, we will call X a *variety with* $\mathbb{P}^n[k]$-*unit.* The main results of this paper can be summarized as follows.

Theorem 1.3. *Let $n+1 = p^\nu$ be a prime power. Then the following are equivalent:*

(i) *there exists a variety with $\mathbb{P}^n[4]$-unit;*
(ii) *there exists a variety with $\mathbb{P}^n[k]$-unit for every even k;*
(iii) *there exists a strict Enriques variety of index $n+1$.*

For $n+1$ arbitrary, the implications (iii)$\Longrightarrow$(ii)$\Longrightarrow$(i) *are still true.*

We do not know whether or not (i)$\Longrightarrow$(iii) is true in general if $n+1$ is not a prime power, but we will prove a slightly weaker statement that holds for arbitrary $n+1$; see Section 5.2. In particular, the universal cover of a variety with $\mathbb{P}^n[4]$-unit, with $n+1$ arbitrary, splits into a product of two hyperkähler varieties; see Proposition 5.3.

Our notion of strict Enriques varieties is inspired by similar notions of higher-dimensional analogues of Enriques surfaces due to Boissière, Nieper-Wißkirchen, and Sarti [BNWS11] and Oguiso and Schröer [OS11]. There are known examples of strict Enriques varieties of index 3 and 4. Hence, we get

Corollary 1.4. *For $n = 2$ and $n = 3$ there are examples of varieties with $\mathbb{P}^n[k]$-units for every even $k \in \mathbb{N}$.*

The motivation for this work comes from questions concerning derived categories and the notions are influenced by this. However, in this paper, with the exception sections 6.5 and 6.6, all results and proofs are also formulated without using the language of derived categories.

The paper is organized as follows. In section 2.1, we fix some notations and conventions. Sections 2.2 and 2.3 are a very brief introduction into derived categories and some types of objects that occur in these categories. In particular, we introduce the notion of $\mathbb{P}^n[k]$-objects.

In section 3.1 we say a few words about compact hyperkähler manifolds. In section 3.2 we discuss automorphisms of Beauville–Bogomolov products and their action on cohomology. This is used in the following section 3.3 in order to give a proof of Observation 1.2. This proof is probably a bit easier than the one in [HNW11, App. A]. More importantly, it allows us to introduce some of the notations and ideas which are used in the later sections. In section 3.4, we discuss a class of varieties which we call strict Enriques varieties. There are two different notions of Enriques varieties in the literature (see [BNWS11] and [OS11]), and our notion is the intersection of these two; see Proposition 3.14(iv). In section 3.5 we quickly mention a generalization; namely strict Enriques stacks.

We give the definition of a variety with a $\mathbb{P}^n[k]$-unit together with some basic remarks in section 4.1. Section 4.2 provides two examples of varieties which look like promising candidates, but ultimately fail to have $\mathbb{P}^n[k]$-units. In section 4.3 we construct series of varieties with $\mathbb{P}^n[k]$-units out of strict Enriques varieties of index $n+1$. In particular, we prove the implication (iii)$\Longrightarrow$(ii) of Theorem 1.3.

In section 5.1 we make some basic observations concerning the fundamental group and the universal cover of varieties with $\mathbb{P}^n[k]$-units. In section 5.2 we specialize to the case $k=4$. We proof that the universal cover of a variety with $\mathbb{P}^n[4]$-unit is the product of two hyperkähler manifolds of dimension $2n$. Then we proceed to proof the implication (i)$\Longrightarrow$(iii) of Theorem 1.3 for $n+1$ a prime power.

Section 6 is a collection of some further observations and ideas. In sections 6.1, 6.2, and 6.3 some further constructions leading to varieties with $\mathbb{P}^n[k]$-units are discussed. We talk briefly about stacks with $\mathbb{P}^n[k]$-units in section 6.4. In section 6.5 we prove that the class of strict Enriques varieties is stable under derived equivalences, and in section 6.6 we study some derived auto-equivalences of varieties with $\mathbb{P}^n[k]$-units. In the final section 6.7, we contemplate a bit about varieties with $\mathbb{P}^n[k]$-units as moduli spaces and constructions of hyperkähler varieties.

2. Notations and preliminaries

2.1. Notations and conventions.

(i) Throughout, X will be a connected compact Kähler manifold (often a smooth projective variety).

(ii) We denote the universal cover by $\widehat{X} \to X$.

(iii) If ω_X is of finite order m, we denote the canonical cover by $\pi\colon \widetilde{X} \to X$. It is defined by the properties that $\omega_{\widetilde{X}}$ is trivial and π is an étale Galois cover of degree m. We have $\pi_*\mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X \oplus \omega_X^{-1} \oplus \omega_X^{-2} \oplus \cdots \oplus \omega_X^{-(m-1)}$, and the covering map $\widetilde{X} \to X$ is the quotient by a cyclic group $G = \langle g \rangle$ with $g \in \mathsf{Aut}(\widetilde{X})$ of order m.

(iv) We will usually write graded vector spaces in the form $V^* = \bigoplus_{i\in\mathbb{Z}} V^i[-i]$. The *Euler characteristic* is given by the alternating sum $\chi(V^*) = \sum_{i\in\mathbb{Z}}(-1)^i \dim V^i$.

(v) Given a sheaf or a complex of sheaves E and an integer $i \in \mathbb{Z}$, we write $\mathrm{H}^i(X,E)$ for the ith derived functor of global sections. In contrast $\mathcal{H}^i(E)$ denotes the cohomology of the complex in the sense kernel modulo image of the differentials.

(vi) We write for short $Y \in \mathsf{HK}_{2d}$ to express the fact that Y is a compact hyperkähler manifold of dimension $2d$. In this case, we denote by y a generator of $\mathrm{H}^2(\mathcal{O}_Y)$, i.e., y is the complex conjugate of a symplectic form on Y. If we just write $Y \in \mathsf{HK}$, this means that Y is a hyperkähler manifold of unspecified dimension. Sometimes, we write $Y \in \mathsf{K3}$ instead of $Y \in \mathsf{HK}_2$.

(vii) We write for short $Z \in \mathsf{CY}_e$ to express the fact that Z is a compact simply connected strict Calabi–Yau variety of dimension $e \geq 3$. In this case, we denote by z a generator of $\mathrm{H}^e(\mathcal{O}_Z)$, i.e., z is the complex conjugate of a volume form on Z. If we just write $Z \in \mathsf{CY}$, this means that Z is a simply connected strict Calabi–Yau variety of unspecified dimension.

(viii) We denote the connected zero-dimensional manifold by pt.

(ix) For $n \in \mathbb{N}$, we denote the symmetric group of permutations of the set $\{1, \dots, n\}$ by $\mathfrak{S}_n$. Given a space X and a permutation $\sigma \in \mathfrak{S}_n$, we denote the automorphism of the cartesian product X^n which is given by the according permutation of components again by $\sigma \in \mathsf{Aut}(X^n)$.

(x) The symbol $\sum_{i_1 \neq i_2 \neq \cdots \neq i_\ell}$ means summation over sets $\{i_1, \dots, i_\ell\}$ of cardinality ℓ (contained in some fixed index set which is, hopefully, clear from the context).

2.2. **Derived categories of coherent sheaves.** As mentioned in the introduction, knowledge of derived categories is not necessary for the understanding of this paper. However, often things can be stated in the language of derived categories in the most convenient way, and questions concerning derived categories motivated this work. Hence, we will give, in a very brief form, some basic definitions and facts.

The derived category $\mathrm{D}(X) := \mathrm{D}^b(\mathrm{Coh}(X))$ is defined as the localization of the homotopy category of bounded complexes of coherent sheaves by the class of quasi-isomorphisms. Hence, the objects of $\mathrm{D}(X)$ are bounded complexes of coherent sheaves. The morphisms are morphisms of complexes together with formal inverses of quasi-isomorphisms. In particular, every quasi-isomorphism between complexes becomes an isomorphism in $\mathrm{D}(X)$. The derived category $\mathrm{D}(X)$ is a triangulated category. In particular, there is the shift auto-equivalence $[1]\colon \mathrm{D}(X) \to \mathrm{D}(X)$. Given two objects $E, F \in \mathrm{D}(X)$, there is a graded Hom-space $\mathrm{Hom}^*(E,F) = \bigoplus_i \mathrm{Hom}_{\mathrm{D}(X)}(E, F[i])[-i]$. For $E = F$, this is a graded algebra by the Yoneda product (composition of morphisms). There is a fully faithful embedding $\mathrm{Coh}(X) \hookrightarrow \mathrm{D}(X)$, $A \mapsto A[0]$ which is given by considering sheaves as complexes concentrated in degree zero. Most of the time, we will denote $A[0]$ simply by A again. For $A, B \in \mathrm{Coh}(X)$, we have $\mathrm{Hom}^*(A,B) \cong \mathrm{Ext}^*(A,B)$. Besides the shift functor, the data of a triangulated category consists of a class of distinguished triangles $E \to F \to G \to E[1]$ consisting of objects and morphisms in $\mathrm{D}(X)$ satisfying certain axioms. In particular, every morphism $f\colon E \to F$ in $\mathrm{D}(X)$ can be completed to a distinguished triangle

$$E \xrightarrow{f} F \to G \to E[1]\,.$$

The object G is determined by f up to isomorphism and denoted by $G = \mathrm{cone}(f)$. There is a long exact cohomology sequence

$$\cdots \to \mathcal{H}^{i-1}(\mathrm{cone}(f)) \to \mathcal{H}^i(E) \to \mathcal{H}^i(F) \to \mathcal{H}^i(\mathrm{cone}(f)) \to \mathcal{H}^{i+1}(E) \to \cdots\,.$$

2.3. **Special objects of the derived category.** In the following we will recall the notions of exceptional, spherical, and $\mathbb{P}$-objects in the derived category $\mathrm{D}(X)$

of coherent sheaves on a compact Kähler manifold X. Exceptional objects can be used in order to decompose derived categories, while spherical and ℙ-objects induce auto-equivalences; see also section 6.6. Our main focus in this paper, however, will be to characterize varieties where $\mathcal{O}_X \in \mathrm{D}(X)$ is an object of one of these types.

Definition 2.1. An object $E \in \mathrm{D}(X)$ is called *exceptional* if $\mathrm{Hom}^*(E,E) \cong \mathbb{C}[0]$.

Let X be a Fano variety, i.e., the anticanonical bundle ω_X^{-1} is ample. Then, by Kodaira vanishing, every line bundle on X is exceptional when considered as an object of the derived category $\mathrm{D}(X)$; see also Remark 2.8. Similarly, every line bundle on an Enriques surface is exceptional. Another typical example of an exceptional object is the structure sheaf $\mathcal{O}_C \in \mathrm{D}(S)$ of a (-1)-curve $\mathbb{P}^1 \cong C \subset S$ on a surface.

Definition 2.2 ([ST01])**.** An object $E \in \mathrm{D}(X)$ is called *spherical* if

(i) $E \otimes \omega_X \cong E$,
(ii) $\mathrm{Hom}^*(E,E) \cong \mathbb{C}[0] \oplus \mathbb{C}[\mathsf{dim}\, X] \cong \mathrm{H}^*(\mathbb{S}^{\mathsf{dim}\, X}, \mathbb{C})$.

Every line bundle on a strict Calabi–Yau variety is spherical. Another typical example of a spherical object is the structure sheaf $\mathcal{O}_C \in \mathrm{D}(S)$ of a (-2)-curve $\mathbb{P}^1 \cong C \subset S$ on a surface.

Definition 2.3 ([HT06])**.** Let $n \in \mathbb{N}$. An object $E \in \mathrm{D}(X)$ is called $\mathbb{P}^n$*-object* if

(i) $E \otimes \omega_X \cong E$,
(ii) there is an isomorphism of $\mathbb{C}$-algebras $\mathrm{Hom}^*(E,E) \cong \mathbb{C}[x]/x^{n+1}$ with $\mathsf{deg}\, x = 2$.

Condition (ii) can be rephrased as $\mathrm{Hom}^*(E,E) \cong \mathrm{H}^*(\mathbb{P}^n, \mathbb{C})$. As we will see in the next subsection, every line bundle on a compact hyperkähler manifold is a ℙ-object. Another typical example is the structure sheaf of the center of a Mukai flop.

Definition 2.4. Let $n, k \in \mathbb{N}$. An object $E \in \mathrm{D}(X)$ is called $\mathbb{P}^n[k]$*-object* if

(i) $E \otimes \omega_X \cong E$,
(ii) there is an isomorphism of $\mathbb{C}$-algebras $\mathrm{Hom}^*(E,E) \cong \mathbb{C}[x]/x^{n+1}$ with $\mathsf{deg}\, x = k$.

Remark 2.5. If there is a $\mathbb{P}^n[k]$-object $E \in \mathrm{D}(X)$, we have $\mathsf{dim}\, X = n \cdot k$ by Serre duality.

Remark 2.6. For $n = 1$, the $\mathbb{P}^1[k]$-objects coincide with the spherical objects. For $k = 2$, the $\mathbb{P}^n[2]$-objects are exactly the $\mathbb{P}^n$-objects in the sense of Huybrechts and Thomas.

The names "spherical" and "ℙ-objects" come from the fact that their graded endomorphism algebra coincides with the cohomology of spheres and projective spaces, respectively. Hence, it would be natural to name a $\mathbb{P}^n[k]$-object by a series of manifolds whose cohomology is of the form $\mathbb{C}[x]/x^{n+1}$ with $\mathsf{deg}\, x = k$. For $k = 4$, there are the quaternionic projective spaces. For $k > 4$, however, there are probably no such series. At least, there are no manifolds M satisfying the possibly stronger condition that $\mathrm{H}^*(M, \mathbb{Z}) \cong \mathbb{Z}[x]/x^{n+1}$ for $\mathsf{deg}\, x > 4$ and $n > 2$; see [Hat02, Cor. 4L.10]. Hence, we will stick to the notion of $\mathbb{P}^n[k]$-objects, which is justified by the following remark.

Remark 2.7. A $\mathbb{P}^n[k]$-object is essentially the same as a $\mathbb{P}$-functor (see [Add16]) $\mathrm{D}(\mathsf{pt}) \to \mathrm{D}(X)$ with $\mathbb{P}$-cotwist $[-k]$. In particular, as we will further discuss in section 6.6, it induces an auto-equivalence of $\mathrm{D}(X)$.

Remark 2.8. Given a compact Kähler manifold X, the following are equivalent:

(i) $\mathcal{O}_X$ is a $\mathbb{P}^n[k]$-object.
(ii) Every line bundle on X is a $\mathbb{P}^n[k]$-object.
(iii) Some line bundle on X is a $\mathbb{P}^n[k]$-object.

The same holds if we replace the property of being a $\mathbb{P}^n[k]$-object by the property of being an exceptional object. Indeed, for every line bundle L on X, we have isomorphisms of $\mathbb{C}$-algebras

$$\mathrm{Hom}^*(L,L) \cong \mathrm{Hom}^*(\mathcal{O}_X,\mathcal{O}_X) \cong \mathrm{H}^*(\mathcal{O}_X),$$

where the latter is an algebra by the cup product. Furthermore, $L \otimes \omega_X \cong L$ holds if and only if ω_X is trivial.

3. Hyperkähler and Enriques varieties

In this section we first review some results on hyperkähler manifolds and their automorphisms. In particular, we give a proof of Observation 1.2, i.e., the fact that hyperkähler manifolds can be characterized by the property that the trivial line bundle is a $\mathbb{P}$-object. Then we introduce and study strict Enriques varieties. They are a generalization of Enriques surfaces to higher dimensions and can be realized as quotients of hyperkähler varieties.

3.1. **Hyperkähler manifolds.** Let X be a compact Kähler manifold of dimension $2n$. We say that X is *hyperkähler* if and only if its Riemannian holonomy group is the symplectic group $\mathsf{Sp}(n)$. A compact Kähler manifold X is hyperkähler if and only if it is *irreducible holomorphic symplectic*, which means that it is simply connected and $\mathrm{H}^2(X,\Omega_X^2)$ is spanned by an everywhere nondegenerate 2-form, called *symplectic form*; see, e.g., [Bea83] or [Huy03].

The structure sheaf of a hyperkähler manifold is a $\mathbb{P}^n$-object; see [HT06, Ex. 1.3(ii)]. This means that the canonical bundle $\omega_X = \Omega_X^{2n}$ is trivial and $\mathrm{H}^*(\mathcal{O}_X) = \mathbb{C}[x]/x^{n+1}$; compare item (vi) of section 2.1. This follows essentially from the holonomy principle together with Bochner's principle. We will see in section 3.3 that also the converse holds, which amounts to Observation 1.2.

3.2. **Automorphisms and their action on cohomology.** In the later sections we will often deal with automorphisms of Beauville–Bogomolov covers. There is the following result of Beauville [Bea83, Sect. 3].

Lemma 3.1. *Let* $X' = \prod_i Y_i^{\lambda_i} \times \prod_j Z_j^{\nu_j}$ *be a finite product with* $Y_i \in \mathsf{HK}_{2d_i}$ *and* $Z_j \in \mathsf{CY}_{e_j}$ *such that the* Y_i *and* Z_j *are pairwise nonisomorphic. Then every automorphism of* X' *preserves the decomposition up to permutation of factors. More concretely, every automorphism* $f \in \mathsf{Aut}(X')$ *is of the form* $f = \prod f_{Y_i^{\lambda_i}} \times \prod f_{Z_j^{\nu_j}}$ *with* $f_{Y_i^{\lambda_i}} \in \mathsf{Aut}(Y_i^{\lambda_i})$ *and* $f_{Z_j^{\nu_j}} \in \mathsf{Aut}(Z_j^{\nu_j})$*. Furthermore,* $f_{Y_i^{\lambda_i}} = (f_{Y_{i1}} \times \cdots \times f_{Y_{i\lambda_i}}) \circ \sigma_{Y_i,f}$ *with* $f_{Y_{i\alpha}} \in \mathsf{Aut}(Y_i)$ *and* $\sigma_{Y_i,f} \in \mathfrak{S}_{\lambda_i}$*. Similarly,* $f_{Z_j^{\nu_i}} = (f_{Z_{j1}} \times \cdots \times f_{Z_{i\nu_i}}) \circ \sigma_{Z_j,f}$ *with* $f_{Z_{j\beta}} \in \mathsf{Aut}(Z_i)$ *and* $\sigma_{Z_j,f} \in \mathfrak{S}_{\nu_i}$.

Let $X' = \prod_i Y_i^{\mu_i} \times \prod_j Z_j^{\nu_j}$ as above. For $\alpha = 1, \dots, \mu_i$ we denote by $y_{i\alpha} \in \mathrm{H}^2(\mathcal{O}_{X'})$ the image of $y_i \in \mathrm{H}^2(\mathcal{O}_{Y_i})$ under pullback along the projection $X' \to Y_i$ to the αth Y_i factor; compare section 2.1(vi). For $\beta = 1, \dots, \nu_j$, the class $z_{j\beta}$ is defined analogously. By the Künneth formula, the $y_{i\alpha}$ and $z_{j\beta}$ together generate the cohomology $\mathrm{H}^*(\mathcal{O}_{X'})$, and we have

$$\mathrm{H}^*(\mathcal{O}_{X'}) = \mathbb{C}[\{y_{i\alpha}\}_{i\alpha}, \{z_{j\beta}\}_{j\beta}]/(y_{i\alpha}^{d_i}, z_{j\beta}^2)\,. \tag{1}$$

Let $Y \in \mathsf{HK}$. The action of automorphisms on $\mathrm{H}^2(\mathcal{O}_Y) \cong \mathbb{C}$ defines a group character which we denote by

$$\rho_Y\colon \mathsf{Aut}(Y) \to \mathbb{C}^*, \quad f \mapsto \rho_{Y,f}\,.$$

In particular, an automorphism $f \in \mathsf{Aut}(Y)$ of finite order $\operatorname{\mathsf{ord}} f = m$ acts on $\mathrm{H}^2(\mathcal{O}_X)$ by multiplication by an mth root of unity $\rho_{Y,f} \in \mu_m$. Similarly, for $Z \in \mathsf{CY}_k$, we have a character $\rho_Z\colon \mathsf{Aut}(Z) \to \mathbb{C}^*$ given by the action of automorphisms on $\mathrm{H}^k(\mathcal{O}_Z)$.

Corollary 3.2. *Let $f \in \mathsf{Aut}(X')$ be of finite order d. Then the induced action of f on the cohomology of the structure sheaf is given by permutations of the $y_{i\alpha}$ with fixed i and the $z_{j\beta}$ with fixed j together with multiplications by dth roots of unity. This means*

$$f\colon \quad y_{i\alpha} \mapsto \rho_{Y_{i\alpha}, f_{Y_{i\alpha}}} \cdot y_{i\sigma_{Y_i,f}(\alpha)}, \quad z_{j\beta} \mapsto \rho_{Z_{j\beta}, f_{Z_{j\beta}}} \cdot z_{j\sigma_{Z_j,f}(\beta)}$$

with $\rho_{Y_{i\alpha},f}, \rho_{Z_{j\beta},f} \in \mu_d$.

The main takeaway for the computations in the latter sections is that the cohomology classes can only be permuted if the corresponding factors of the product coincide.

Definition 3.3. Let $Y \in \mathsf{HK}$ and $f \in \mathsf{Aut}\, Y$ be of finite order. We call the order of $\rho_{Y,f} \in \mathbb{C}$ the *symplectic order* of f. The reason for the name is that f acts by a root of unity of the same order, namely $\bar{\rho}_{Y,f}$, on $\mathrm{H}^0(\Omega_X^2)$, i.e., on the symplectic forms. In general, the symplectic order divides the order of f in $\mathsf{Aut}(X)$. We say that f is *purely nonsymplectic* if its symplectic order is equal to $\operatorname{\mathsf{ord}} f$.

Lemma 3.4. *Let $Y \in \mathsf{HK}_{2n}$, and let $f \in \mathsf{Aut}(Y)$ be an automorphism of finite order m such that the generated group $\langle f \rangle$ acts freely on Y. Then f is purely nonsymplectic and $m \mid n+1$. Similarly, every fixed-point-free automorphism of finite order of a strict Calabi–Yau variety is a nonsymplectic involution.*

Proof. This follows from the holomorphic Lefschetz fixed point theorem; compare [BNWS11, Sect. 2.2]. □

Corollary 3.5. *Let $Y \in \mathsf{HK}_{2n}$, and let $X = Y/\langle f \rangle$ be the quotient by a cyclic group of automorphisms acting freely. Then ω_X is nontrivial and of finite order.*

Proof. The order of ω_X is exactly the order of the action of f on $\mathrm{H}^{2n}(\mathcal{O}_X)$, i.e., the order of $\rho^n_{Y,f} \in \mathbb{C}^*$. By the previous lemma this order is finite and greater than one. □

Here is a simple criterion for automorphisms of products to be fixed point free.

Lemma 3.6.

(i) *Let $X_1, \dots, X_k$ be manifolds, and let $f_i \in \mathsf{Aut}(X_i)$. Then*

$$f_1 \times \cdots \times f_k \in \mathsf{Aut}(X_1 \times \cdots \times X_k)$$

is fixed point free if and only if at least one of the f_i is fixed point free.

(ii) *Let X be a manifold, and let $g_1, \dots, g_k \in \mathsf{Aut}(X)$. Consider the automorphism*

$$\varphi = (g_1 \times \cdots \times g_k) \circ (1\ 2\ \dots\ k) \in \mathsf{Aut}(X^k)$$

given by $(p_1, p_2, \dots, p_k) \mapsto (g_1(p_k), g_2(p_1), \dots, g_k(p_{k-1}))$. Then φ is fixed point free if and only if the composition $g_k \circ g_{k-1} \circ \cdots \circ g_1$ (or, equivalently, $g_i \circ g_{i-1} \circ \cdots \circ g_{i+1}$ for some $i = 1, \dots, k$) is fixed point free.

We also will frequently use the following well-known fact.

Lemma 3.7. *Let X' be a smooth projective variety, and let $G \subset \mathsf{Aut}(X')$ be a finite subgroup which acts freely. Then, the quotient variety $X := X'/G$ is again smooth projective and*

$$\chi(\mathcal{O}_{X'}) = \chi(\mathcal{O}_X) \cdot \mathsf{ord}\, G\,.$$

Furthermore, $\mathrm{H}^*(\mathcal{O}_X) = \mathrm{H}^*(\mathcal{O}_{X'})^G$.

3.3. **Proof of Observation 1.2.** We already remarked in section 3.1 that the structure sheaf of a hyperkähler manifold is a $\mathbb{P}$-object. Hence, for the verification of Observation 1.2 we only need to prove the following

Proposition 3.8. *Let X be a compact Kähler manifold such that $\mathcal{O}_X \in \mathrm{D}(X)$ is a $\mathbb{P}^n[2]$-object. Then X is hyperkähler of dimension $2n$.*

Proof. As already mentioned in the introduction, this follows immediately from [HNW11, Prop. A.1]. We will give a slightly different proof.

Recall that the assumption that $\mathcal{O}_X$ is a $\mathbb{P}^n$-object means

(i) ω_X is trivial,
(ii) $\mathrm{H}^*(\mathcal{O}_X) \cong \mathbb{C}[x]/x^{n+1}$ with $\mathsf{deg}\, x = 2$.

For $n = 1$, it follows easily by the Kodaira classification of surfaces that $X \in \mathsf{K3} = \mathsf{HK}_2$. Hence, we may assume that $n \geq 2$.

Assumption (i) says that, in particular, $c_1(X) = 0$. Hence, we have a finite étale covering $X' \to X$ and a Beauville–Bogomolov decomposition

$$X' = T \times \prod_i Y_i \times \prod_j Z_j\,. \tag{2}$$

The plan is to show that X' is hyperkähler and the covering map is an isomorphism.

Convention 3.9. Whenever we have a Beauville–Bogomolov decomposition of the form (2), T is a complex torus, $Y_i \in \mathsf{HK}_{2d_i}$ is hyperkähler of dimension $2d_i$, and $Z_j \in \mathsf{CY}_{e_j}$ is a strict simply connected Calabi–Yau variety of dimension $e_j \geq 3$. Furthermore, $\mathrm{H}^2(\mathcal{O}_{Y_i}) = \langle y_i \rangle$ and $\mathrm{H}^{e_j}(\mathcal{O}_{Z_j}) = \langle z_j \rangle$.

By assumption (ii), we have $\chi(\mathcal{O}_X) = n+1$. On the other hand, since $X' \to X$ is étale, say of degree m, we have

$$m(n+1) = m \cdot \chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X'}) = \chi(T) \cdot \prod_i \chi(\mathcal{O}_{Y_i}) \cdot \prod_j \chi(\mathcal{O}_{Z_j})\,. \tag{3}$$

This implies that $T = \mathsf{pt}$ and all e_j are even. Otherwise, the right-hand side of (3) would be zero. Since the torus part is trivial, X' is simply connected. Hence, $X' = \widehat{X}$ is the universal cover of $X = \widehat{X}/G$ where $\pi_1(X) \cong G \subset \mathsf{Aut}(X)$. It follows by Lemma 3.7 that

$$\mathbb{C}[x]/x^{n+1} \cong \mathrm{H}^*(\mathcal{O}_X) \cong \mathrm{H}^*(\mathcal{O}_{\widehat{X}})^G \subset \mathrm{H}^*(\mathcal{O}_{\widehat{X}})\,.$$

In particular, there must be an $x \in \mathrm{H}^2(\mathcal{O}_{\widehat{X}})^G \subset \mathrm{H}^2(\mathcal{O}_{\widehat{X}})$ such that $x^n \neq 0$ in $\mathrm{H}^{2n}(\mathcal{O}_{\widehat{X}})$. Since

$$2n = \dim X = \dim \widehat{X} = \sum_i 2d_i + \sum_j e_j\,,$$

we have $\mathrm{H}^{2n}(\mathcal{O}_{\widehat{X}}) = \langle s \rangle$ with $s = \prod_i y_i^{d_i} \cdot \prod_j z_j$. As $\deg x = 2$ and $\deg z_j = e_j \geq 3$, it follows that x is a linear combination of some y_i. Hence, x^n can be a nonzero multiple of s only if no z_j occurs in the expression of s as above. In other words, X' does not have Calabi–Yau factors. This means that $X = \prod_i Y_i$ and $x = \sum_i y_i$ (up to coefficients which we can absorb by the choice of the generators y_i of $\mathrm{H}^2(\mathcal{O}_{Y_i})$). Every element of G acts by some permutation on the y_i; see Corollary 3.2. By assumption, $\mathrm{H}^2(\mathcal{O}_X)$ is of dimension one. Hence, $\mathrm{H}^2(\mathcal{O}_{\widehat{X}})^G = \langle x \rangle$. It follows that the action of G on the y_i is transitive. Otherwise, there would be G-invariant summands of $x = \sum_i y_i$ which would be linearly independent. Hence, again by Corollary 3.2, we have $\widehat{X} \cong Y^\ell$ for some $Y \in \mathsf{HK}_{2d}$. For dimension reasons, $d \cdot \ell = n$.

We assume for a contradiction that $\ell > 1$. We have the G-invariant class

$$x^2 = \sum_\alpha y_\alpha^2 + 2 \sum_{\alpha \neq \beta} y_\alpha y_\beta \in \mathrm{H}^4(\mathcal{O}_{X'})^G = \mathrm{H}^4(\mathcal{O}_X)\,. \tag{4}$$

It follows by Corollary 3.2 that the two summands in (4) are again G-invariant. But, by assumption, $h^4(\mathcal{O}_X) = 1$. Thus, one of the two summands must be zero. By (1), we see that the only possibility for this to happen is $d = 1$, i.e., $Y \in \mathsf{K3}$. Thus, $\ell = n$. Note that $\mathsf{ord}\, G = \mathsf{deg}(X' \to X) = m$. By (3) or Lemma 3.7, we have $m \mid \chi(X') = \chi(Y)^n = 2^n$. As G acts transitively on $\{y_1, \dots, y_n\}$, we get $n \mid m \mid 2^n$. Again by (3), also $n+1 \mid 2^n$. For $n \geq 2$, this is a contradiction.

Hence, we are in the case $\ell = 1$ which means that $\hat{X} = Y \in \mathsf{HK}_{2n}$. In particular, $\chi(\mathcal{O}_{\hat{X}}) = n+1 = \chi(X)$. By (3) we get $m = 1$ which means that we have an isomorphism $Y \cong X$. □

3.4. **Enriques varieties.** In this section we will consider a certain class of compact Kähler manifolds with the property that $\mathcal{O}_X \in \mathrm{D}(X)$ is exceptional; see Definition 2.1. These manifolds are automatically algebraic by the following result; see, e.g., [Voi07, Exc. 7.1].

Lemma 3.10. *Let X be a compact Kähler manifold with $\mathrm{H}^2(\mathcal{O}_X) = 0$. Then X is projective.*

From now on, let E be a smooth projective variety.

Definition 3.11. We call E a *strict Enriques variety* if the following three conditions hold:

(S1) The trivial line bundle $\mathcal{O}_E$ is exceptional.
(S2) The canonical line bundle ω_E is nontrivial and of finite order $m := \mathsf{ord}(\omega_E)$ in $\mathsf{Pic}\, E$ (this order is called the *index* of E).
(S3) The canonical cover $\widetilde{E}$ of E is hyperkähler.

This definition is inspired by similar, but different, notions of higher-dimensional Enriques varieties which are as follows.

Definition 3.12 ([BNWS11])**.** We call E a *BNWS* (*Boissière–Nieper-Wißkirchen–Sarti*) *Enriques variety* if the following three conditions hold:

(BNWS1) $\chi(\mathcal{O}_E) = 1$.
(BNWS2) The canonical line bundle ω_E is nontrivial and of finite order $m := \mathsf{ord}(\omega_E)$ in $\mathsf{Pic}\, E$ (this order is called the *index* of E).
(BNWS3) The fundamental group of E is cyclic of the same order, i.e., $\pi_1(E) \cong \mu_m$.

Definition 3.13 ([OS11])**.** We call E an *OS* (*Oguiso–Schröer*) *Enriques variety* if E is not simply connected and its universal cover $\widehat{E}$ is a compact hyperkähler manifold.

Proposition 3.14.

(i) *Let E be a strict Enriques variety of index $n+1$. Then $\mathsf{dim}\, E = 2n$.*
(ii) *Conversely, every smooth projective variety E satisfying* (S2) *with $m = n+1$,* (S3)*, and $\mathsf{dim}\, E = 2n$ is already a strict Enriques variety.*
(iii) *Strict Enriques varieties of index $n+1$ are exactly the quotient varieties of the form $E = Y/\langle g\rangle$, where $Y \in \mathsf{HK}_{2n}$ and $g \in \mathsf{Aut}(Y)$ is purely nonsymplectic of order $n+1$ such that $\langle g\rangle$ acts freely on Y.*
(iv) *X is a strict Enriques variety if and only if it is BNWS Enriques and OS Enriques.*

Proof. Let E be a strict Enriques variety of index $n+1$ with canonical cover $\widetilde{E} \in \mathsf{HK}_{2d}$. To verify (i) we have to show that $d = n$. By definition of the canonical cover (see section 2.1(iii)), the covering map $\widetilde{E} \to E$ is the quotient by a cyclic group G of order $n+1$. As $\widetilde{E} \in \mathsf{HK}_{2d}$, we have $\chi(\mathcal{O}_Y) = d+1$. Also, $\chi(\mathcal{O}_E) = 1$ by (S1). We get $d = n$ by Lemma 3.7.

Consider now a smooth projective variety E with $\mathsf{ord}\, \omega_E = n+1$ and $\mathsf{dim}\, E = 2n$ such that its canonical cover $\widetilde{E}$ is hyperkähler, necessarily of $\mathsf{dim}\, \widetilde{E} = \mathsf{dim}\, E = 2n$. Then, again by Lemma 3.7, we have $\chi(\mathcal{O}_E) = 1$. Furthermore,

$$\mathbb{C}[0] \subset \mathrm{H}^*(\mathcal{O}_E) \cong \mathrm{H}^*(\mathcal{O}_{\widetilde{E}})^{\mu_{n+1}} \subset \mathrm{H}^*(\mathcal{O}_{\widetilde{E}}) \cong \mathbb{C}[y]/y^{n+1} \tag{5}$$

with $\mathsf{deg}\, y = 2$. Since $\chi(\mathcal{O}_E) = 1$ and $\mathrm{H}^*(\mathcal{O}_{\widetilde{E}})$ is concentrated in even degrees, the first inclusion must be an equality which means that $\mathcal{O}_E$ is exceptional.

Let us prove part (iii). Given a strict Enriques variety E of index $n+1$, the canonical cover $Y := \widetilde{E}$ has the desired properties.

Conversely, let $Y \in \mathsf{HK}_{2n}$ together with a purely nonsymplectic $g \in \mathsf{Aut}(Y)$ of order $n+1$ such that $\langle g\rangle$ acts freely on Y, and set $E := Y/\langle g\rangle$. The action of g on the cohomology $\mathrm{H}^*(\mathcal{O}_Y) = \mathbb{C}[y]/y^{n+1}$ is given by $g \cdot y^i = \rho_{Y,g}^i y^i$. Since, by assumption, $\rho_{Y,g}$ is a primitive $(n+1)$-th root of unity, we get $\mathrm{H}^*(\mathcal{O}_E) \cong \mathrm{H}^*(\mathcal{O}_Y)^G \cong \mathbb{C}[0]$, hence (S1). The action of g on the nth power of a symplectic form, hence on the canonical bundle ω_Y, is also given by multiplication by $\rho_{Y,g}$. It follows that the canonical bundle ω_E of the quotient is of order $n+1$ and $Y \to E$ is the canonical cover.

For the proof of (iv), first note that (S1) implies (BNWS1). Furthermore, given a strict Enriques variety E, the canonical cover $Y = \widetilde{E}$ of E is also the universal cover, since Y is connected. From this, we get (BNWS2) and (BNWS3). Furthermore, E is OS Enriques, since Y is hyperkähler.

Conversely, if E is BNWS and OS Enriques, its canonical and universal cover coincide and is given by a hyperkähler manifold Y with the properties as in (iii). □

Note that the variety $Y \in \mathsf{HK}_{2n}$ from part (iii) of Proposition 3.14 is the universal as well as the canonical cover of E. We call Y the *hyperkähler cover* of E.

Another way to characterize strict Enriques varieties is as OS Enriques varieties whose fundamental group have the maximal possible order; see [OS11, Prop. 2.4].

Strict Enriques varieties of index 2 are exactly the Enriques surfaces. To get examples of higher index, by Proposition 3.14(iv), we just have to look for examples which occur in [BNWS11] as well as in [OS11].

Theorem 3.15 ([BNWS11],[OS11]). *There are strict Enriques varieties of index* 2, 3, *and* 4.

Note that the statement does not exclude the existence of strict Enriques varieties of index greater than 4, but, for the time being, there are no known examples.

In the known examples of index $n+1 = 3$ or $n+1 = 4$, the hyperkähler cover Y is given by a generalized Kummer variety $K_nA \subset A^{[n+1]}$. More concretely, in these examples A is an abelian surface isogenous to a product of elliptic curves with complex multiplication, and there is a nonsymplectic automorphism $f \in \mathsf{Aut}(A)$ of order $n+1$ which induces a nonsymplectic fixed-point-free automorphism $K_n(f) \in \mathsf{Aut}(K_nA)$ of the same order.

Note that there are examples of varieties which are BNWS Enriques but not OS Enriques [BNWS11, Sect. 4.3] and of the converse [OS11, Sect. 4].

We will use the following lemma in the proof of Theorem 4.5.

Lemma 3.16. *Let E be a strict Enriques variety of index $n+1$ with hyperkähler cover Y. Then there is an isomorphism of algebras $\bigoplus_{s=0}^{n} \mathrm{H}^*(\omega_E^{-s}) \cong \mathrm{H}^*(\mathcal{O}_Y) = \mathbb{C}[y]/y^{n+1}$. Under this isomorphism, $\mathrm{H}^*(\omega_E^{-s}) \cong \mathbb{C}\cdot y^s \cong \mathbb{C}[-2s]$.*

Proof. Let $\pi\colon Y \to E$ be the morphism which realizes Y as the universal and canonical cover of E. By the construction of the canonical cover (see section 2.1(iii)), we have an isomorphism of $\mathcal{O}_E$-algebras $\pi_*\mathcal{O}_Y \cong \mathcal{O}_E \oplus \omega_E^{-1} \oplus \cdots \oplus \omega_E^{-n}$. Hence, we get an isomorphism of graded $\mathbb{C}$-algebras

$$\mathbb{C}[y]/y^{n+1} \cong \mathrm{H}^*(\mathcal{O}_Y) \cong \mathrm{H}^*(\mathcal{O}_E) \oplus \mathrm{H}^*(\omega_E^{-1}) \oplus \cdots \oplus \mathrm{H}^*(\omega_E^{-n}) \tag{6}$$

with $\mathsf{deg}\, y = 2$. Hence, for the proof of the assertion, it is only left to show that the generator y lives in the direct summand $\mathrm{H}^2(\omega_E^{-1})$ under the decomposition (6). We have $y \in \mathrm{H}^2(\omega_E^{s})$ for some $s \in \mathbb{Z}/(n+1)\mathbb{Z}$. By Serre duality, we have $\mathrm{H}^*(\omega_E) = \mathbb{C}[-2n]$, hence $y^n \in \mathrm{H}^{2n}(\omega_E)$. It follows that

$$-s \equiv n \cdot s \equiv 1 \mod n+1\,.$$

□

3.5. **Enriques stacks.** The main difficulty in finding pairs $Y \in \mathsf{HK}$ and $f \in \mathsf{Aut}(Y)$ which, by Proposition 3.14(iii), induce strict Enriques varieties, is the condition that $\langle f \rangle$ acts freely.

Let us drop this assumption and consider a $Y \in \mathsf{HK}_{2n}$ together with a purely nonsymplectic automorphism $f \in \mathsf{Aut}(Y)$ which may have fixed points. Then we call the corresponding quotient stack $\mathcal{E} = [Y/\langle f \rangle]$ a *strict Enriques stack*. In analogy to the proof of Proposition 3.14, one can show that there is also the following equivalent.

Definition 3.17. A *strict Enriques stack* is a smooth projective stack $\mathcal{E}$ such that the following hold.

- (S1') The trivial line bundle $\mathcal{O}_{\mathcal{E}}$ is exceptional.
- (S2') The canonical bundle $\omega_{\mathcal{E}}$ is nontrivial and of finite order $m := \mathsf{ord}(\omega_{\mathcal{E}})$ in $\mathsf{Pic}\,\mathcal{E}$ (this order is called the *index* of $\mathcal{E}$).
- (S3') The canonical cover $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ is a hyperkähler manifold of dimension $\mathsf{dim}\,\widetilde{E} = \mathsf{dim}\,E = 2(m-1)$.

Note that, in contrast to the case of strict Enriques varieties, the formula relating index and dimension is not a consequence of the other conditions but is part of the assumptions.

As alluded to above it is much easier to find examples of strict Enriques stacks compared to strict Enriques varieties. Let $S \in \mathsf{K3}$ together with a purely nonsymplectic automorphism $f \in \mathsf{Aut}(S)$ of order $n+1$ (which may, and, for $n+1>2$, will have fixed points). Then the quotient of the associated Hilbert scheme of points by the induced automorphism $[X^{[n]}/f^{[n]}]$ is a strict Enriques stack. There are also examples of strict Enriques stacks whose hyperkähler cover is $K_5(A)$; compare [BNWS11, Rem. 4.1].

4. Construction of varieties with $\mathbb{P}^n[k]$-units

4.1. Definition and basic properties.

Definition 4.1. Let X be a compact Kähler manifold. We say that X has a $\mathbb{P}^n[k]$-*unit* if $\mathcal{O}_X$ is a $\mathbb{P}^n[k]$-object in $\mathrm{D}(X)$. This means that the following two conditions are satisfied:

- (C1) the canonical bundle ω_X is trivial;
- (C2) there is an isomorphism of $\mathbb{C}$-algebras $\mathrm{H}^*(\mathcal{O}_X) \cong \mathbb{C}[x]/x^{n+1}$ with $\mathsf{deg}\,x = k$.

Remark 4.2. If X has a $\mathbb{P}^n[k]$-unit, we have $\mathsf{dim}\,X = n \cdot k$. This follows by Serre duality.

Remark 4.3. For $n=1$, compact Kähler manifolds with $\mathbb{P}^1[k]$-units are exactly the strict Calabi–Yau manifolds of dimension k. For $k=2$, compact Kähler manifolds with $\mathbb{P}^n[2]$-units are exactly the compact hyperkähler manifolds of dimension $2n$; see Observations 1.1 and 1.2 and Remark 2.5.

Remark 4.4. If $n \geq 2$, the number k must be even. The reason is that the algebra $\mathrm{H}^*(\mathcal{O}_X)$ is graded-commutative. Hence, every $x \in \mathrm{H}^k(\mathcal{O}_X)$ with k odd satisfies $x^2 = 0$.

Since, in the following, we usually consider the case that $k > 2$, we will speak about *varieties with* $\mathbb{P}^n[k]$*-units*; compare Lemma 3.10.

4.2. **Nonexamples.** In order to get a better understanding of the notion of varieties with $\mathbb{P}^n[k]$-units, it might be instructive to start with some examples which satisfy some of the conditions but fail to satisfy others.

4.2.1. *Products of Calabi–Yau varieties.* Let $Z \in \mathsf{CY}_8$ and $Z' \in \mathsf{CY}_4$, and set $X := Z \times Z'$. Then ω_X is trivial and by the Künneth formula

$$\mathrm{H}^*(\mathcal{O}_X) \cong \mathbb{C}[0] \oplus \mathbb{C}[-4] \oplus \mathbb{C}[-8] \oplus \mathbb{C}[-12]\,.$$

Hence, as a graded vector space, $\mathrm{H}^*(\mathcal{O}_X)$ has the right shape for a $\mathbb{P}^3[4]$-unit. As an isomorphism of graded algebras, however, the Künneth formula gives

$$\mathrm{H}^*(\mathcal{O}_X) \cong \mathbb{C}[z]/z^2 \otimes \mathbb{C}[z']/z'^2 \cong \mathbb{C}[z,z']/(z^2, z'^2), \quad \mathsf{deg}\, z = 8\,, \, \mathsf{deg}\, z' = 4\,.$$

This means that, as a $\mathbb{C}$-algebra, $\mathrm{H}^*(\mathcal{O}_X)$ is not generated in degree 4 so that $\mathcal{O}_X$ is not a $\mathbb{P}^3[4]$-object.

4.2.2. *Hilbert schemes of points on Calabi–Yau varieties.* For every smooth projective variety X and $n = 2, 3$, the Hilbert schemes $X^{[n]}$ of n points on X are smooth and projective of dimension $n \cdot \mathsf{dim}\, X$. If $\mathsf{dim}\, X \geq 3$ and $n \geq 4$, the Hilbert scheme $X^{[n]}$ is not smooth; see [Che98].

Let now X be a Calabi–Yau variety of even dimension k and $n = 2$ or $n = 3$. Then there is an isomorphism of algebras $\mathrm{H}^*(\mathcal{O}_{X^{[n]}}) \cong \mathbb{C}[x]/(x^{n+1})$ with $\mathsf{deg}\, x = k$. The reason is that $X^{[n]}$ is a resolution of the singularities of the symmetric quotient variety $X^n/\mathfrak{S}_n$, which has rational singularities, by means of the Hilbert–Chow morphism $X^{[n]} \to X^n/\mathfrak{S}_n$. For $k = 2$, the Hilbert scheme of points on a K3 surface is one of the few known examples of a compact hyperkähler manifold which means that $X^{[n]}$ has a $\mathbb{P}^n[2]$-unit for $X \in \mathsf{K3}$. For $\mathsf{dim}\, X = k > 2$, however, the canonical bundle $\omega_{X^{[n]}}$ is not trivial as this resolution is not crepant.

In contrast, the symmetric quotient stack $[X^n/\mathfrak{S}_n]$ has a trivial canonical bundle for $\mathsf{dim}\, X = k$ an arbitrary even number, and is, in fact, a stack with $\mathbb{P}^n[k]$-unit; see section 6.4 for some further details.

4.3. **Main construction method.** In this section given strict Enriques varieties of index $n + 1$, we construct a series of varieties with $\mathbb{P}^n[2k]$-units. In other words we prove the implication (iii)$\Longrightarrow$(ii) of Theorem 1.3.

Let $E_1, \dots, E_k$ be strict Enriques varieties of index $n + 1$. We do not assume that the E_i are nonisomorphic. For the time being, there are known examples of such E_i for $n = 1, 2, 3$; see Theorem 3.15. We set $F := E_1 \times \cdots \times E_k$.

Theorem 4.5. *The canonical cover $X := \widetilde{F}$ of F has a $\mathbb{P}^n[2k]$-unit.*

Proof. By definition of the canonical cover, ω_X is trivial. Hence, condition (C1) of Definition 4.1 is satisfied. It is left to show that $\mathrm{H}^*(\mathcal{O}_X) \cong \mathbb{C}[x]/x^{n+1}$ with $\mathsf{deg}\, x = 2k$. Let $\pi\colon X \to F$ be the étale cover with $\pi_*\mathcal{O}_X \cong \mathcal{O}_F \oplus \omega_F^{-1} \oplus \cdots \oplus \omega_F^{-n}$.

Note that $\omega_F \cong \omega_{E_1} \boxtimes \cdots \boxtimes \omega_{E_k}$. By the Künneth formula together with Lemma 3.16, we get

$$\begin{aligned}
\mathrm{H}^*(\mathcal{O}_X) &\cong \mathrm{H}^*(\mathcal{O}_F) \oplus \mathrm{H}^*(\omega_F^{-1}) \oplus \cdots \oplus \mathrm{H}^*(\omega_F^{-n}) \\
&\cong \Big(\bigotimes_{i=1}^{k} \mathrm{H}^*(\mathcal{O}_{E_i})\Big) \oplus \Big(\bigotimes_{i=1}^{k} \mathrm{H}^*(\omega_{E_i}^{-1})\Big) \oplus \cdots \oplus \Big(\bigotimes_{i=1}^{k} \mathrm{H}^*(\omega_{E_i}^{-n})\Big) \\
&\cong \mathbb{C} \oplus \mathbb{C} \cdot y_1 \cdots y_k \oplus \cdots \oplus \mathbb{C} \cdot y_1^n \cdots y_k^n \\
&\cong \mathbb{C}[x]/x^{n+1},
\end{aligned}$$

where $x := y_1 \cdots y_n$ is of degree $2k$. $\square$

Remark 4.6. Let $f_i \in \mathsf{Aut}(Y_i)$ be a generator of the group of deck transformations of the cover $Y_i \to E_i$. In other words $E_i = Y_i/\langle f_i \rangle$. Then we can describe X alternatively as $X = (Y_1 \times \cdots \times Y_k)/G$, where

$$\mu_{n+1}^{k-1} \cong G = \big\{ f_1^{a_1} \times \cdots \times f_k^{a_k} \mid a_1 + \cdots + a_k \equiv 0 \mod n+1 \big\} \subset \mathsf{Aut}(Y_1 \times \cdots \times Y_n)\,.$$

Remark 4.7. In the case $n = 1$, one can replace the $Y_i \in \mathsf{K3}$ by strict Calabi–Yau varieties Z_i of dimension $\dim Z_i = d_i$ together with fixed-point-free involutions $f_i \in \mathsf{Aut}(Z_i)$. Then the same construction gives a variety X with $\mathbb{P}^1[d_1 + \cdots + d_k]$-unit, i.e., a strict Calabi–Yau variety of dimension $\dim X = d_1 + \cdots + d_k$. This coincides with a construction of Calabi–Yau varieties by Cynk and Hulek [CH07].

Remark 4.8. The construction still works if we replace one of the strict Enriques varieties E_i by an Enriques stack. The reason is that the group G still acts freely on $Y_1 \times \cdots \times Y_k$, even if one of the f_i has fixed points; see Lemma 3.6.

5. Structure of varieties with $\mathbb{P}^n[k]$-units

5.1. **General properties.** As mentioned in Remark 4.3, varieties with $\mathbb{P}^1[k]$-units are exactly the strict Calabi–Yau varieties (not necessarily simply connected) and manifolds with $\mathbb{P}^n[2]$-units are exactly the compact hyperkähler manifolds. From now on, we will concentrate on the other cases; i.e., we assume that $n > 1$ and $k > 2$. By Remark 4.4, this means that k is even.

Lemma 5.1. *Let X be a variety with a $\mathbb{P}^n[k]$-unit. Then there is an étale cover $X' \to X$ of the form $X' = \prod_i Y_i \times \prod_j Z_j$ with $Y_i \in \mathsf{HK}$ and $Z_i \in \mathsf{CY}$ of even dimension.*

Proof. Let $X' = T \times Y_i \times \prod_j Z_j$ be a Beauville–Bogomolov cover of X as in Convention 3.9. The proof is the same as the first part of the proof of Proposition 3.8: We have $\chi(\mathcal{O}_X) = n + 1 \neq 0$, hence $\chi(\mathcal{O}_{X'}) \neq 0$. It follows that there cannot be a torus or an odd-dimensional Calabi–Yau factor occurring in the decomposition on X'. $\square$

In particular, X' is simply connected and hence agrees with the universal cover,

$$\widehat{X} = X' = \prod_i Y_i \times \prod_j Z_j\,.$$

Since $\mathrm{H}^*(\mathcal{O}_X) = \mathbb{C}[x]/x^{n+1}$ with $\deg x = k \geq 4$, we see by the Künneth formula that $\widehat{X} \to X$ cannot be an isomorphism; compare (1).

Corollary 5.2. *For X a variety with a $\mathbb{P}^n[k]$-unit, $\pi_1(X)$ is a nontrivial finite group.*

5.2. **The case $k = 4$.** Now we focus on the case $k = 4$ where we can determine the decomposition of the universal cover concretely.

Proposition 5.3. *Let $n \geq 3$, and let X be a variety with $\mathbb{P}^n[4]$-unit. Then the universal cover $\widehat{X}$ is a product of two hyperkähler varieties of dimension $2n$.*

We divide the proof of this statement into several lemmas. So in the following let X be a variety with a $\mathbb{P}^n[4]$-unit where $n \geq 3$.

Lemma 5.4. *The universal cover $\widehat{X}$ of X is a product of compact hyperkähler manifolds.*

Proof. By Lemma 5.1 we have $\widehat{X} = \prod_i Y_i \times \prod_j Z_j$ with $Y_i \in \mathsf{HK}_{2d_i}$ and $Z_j \in \mathsf{CY}_{e_j}$ with $e_i \geq 4$ even. Let $\pi_1(X) \cong G \subset \mathsf{Aut}(\widehat{X})$ such that $X = \widehat{X}/G$. Analogously to the proof of Proposition 3.8, we see that there is an $x \in \mathrm{H}^4(\mathcal{O}_{\widehat{X}})^G \cong \mathrm{H}^4(\mathcal{O}_X)$ such that x^n is a nonzero multiple of the generator $\prod_i y_i^{d_i} \cdot \prod_j z_j$ of $\mathrm{H}^{4n}(\mathcal{O}_{\widehat{X}})$. In particular, all the z_j have to occur in the expression of $x \in \mathrm{H}^4(\mathcal{O}_{\widehat{X}})$ in terms of the Künneth formula. Hence, $e_j = 4$ for all j. We get

$$x = \sum_j z_j + \text{ terms involving the } y_i, \tag{7}$$

where we absorb possible coefficients in the choice of the generators z_j of $\mathrm{H}^4(\mathcal{O}_{Z_j})$. Both summands of (7) are G-invariant. This follows by the G-invariance of x together with Corollary 3.2. Hence, one of the two summands must vanish. Consequently, $\widehat{X}$ either has no Calabi–Yau or no hyperkähler factors, i.e., $\widehat{X} = \prod Y_i$ or $\widehat{X} = \prod Z_j$.

Let us assume for a contradiction that the latter is the case. We have $e_j = \mathsf{dim}\, Z_j = 4$ for all j. Since $\mathsf{dim}\, \widehat{X} = \mathsf{dim}\, X = 4n$, there must be n factors $Z_j \in \mathsf{CY}_4$ of $\widehat{X}$. Hence, $\chi(\mathcal{O}_{\widehat{X}}) = 2^n$. By Lemma 3.7 we have

$$\chi(\mathcal{O}_{\widehat{X}}) = \chi(\mathcal{O}_X) \cdot \mathsf{ord}(G)\,. \tag{8}$$

Hence, $\chi(\mathcal{O}_X) = n + 1 \mid 2^n$. Furthermore, G must act transitively on $\{z_1, \dots, z_n\}$. Otherwise, there would be G-invariant summands of $x = \sum z_j$ contradicting the assumption that $h^4(\mathcal{O}_X) = 1$. Hence, $n \mid \mathsf{ord}\, G \mid 2^n$ which, for $n \geq 2$, is not consistent with $n + 1 \mid 2^n$. $\square$

Hence, we have $\widehat{X} = \prod_{i \in I} Y_i$ with $Y_i \in \mathsf{HK}_{2d_i}$ for some finite index set I and there is a G-invariant

$$0 \neq x = \sum_i c_{ii} y_i^2 + \sum_{i \neq j} c_{ij} y_i y_j \in \mathrm{H}^4(\mathcal{O}_{\widehat{X}}), \quad c_{ij} \in \mathbb{C}\,. \tag{9}$$

Again by Corollary 3.2 both summands in (9) are G-invariant so that one of them must be zero.

Lemma 5.5. *There is a nonzero G-invariant $x \in \mathrm{H}^4(\mathcal{O}_{\widehat{X}})$ of the form $x = \sum_{i \neq j} c_{ij} \cdot y_i y_j$.*

Proof. Let us assume for a contradiction that we are in the case that $x = \sum_i y_i^2$, where we hide the coefficients c_{ii} in the choice of the y_i. By the same arguments as above, G must act transitively on the set of y_i. Hence, by Corollary 3.2 we have $\widehat{X} = Y^\ell$, $Y \in \mathsf{HK}_{2d}$ with $d\ell = 2n$. We must have $\ell \geq 2$ by Corollary 3.5. Then

$$x^2 = \sum_i y_i^4 + 2\sum_{i\neq j} y_i^2 y_j^2 \in \mathrm{H}^8(\mathcal{O}_{\widehat{X}})^G,$$

and both summands are G-invariant. Hence, one of them must be zero, and the only possibility for that to happen is that $d < 4$. Since x^n is a scalar multiple of the generator $y_1^d y_2^d \cdots y_\ell^d$ of $\mathrm{H}^{4n}(\mathcal{O}_{\widehat{X}})$, we must have $d = 2$. Hence, $\ell = n$ and $\chi(\mathcal{O}_{\widehat{X}}) = 3^n$. By (8) and the fact that G acts transitively on $\{y_1, \dots, y_n\}$, we get the contradiction $n \mid 3^n$ and $n+1 \mid 3^n$. $\square$

Lemma 5.6. *We have $|I| = 2$ which means that $\widehat{X} = Y \times Y'$ with $Y, Y' \in \mathsf{HK}$.*

Proof. Let $0 \neq x = \sum_{i\neq j} c_{ij} y_i y_j \in \mathrm{H}^4(\mathcal{O}_{\widehat{X}})^G$ with $c_{ij} \in \mathbb{C}$, some of which might be zero, as in Lemma 5.5. As already noted above, we have $|I| \geq 2$ by Corollary 3.5. Let us assume that $|I| \geq 3$. This assumption will be divided into several subcases, each of which leads to a contradiction. We have

$$x^2 = \sum_{i\neq j} c_{ij}^2 \cdot y_i^2 y_j^2 + \sum_{h\neq i\neq j} c_{hi} c_{ij} \cdot y_h y_i^2 y_j + \sum_{g\neq h\neq i\neq j} \hat{c}_{ghij} \cdot y_g y_h y_i y_j\,, \qquad \hat{c}_{ghij} = c_{gh} c_{ij} + \cdots\,. \tag{10}$$

All three summands are G-invariant by Corollary 3.2, hence two of them must be zero. For one of the first two summands of (10) to be zero, the square of some y_i must be zero, i.e., some Y_{i_0} must be a K3 surface. Write the index set I of the decomposition $\widehat{X} = \prod_{i\in I} Y_i$ as $I = N \uplus M$ where $N = G \cdot i_0$ is the orbit of i_0. Here we consider the G-action on I given by the permutation part of the auto-equivalences in $G \subset \mathsf{Aut}(\widehat{X})$; see Lemma 3.1. With this notation, $Y_j = Y_{i_0} \in \mathsf{K3}$ for $j \in N$.

Let us first consider the case that G acts transitively on the factors of the decomposition of $\widehat{X}$, i.e., $I = N$. Then, by dimension reasons, $|I| = 2n$. In other words, $\widehat{X} = Y^{2n}$ with $Y \in \mathsf{K3}$. Hence, $\chi(\mathcal{O}_{\widehat{X}}) = 2^{2n}$. By (8) we get the contradiction $2n \mid 2^{2n}$ and $n+1 \mid 2^{2n}$.

In the case that $M \neq 0$, all the nonzero coefficients c_{ij} in the G-invariant $x = \sum_{i\neq j} c_{ij} y_i y_j$ must be of the form $i \in N$ and $j \in M$ (or the other way around). Indeed, otherwise we would have G-invariant proper summands of x in contradiction to the assumption $\mathrm{H}^4(\mathcal{O}_{\widehat{X}})^G = \langle x \rangle$. Furthermore, for all $i \in N$ there must be a nonzero $c_{ii'}$ and for all $j' \in M$ there must be a nonzero $c_{jj'}$ since x^n is a nonzero multiple of the generator $\prod_{i\in N} y_i \cdot \prod_{j\in M} y_j^{d_j}$ of $\mathrm{H}^{4n}(\mathcal{O}_{\widehat{X}})$. Hence, to avoid proper G-invariant summands of x, the group G must also act transitively on M. It follows that $\widehat{X} = Y^\ell \times (Y')^{\ell'}$ where $\ell = |N|$, $\ell' = |M|$, $Y \in \mathsf{K3}$, and $Y' \in \mathsf{HK}_{2d'}$ for some d'. Now, x^n is a nonzero multiple of

$$\prod_{i=1}^{\ell} y_i \cdot \prod_{j=1}^{\ell'} (y_j')^{d'} \in \mathrm{H}^{4n}(\mathcal{O}_{\widehat{X}})\,.$$

Since all the nonzero summands of x are of the form $c_{ij} y_i y_j'$, we get that $\ell = n = \ell' \cdot d'$. In particular,

$$\widehat{X} = Y^n \times (Y')^{\ell'}\,. \tag{11}$$

First, we consider for a contradiction the case that $\ell' = 1$, hence $\widehat{X} = Y^n \times Y'$ with $Y \in \mathsf{K3}$ and $Y' \in \mathsf{HK}_{2n}$. Then, by (8), we get $\operatorname{ord} G = 2^n$. We have (up to coefficients which we avoid by the correct choice of the y_i), $x = \sum_{i=1}^n y_i y'$. Accordingly, $x^2 = \sum_{i\neq j} y_i y_j (y')^2$. Hence, G acts transitively on $\{y_1, \dots, y_n\}$ as well as on $\{y_i y_j \mid 1 \le i < j \le n\}$. We get the contradiction $n \mid 2^n$ and $\binom{n}{2} \mid 2^n$.

Note that, for this to be a contradiction, we need the assumption $n \geq 3$. Indeed, in section 6.2, we will see examples of a variety X with a $\mathbb{P}^2[4]$-unit whose canonical covers are of the form $\widehat{X} = Y^2 \times Y'$ with $Y \in \mathsf{K3}$ and $Y' \in \mathsf{HK}_4$.

Now, let $\ell' > 1$ in (11). Then, we get

(12)
$$x^2 = \sum_{i\neq j, i'} c_{ii'} c_{ji'} \cdot y_i y_j (y'_{i'})^2 + \sum_{i\neq j, i'\neq j'} \tilde{c}_{iji'j'} \cdot y_i y_j y'_{i'} y'_{j'}, \quad \tilde{c}_{iji'j'} = c_{ii'}c_{jj'} + c_{ij'}c_{ji'},$$

where both summands are G-invariant. Hence, in order to avoid linearly independent classes in $\mathrm{H}^8(\mathcal{O}_{\widehat{X}})^G$, one of them must be zero.

Let us assume for a contradiction that all the $\tilde{c}_{iji'j'}$ are zero. Then all the $c_{ii'}$ with $i \in N$ and $i' \in M$ are nonzero. Indeed, as mentioned above, given $i \in N$ and $i' \in M$, there exist $j \in N$ and $j' \in M$ such that $c_{ij'} \neq 0 \neq c_{ji'}$. By $\tilde{c}_{iji'j'} = 0$, it follows that also $c_{ii'} \neq 0 \neq c_{jj'}$. Given pairwise distinct $h, i, j \in N$ and $i', j' \in M$ we consider the following term, which is the coefficient of $y_h y_i y_j (y'_{i'})^2 y_{j'}$ in x^3,

$$\begin{aligned} C := {} & c_{hi'}c_{ii'}c_{jj'} + c_{hi'}c_{ij'}c_{ji'} + c_{hj'}c_{ii'}c_{ji'} \\ = {} & c_{hi'}\tilde{c}_{iji'j'} + c_{hj'}c_{ii'}c_{ji'} \\ = {} & c_{ii'}\tilde{c}_{hji'j'} + c_{hi'}c_{ij'}c_{ji'} \\ = {} & c_{ji'}\tilde{c}_{hii'j'} + c_{hi'}c_{ii'}c_{jj'}\,. \end{aligned} \tag{13}$$

By the vanishing of the $\tilde{c}$, we get

$$C = c_{hi'}c_{ii'}c_{jj'} = c_{hi'}c_{ij'}c_{ji'} = c_{hj'}c_{ii'}c_{ji'}\,.$$

By the nonvanishing of all the c, we get $C \neq 0$. But at the same time by (13) we have $3C = C$; a contradiction.

We conclude that the first summand of (12) is zero. This can only happen for $(y'_{i'})^2 = 0$, hence $Y' \in \mathsf{K3}$. Then $\chi(\mathcal{O}_{\widehat{X}}) = 2^{2n}$ and, as before, we get the contradiction that $n \mid 2^{2n}$ and $n+1 \mid 2^{2n}$. □

Proof of Proposition 5.3. By now we know that $\widehat{X} = Y \times Y'$ with $Y \in \mathsf{HK}_{2d}$ and $Y \in \mathsf{HK}_{2d'}$, and $x = yy'$. We have $d + d' = 2n$. Furthermore, $0 \neq x^n = y^n (y')^n$. Hence, $d = n = d'$. □

Remark 5.7. The proof of Proposition 5.3 becomes considerably simpler if one assumes that $n+1$ is a prime number. In this case it follows directly by Lemma 3.7 that the universal cover must have a factor $Y \in \mathsf{HK}_{2n}$. Hence, there are much fewer cases one has to deal with.

Theorem 5.8. *Let $n \geq 3$, and let X be a variety with a $\mathbb{P}^n[4]$-unit.*

- (i) *We have $X = (Y \times Y')/G$ with $Y, Y' \in \mathsf{HK}_{2n}$. The group $\pi_1(X) \cong G \subset \mathsf{Aut}(Y \times Y')$ acts freely and is of the form $G = \langle f \times f' \rangle$ with $f \in \mathsf{Aut}(Y)$ and $f \in \mathsf{Aut}(Y')$ purely symplectic of order $n+1$.*
- (ii) *If $n + 1 = p^\nu$ is a prime power, at least one of the cyclic groups $\langle f \rangle \subset \mathsf{Aut}(Y)$ and $\langle f' \rangle \subset \mathsf{Aut}(Y')$ acts freely.*

Before giving the proof of the theorem, let us restate, for convenience, the special case of Lemma 3.6 for automorphisms of products with two factors.

Lemma 5.9. *Let X and Y be manifolds, and let $g, f \in \mathsf{Aut}(X)$ and $h \in \mathsf{Aut}(Y)$.*

(i) *$g \times h \in \mathsf{Aut}(X \times Y)$ is fixed point free if and only if at least one of g and h is fixed point free.*

(ii) *Let $\varphi := (f \times g) \circ (1\ 2) \in \mathsf{Aut}(X^2)$ be given by $(a, b) \mapsto (f(b), g(a))$. Then, φ is fixed point free if and only if $f \circ g$ and $g \circ f$ are fixed point free.*

Proof of Theorem 5.8. The fact that $X = (Y \times Y')/G$ with $Y, Y' \in \mathsf{HK}_{2n}$ and $G \cong \pi_1(X)$ is just a reformulation of Proposition 5.3. By the proof of this proposition we see that $\mathrm{H}^*(\mathcal{O}_{Y\times Y'})^G \cong \mathrm{H}^*(\mathcal{O}_X)$ is generated by $x = yy'$ in degree 4.

Let us assume for a contradiction that G contains an element which permutes the factors Y and Y', in which case we have $Y = Y'$ by Lemma 3.1. In other words there exists an $\varphi = (f \times g) \circ (1\ 2) \in G$ as in Lemma 5.9(ii). Hence, $f \circ g$ is fixed point free. By Lemma 3.4 the composition $f \circ g$ is nonsymplectic, i.e., $\rho_{f\circ g} \neq 1$. But $\rho_{f\circ g} = \rho_f \cdot \rho_g$ so that φ acts nontrivially on $x = yy'$ in contradiction to the G-invariance of x.

Hence, every element of G is of the form $g \times h$ as in Lemma 3.6(i). We consider the group homomorphisms $\rho_Y : G \to \mathbb{C}^*$ and $\rho_{Y'} : G \to \mathbb{C}^*$. Their images are of the form μ_m and $\mu_{m'}$, respectively. We must have $m, m' \geq n + 1$. Indeed, y^m and $(y')^{m'}$ are G-invariant but, for $m \leq n$ or $m' \leq n$, are not contained in the algebra generated by $x = yy'$. Since $|G| = n + 1$, assertion (i) follows.

Let now $n + 1 = p^\nu$ be a prime power, and let $G = \langle f\rangle$. Let us assume for a contradiction that there exist $a, b \in \mathbb{N}$ with $n+1 = p^\nu \nmid a, b$ such that f^a and $(f')^b$ have fixed points. Note that, in general, if an automorphism g has fixed points, also all of its powers have fixed points. Furthermore, for two elements $a, b \in \mathbb{Z}/(p^\nu)$ we have $a \in \langle b\rangle$ or $b \in \langle a\rangle$. Hence, $(f \times f')^a$ or $(f \times f')^b$ has fixed points in contradiction to part (i). □

This proves the implication (i)$\Longrightarrow$(iii) of Theorem 1.3. Indeed, for $n+1$ a prime power, the above Theorem says that $Y/\langle f\rangle$ or $Y'/\langle f'\rangle$ is a strict Enriques variety; see Proposition 3.14(iii). Note that Theorem 5.8 above does not hold for $n = 2$; see section 6.2. However, both conditions (i) and (iii) of Theorem 1.3 hold true for $n = 2$; see Theorem 3.15 and Corollary 1.4.

Remark 5.10. The proof of part (ii) of Theorem 5.8 does not work if $n + 1$ is not a prime power. For example, if $n + 1 = 6$, one could obtain a variety with $\mathbb{P}^5[4]$-unit as a quotient $X = (Y \times Y')/\langle f \times g\rangle$ with $Y, Y' \in \mathsf{HK}_{10}$ such that f and g are purely nonsymplectic of order 6, and f, f^2, f^4, f^5, g, g^3, g^5 are fixed point free but f^3, g^2, and g^4 are not. The author does not know whether hyperkähler manifolds together with these kinds of automorphisms exist.

6. Further remarks

6.1. **Further constructions using strict Enriques varieties.** Given strict Enriques varieties of index $n + 1$, there are, for $k \geq 6$, further constructions of varieties with $\mathbb{P}^n[k]$-units besides the one of section 4.3. Let $Y \in \mathsf{HK}_{2n}$ and $f \in \mathsf{Aut}(Y)$ purely symplectic of order $n + 1$ such that $\langle f\rangle$ acts freely; i.e., the quotient $E = Y/\langle f\rangle$ is a strict Enriques variety. We consider the $(n + 1)$-cycle

$\sigma := (1\ 2 \cdots n+1) \in \mathfrak{S}_{n+1}$ and the subgroup $G(Y) \subset \mathsf{Aut}(Y^{n+1})$ given by

$$G(Y) := \left\{ (f^{a_1} \times \cdots \times f^{a_{n+1}}) \circ \sigma^a \mid a_1 + \cdots + a_{n+1} \equiv a \mod n+1 \right\}.$$

Every nontrivial element of $G(Y)$ acts without fixed points on Y^{n+1} by Lemma 3.6. There is the surjective homomorphism

$$G(Y) \to \mathbb{Z}/(n+1)\mathbb{Z}, \quad (f^{a_1} \times \cdots \times f^{a_{n+1}}) \circ \sigma^a \mapsto a \mod n+1,$$

and we denote the fibers of this homomorphism by $G_a(Y)$.

Now, consider further $Z_1, \dots, Z_k \in \mathsf{HK}_{2n}$ together with purely nonsymplectic $g_i \in \mathsf{Aut}(Z_i)$ of order $n+1$ such that $\langle g_i \rangle$ acts freely and

$$\rho_{Z_i, g_i} = \rho_{Y,f} \quad \text{for all } i = 1, \dots, k. \tag{14}$$

The equality (14) can be achieved as soon as we have any purely nonsymplectic automorphisms $g_i \in \mathsf{Aut}(Z_i)$ of order $n+1$ by replacing the g_i by an appropriate power g_i^ν with $\mathsf{gcd}(\nu, n+1) = 1$. We consider the subgroup $G(Y; Z_1, \dots, Z_k) \subset \mathsf{Aut}(Y^{n+1} \times Z_1 \times \cdots \times Z_k)$ given by

$$\begin{aligned} &G(Y; Z_1, \dots, Z_k) \\ &\quad := \left\{ F \times g_1^{b_1} \times \cdots \times g_k^{b_k} \mid F \in G_a(Y),\, a + b_1 + \cdots + b_k \equiv 0 \mod n+1 \right\}. \end{aligned}$$

Proposition 6.1. *The quotient* $X := (Y^{n+1} \times Z_1 \times \cdots \times Z_k)/G(Y; Z_1, \dots, Z_k)$ *is a smooth projective variety with* $\mathbb{P}^n[2(n+1+k)]$*-unit.*

Proof. One can check using Lemma 3.6 that the group $G := G(Y; Z_1, \dots, Z_k)$ acts freely on $X' := Y^{n+1} \times Z_1 \times \cdots \times Z_k$. Hence, X is indeed smooth.

By the defining property of the elements of $G(Y; Z_1, \dots, Z_k)$ together with (14), we see that $x := y_1 y_2 \cdots y_{n+1} z_1 z_2 \cdots z_k$ is G-invariant. Hence, as $x^i \neq 0$ for $0 \leq i \leq n$, we get the inclusion

$$\mathbb{C}[x]/x^{n+1} \subset \mathrm{H}^*(\mathcal{O}_{X'})^G \cong \mathrm{H}^*(\mathcal{O}_X), \quad \mathsf{deg}\, x = 2(n+1+k). \tag{15}$$

Also, $\mathsf{ord}\, G(Z_1, \dots, Z_k) = (n+1)^{n+1+k-1}$. By Lemma 3.7, we get $\chi(\mathcal{O}_{X'}) = n+1$ so that the inclusion (15) must be an equality which is (C2). Finally, the canonical bundle ω_X is trivial since G acts trivially on $\langle x^n \rangle = \mathrm{H}^{\mathsf{dim}\, X'}(\mathcal{O}_{X'}) \cong \mathrm{H}^0(\omega_{X'})$. □

Remark 6.2. For $n \geq 2$, the group $G(Y; Z_1, \dots, Z_k)$ is not abelian. Since $X' \to X$ is the universal cover, we see that, for $k \geq 4$, there are examples of varieties with $\mathbb{P}^n[k]$-units which have a nonabelian fundamental group.

Remark 6.3. Again, for one $i \in \{1, \dots k\}$, we may drop the assumption that $\langle g_i \rangle$ acts freely; compare Remark 4.8.

Remark 6.4. One can further generalize the above construction as follows. Consider hyperkähler manifolds $Y_1, \dots, Y_m, Z_1, \dots, Z_k \in \mathsf{HK}_{2n}$ together with $f_i \in \mathsf{Aut}(Y_i)$ and $g_j \in \mathsf{Aut}(Z_j)$ purely nonsymplectic of order $n+1$ such that the generated cyclic groups act freely. Set $X' := Y_1^{n+1} \times \cdots \times Y_m^{n+1} \times Z_1 \times \cdots \times Z_k$ and consider $G := G(Y_1, \dots, Y_m; Z_1, \dots, Z_k) \subset \mathsf{Aut}(X')$ given by

$$\begin{aligned} G = \big\{ F_1 \times \cdots \times F_m \times g_1^{b_1} \times \cdots \times g_k^{b_k} \mid F_i \in G_{a_i}(Y),\, a_1 + \cdots + a_m + b_1 + \cdots + b_k \equiv 0 \\ \mod n+1 \big\}. \end{aligned}$$

Then, $X := X'/G$ has a $\mathbb{P}^n[2(m(n+1)+k)]$-unit.

Remark 6.5. In the case $n = 1$, one may replace the K3 surfaces Y_i and Z_j by strict Calabi–Yau varieties of arbitrary dimensions. Still, the quotient X will be a strict Calabi–Yau variety.

6.2. **A construction not involving strict Enriques varieties.** As mentioned in section 5.2, there is a variety X with $\mathbb{P}^2[4]$-unit whose universal cover $\widehat{X}$ is not a product of two hyperkähler varieties of dimension 4. This shows that the assumption $n \geq 3$ in Proposition 5.3 is really necessary.

For the construction let Z be a strict Calabi–Yau variety of dimension $\mathsf{dim}\, Z = e$ together with a fixed-point-free involution $\iota \in \mathsf{Aut}(Z)$. Necessarily, $\rho_{Z,\iota} = -1$; see Lemma 3.4. Furthermore, let $Y \in \mathsf{HK}_4$ together with a purely nonsymplectic $f \in \mathsf{Aut}(Y)$ of order 4. Note that g must have fixed points on Y. Such pairs (Y, f) exist. Take a K3 surface S (an abelian surface A) together with a purely nonsymplectic automorphism of order 4 and $Y = S^{[2]}$ ($Y = K_2A$) together with the induced automorphism.

Now, consider $G(Z) \subset \mathsf{Aut}(Z^2)$ as in the previous section. It is a cyclic group of order 4 with generator $g = (\iota \times \mathrm{id}) \circ (1\ 2)$. Set $X' = Y \times Z^2$ and $G := \langle f \times g \rangle \subset \mathsf{Aut}(X')$. The group G acts freely, since $G(Z)$ does; see Lemma 5.9. One can check that $x = yz_1 + i \cdot yz_2 \in \mathrm{H}^{2+e}(\mathcal{O}_{X'})$ is G-invariant. By the same argument as in the proof of Proposition 6.1, we conclude that X has a $\mathbb{P}^2[2+e]$-unit. In particular, in the case that $Z \in \mathsf{K3}$, we get a variety with $\mathbb{P}^2[4]$-unit.

6.3. **Possible construction for $k = 6$.** In contrast to the case $k = 4$ and $n + 1$ a prime power (see Theorem 1.3), there might be a variety with $\mathbb{P}^n[6]$-unit even if there is no Enriques variety of index $n+1$ but one of index $2n+1$. Of course, since there are at the moment only known examples of strict Enriques varieties of index 2, 3, and 4, this is only hypothetical.

Indeed, let $Y \in \mathsf{HK}_{4n}$ together with subgroup $\langle f \rangle \subset \mathsf{Aut}(Y)$ acting freely, where f is purely nonsymplectic of order $2n + 1$, and let $Y' \in \mathsf{HK}_{2n}$ together with $f' \in \mathsf{Aut}(Y')$ nonsymplectic of order $n + 1$ with $\rho_{Y,f} = \rho^{-1}_{Y',f'}$. Necessarily, f' has fixed points; see Lemma 3.4. Then $G = \langle f \times f'^2 \rangle$ acts freely on Y and $x = y^2 \cdot y'$ is G-invariant. It follows that $X = (Y \times Y')/G$ has a $\mathbb{P}^n[6]$-unit.

6.4. **Stacks with $\mathbb{P}^n[k]$-units.** Let $\mathcal{X}$ be a smooth projective stack. In complete analogy to the case of varieties, we say that $\mathcal{X}$ *has a* $\mathbb{P}^n[k]$*-unit* if $\mathcal{O}_{\mathcal{X}} \in \mathrm{D}(\mathcal{X})$ is a $\mathbb{P}^n[k]$-object. Again, this means

(C1') the canonical bundle $\omega_{\mathcal{X}}$ is trivial,

(C2') there is an isomorphism of $\mathbb{C}$-algebras $\mathrm{H}^*(\mathcal{O}_{\mathcal{X}}) \cong \mathbb{C}[x]/x^{n+1}$ with $\mathsf{deg}\, x = k$.

In contrast to the case of varieties, it is very easy to construct stacks with $\mathbb{P}^n[k]$-units.

Let $Z \in \mathsf{CY}_k$ with k even. Then, the symmetric group $\mathfrak{S}_n$ acts on Z^n by permutation of the factors, and we call the associated quotient stack $\mathcal{X} = [Z^n/\mathfrak{S}_n]$ the *symmetric quotient stack*. Then, as $k = \mathsf{dim}\, Z$ is even, the canonical bundle of $\mathcal{X}$ is trivial; see [KS15a, Sect. 5.4]. Condition (C2') follows by the Künneth formula

$$\mathrm{H}^*(\mathcal{O}_{\mathcal{X}}) \cong \mathrm{H}^*(\mathcal{O}_{Z^n})^{\mathfrak{S}_n} \cong (\mathrm{H}^*(\mathcal{O}_Z)^{\otimes n})^{\mathfrak{S}_n} \cong S^n(\mathrm{H}^*(\mathcal{O}_Z)) .$$

There are also plenty of other examples of stacks with $\mathbb{P}^n[k]$-units. Let $S \in \mathsf{K3}$ with $\iota \in S$ a nonsymplectic involution and $\iota^{[n]} \in \mathsf{Aut}(S^{[n]})$ the induced automorphism on the Hilbert scheme of n points on S. Then, for n even, the associated quotient stack $[X^{[n]}/\iota^{[n]}]$ has a $\mathbb{P}^{n/2}[4]$-unit. In contrast, if ι is fixed point free and

n is odd, $\iota^{[n]}$ is again fixed point free and the quotient $X^{[n]}/\iota^{[n]}$ is an OS Enriques variety; see [OS11, Prop. 4.1].

Also, all the constructions of the earlier sections lead to stacks with $\mathbb{P}^n[k]$-units if we replace the strict Enriques varieties by strict Enriques stacks.

6.5. **Derived invariance of strict Enriques varieties.** In [Abu15] Abuaf conjectured that the homological unit is a derived invariant of smooth projective varieties. This means that for two varieties X_1, X_2 with $\mathrm{D}(X_1) \cong \mathrm{D}(X_2)$, we should have an isomorphisms of $\mathbb{C}$-algebras $\mathrm{H}^*(\mathcal{O}_{X_1}) \cong \mathrm{H}^*(\mathcal{O}_{X_2})$.

In regard to this conjecture, one would like to prove that the class of varieties with $\mathbb{P}^n[k]$-units is stable under derived equivalences. This is true for $k = 2$: In [HNW11] it is shown that the class of compact hyperkähler manifolds is stable under derived equivalence. However, the methods of the proof do not seem to generalize to higher k. At least, we can use the result of [HNW11] in order to show that the class of strict Enrqiues varieties is derived stable.

Lemma 6.6. *Let E_1 be a strict Enriques variety of index $n+1$, and let E_2 be a Fourier–Mukai partner of E_2; i.e., E_2 is a smooth projective variety with* $\mathrm{D}(E_1) \cong \mathrm{D}(E_2)$. *Then E_2 is also a strict Enriques variety of the same index $n+1$.*

Proof. By Proposition 3.14, condition (S1) of a strict Enriques variety of index $n+1$ can be replaced by the condition $\mathsf{dim}\, E_1 = 2n$. The dimension of a variety and the order of its canonical bundle are derived invariants; see, e.g., [Huy06, Prop. 4.1]. Hence, also $\mathsf{dim}\, E_2 = 2n$ and $\mathsf{ord}\, \omega_{E_2} = n+1$.

It remains to show that the canonical cover $\widetilde{E_2}$ is again hyperkähler. Indeed, the equivalence $\mathrm{D}(E_1) \cong \mathrm{D}(E_2)$ lifts to an equivalence of the canonical covers $\mathrm{D}(\widetilde{E}_1) \cong \mathrm{D}(\widetilde{E_2})$ and the class of hyperkähler varieties is stable under derived equivalences; see [BM98] and [HNW11], respectively. $\square$

6.6. **Auto-equivalences of varieties with $\mathbb{P}^n[k]$-unit.** As mentioned in Remark 2.7, every $\mathbb{P}^n[k]$-object $E \in \mathrm{D}(X)$ induces an auto-equivalence, called $\mathbb{P}$-twist, $P_E \in \mathsf{Aut}(\mathrm{D}(X))$. This can be seen as a special case of [Add16, Thm. 3] or as a straightforward generalization of [HT06, Prop. 2.6]. We will describe the twist only in the special case $E = \mathcal{O}_X$. In particular, we assume that X has a $\mathbb{P}^n[k]$-unit. Then, by Remark 2.8, every line bundle $L \in \mathsf{Pic}\, X$ is a $\mathbb{P}^n[k]$-object too. However, it suffices to understand the twist $P_X := P_{\mathcal{O}_X}$ as we have $P_L = M_L P_X M_L^{-1}$ where $M_L = (\_) \otimes L$ is the auto-equivalence given by tensor product with L; see [Kru15, Lem. 2.4].

The $\mathbb{P}$-twist along $\mathcal{O}_X$ is constructed as the Fourier–Mukai transform $P_X := \mathsf{FM}_{\mathcal{Q}} \colon \mathrm{D}(X) \to \mathrm{D}(X)$ where

$$\mathcal{Q} = \mathrm{cone}\big(\mathrm{cone}(\mathcal{O}_{X\times X} \xrightarrow{x\boxtimes \mathrm{id} - \mathrm{id}\boxtimes x} \mathcal{O}_{X\times X}) \xrightarrow{r} \mathcal{O}_\Delta\big) \in \mathrm{D}(X \times X)\,.$$

Here, x is a generator of $\mathrm{H}^k(\mathcal{O}_X) \cong \mathrm{Hom}(\mathcal{O}_X[-k], \mathcal{O}_X)$ and $r\colon \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta$ is the restriction of sections to the diagonal. The double cone makes sense, since $r \circ (x \boxtimes \mathrm{id} - \mathrm{id} \boxtimes x) = 0$; see [HT06, Sect. 2] for details. On the level of objects $F \in \mathrm{D}(X)$, the twist P_X is given by

$$P_X(F) = \mathrm{cone}\Big(\mathrm{cone}\big(\mathrm{H}^*(F) \otimes \mathcal{O}_X[-k] \to \mathrm{H}^*(F) \otimes \mathcal{O}_X\big) \to F\Big)\,. \tag{16}$$

We summarize the main properties of the twist P_X in the following.

Proposition 6.7. *The $\mathbb{P}$-twist $P_X\colon \mathrm{D}(X) \to \mathrm{D}(X)$ is an auto-equivalence with the following properties.*

(i) $P_X(\mathcal{O}_X) = \mathcal{O}_X[-k(n+1)+2]$.
(ii) $P_X(F) = F$ *for* $F \in \mathcal{O}_X^{\perp} = \{F \in \mathrm{D}(X) \mid \mathrm{Hom}^*(\mathcal{O}_X, F) = 0\}$.
(iii) *Let* $\Phi \in \mathsf{Aut}(\mathrm{D}(X))$ *with* $\Phi(\mathcal{O}_X) = \mathcal{O}_X[m]$ *for some* $m \in \mathbb{Z}$*. Then the auto-equivalences* Φ *and* P_X *commute.*

Proof. For the first two properties, see [HT06, Sect. 2] or [Add16, Sect. 3.4 & 3.5]. Part (iii) follows from [Kru15, Lem. 2.4]. □

Lemma 6.8. *Let X be a variety with $\mathbb{P}^n[k]$-unit with $k \geq 2$ (not an elliptic curve). Let $Z_1, Z_2 \subset X$ be two disjoint closed subvarieties. and set*

$$F := R\mathcal{H}\mathrm{om}\big(P_X(\mathcal{O}_{Z_1}), P_X(\mathcal{O}_{Z_2})\big).$$

Then $\mathrm{Hom}^*(\mathcal{O}_X, F) = \mathrm{H}^*(F) = 0$ *and* $F \neq 0$*. In particular, the orthogonal complement of* $\mathcal{O}_X$ *is nontrivial.*

Proof. Clearly, $\mathrm{Hom}^*(\mathcal{O}_{Z_1}, \mathcal{O}_{Z_2}) = 0$. Using the fact that the equivalence P_X is, in particular, fully faithful and standard compatibilities between derived functors, we get

$$0 = \mathrm{Hom}^*\big(P_X(\mathcal{O}_{Z_1}), P_X(\mathcal{O}_{Z_2})\big) = \mathrm{Hom}^*\big(\mathcal{O}_X, R\mathcal{H}\mathrm{om}\big(P_X(\mathcal{O}_{Z_1}), P_X(\mathcal{O}_{Z_2})\big)\big)\,.$$

It is left to show that $F := R\mathcal{H}\mathrm{om}\big(P_X(\mathcal{O}_{Z_1}), P_X(\mathcal{O}_{Z_2})\big) \neq 0$. We denote by α_i the top nonzero degree of $\mathrm{H}^*(\mathcal{O}_{Z_i})$ for $i = 1, 2$. Let $V := X \setminus (Z_1 \cup Z_2)$. Then by (16) the cohomology of $P_X(\mathcal{O}_{Z_i})$ is concentrated in degrees between -1 and $\alpha_i + k - 2$ with $\mathcal{H}^{-1}(P_X(\mathcal{O}_{Z_i}))_{|V} \cong \mathcal{O}_V$ and $\mathcal{H}^{\alpha_i+k-2}(P_X(\mathcal{O}_{Z_i}))_V \cong \mathcal{O}_V \otimes \mathrm{H}^{\alpha_i}(\mathcal{O}_{Z_i})$. Hence, the spectral sequence

$$E_2^{p,q} = \oplus_i \mathcal{E}\mathrm{xt}^p\big(\mathcal{H}^i(P(\mathcal{O}_{Z_1})), \mathcal{H}^{i+q}(P(\mathcal{O}_{Z_1}))\big)_{|V} \quad\Longrightarrow\quad E^{p+q} = \mathcal{H}^{p+q}(F)_{|V}$$

is concentrated in the quadrant to the upper right of $(0, -\alpha_1 - k + 1)$. Furthermore, we have $E_2^{0,-\alpha_1-k+1} \cong \mathcal{O}_V \otimes \mathrm{H}^{\alpha_1}(\mathcal{O}_{Z_1}) \neq 0$. Hence $\mathcal{H}^{-\alpha_1-k+1}(F) \neq 0$. □

Let now X be obtained from strict Enriques varieties via the construction of section 4.3. This means that $X = (Y_1 \times \cdots \times Y_k)/G$ with $Y_i \in \mathsf{HK}_{2n}$ and

$$G = \big\{ f_1^{a_1} \times \cdots \times f_k^{a_k} \mid a_1 + \cdots + a_k \equiv 0 \mod n+1 \big\},$$

where the $f_i \in \mathsf{Aut}(Y_i)$ are purely nonsymplectic of order $n+1$. There are the $\mathbb{P}$-twists $P_{Y_i} := P_{\mathcal{O}_{Y_i}} \in \mathsf{Aut}(\mathrm{D}(Y_i))$ whose Fourier–Mukai kernels we denote by $\mathcal{Q}_i$. These induce auto-equivalences $P'_{Y_i} := \mathsf{FM}_{\mathcal{Q}'_i} \in \mathsf{Aut}(\mathrm{D}(Y_1 \times \cdots \times Y_k))$, where

$$\mathcal{Q}'_i = \mathcal{O}_{\Delta Y_1} \boxtimes \cdots \boxtimes \mathcal{Q}_i \boxtimes \cdots \boxtimes \mathcal{O}_{\Delta Y_k} \in \mathrm{D}\big((Y_1 \times Y_1) \times \cdots \times (Y_i \times Y_i) \times \cdots \times (Y_k \times Y_k)\big)\,.$$

We have

$$P'_{Y_i}(F_1 \boxtimes \cdots \boxtimes F_k) = F_1 \boxtimes \cdots \boxtimes P_{Y_i}(F_i) \boxtimes \cdots \boxtimes F_k\,. \tag{17}$$

In the following we will use the identification $\mathrm{D}(X) \cong \mathrm{D}_G(X')$ of the derived category of X with the derived category of G-linearized coherent sheaves on the cover $X' = Y_1 \times \cdots \times Y_k$; see, e.g., [BKR01, Sect. 4] or [KS15a] for details. One can check that the $\mathcal{Q}_i$ are $\langle f_i \rangle$-linearizable, hence the $\mathcal{Q}'_i$ are G-linearizable. It follows that the auto-equivalences P'_{Y_i} descend to auto-equivalences $\check{P}_{Y_i} \in \mathsf{Aut}(\mathrm{D}_G(X')) \cong \mathsf{Aut}(\mathrm{D}(X))$; see [KS15a, Thm. 1.1]. One might expect that the composition of the $\check{P}_{Y_i}$ equals P_X but this is not the case.

Proposition 6.9. *There is an injective group homomorphism* $\mathbb{Z}^{\oplus k+2} \hookrightarrow \mathsf{Aut}(\mathrm{D}(X))$ *given by*

$$e_{k+1} \mapsto P_X, \quad e_{k+2} \mapsto [1], \quad e_i \mapsto \check{P}_{Y_i} \quad \textit{for } i = 1, \dots, k.$$

Proof. Under the equivalence $\mathrm{D}(X) \cong \mathrm{D}_G(X')$, the structure sheaf $\mathcal{O}_X \in \mathrm{D}(X)$ corresponds to $\mathcal{O}_{X'} = \mathcal{O}_{Y_1} \boxtimes \cdots \boxtimes \mathcal{O}_{Y_k}$ equipped with the natural linearization. By (17) and Proposition 6.7(1), we get

$$\check{P}_{Y_i}(\mathcal{O}_X) \cong \mathcal{O}_{Y_1} \boxtimes \cdots \boxtimes (\mathcal{O}_{Y_i}[-2n]) \boxtimes \cdots \boxtimes \mathcal{O}_{Y_k} \cong \mathcal{O}_X[-2n]\,.$$

Hence, by Proposition 6.7(iii), the $\check{P}_{Y_i}$ commute with P_X. By a similar argument, one can see that the $\check{P}_i$ commute with one another. The shift functor [1] commutes with every auto-equivalence of the triangulated category $\mathrm{D}(X)$. In summary, we have shown by now that the homomorphism $\mathbb{Z}^{\oplus k+2} \to \mathsf{Aut}(\mathrm{D}(X))$ is well-defined.

For the injectivity, let us fix for every $i = 1, \dots, n$ a G-linearizable $F_i \in \mathcal{O}_{Y_i}^{\perp}$. For example, let Z_1 and Z_2 in Lemma 6.8 be two different $\langle f_i \rangle$-orbits in Y_i. Let $a_1, \dots, a_k, b, c \in \mathbb{Z}$, and set $\Psi := \check{P}_{Y_1}^{a_1} \circ \cdots \circ \check{P}_{Y_k}^{a_k} \circ P_X^b[c]$. By plugging various box-products of the $\mathcal{O}_{Y_i}$ and F_i into Ψ, we can show that $\Psi \cong \mathrm{id}$ implies $0 = a_1 = a_2 = \cdots = a_k = b = c$; this is very similar to computations done in [Add16, Sect. 1.4] or the proof of [KS15b, Prop. 3.18]. □

Remark 6.10. In the known examples, the Y_i are generalized Kummer varieties; compare section 3.4. In these cases, there are many more ℙ-objects in $\mathrm{D}(Y_i)$ which induce further auto-equivalences on X; see [Kru15, Sect. 6].

Corollary 6.11. *Let X be a variety with $\mathbb{P}^n[4]$-unit for $n \geq 3$. Then, there is an embedding $\mathbb{Z}^4 \subset \mathsf{Aut}(\mathrm{D}(X))$.*

Proof. By Theorem 5.8, we are in the situation of the above proposition. □

6.7. **Varieties with $\mathbb{P}^n[k]$-units as moduli spaces.** All the examples of varieties with $\mathbb{P}^n[k]$-units presented in this article are constructed out of examples of hyperkähler manifolds with special auto-equivalences, usually with the property that the quotients are strict Enriques varieties. Then the varieties with $\mathbb{P}^n[k]$-units are constructed as intermediate quotients between the product of the hyperkähler manifolds and the product of the quotients.

It would be very interesting to find ways to construct varieties X with $\mathbb{P}^n[k]$-units directly. In the case $k = 4$, by Proposition 5.3, the universal cover of such an X decomposes into two hyperkähler manifolds. Hence, one could hope to find in this way new examples of Enriques or even hyperkähler varieties.

For example, one could try to construct varieties with $\mathbb{P}^n[k]$ units as moduli spaces of sheaves (or objects) on varieties with trivial canonical bundle (or Calabi–Yau categories) of dimension k. Indeed, all of the examples that we found in this paper can be realized as moduli spaces.

For example, let A, B be abelian surfaces together with automorphisms $a \in \mathsf{Aut}(A)$ and $b \in \mathsf{Aut}(B)$. We set $Y := K_2A$, $Z := K_2B$, $f := K_2a$, $g := K_2b$, and assume that $Y/\langle f \rangle$ and $Z/\langle g \rangle$ are strict Enriques varieties of index 3. This implies that $X := (Y \times Z)/\langle f \times g \rangle$ has a $\mathbb{P}^2[4]$-unit; see Remark 4.6. As $Y = K_2A$ and $Z = K_2B$ are moduli spaces of sheaves on A and B, respectively, the product $Y \times Z$ is a moduli space of sheaves on $A \times B$. We denote the universal family by $\mathcal{F} \in \mathrm{Coh}(A \times B \times Y \times Z)$. This descends to a sheaf $\check{\mathcal{F}} \in \mathrm{Coh}((A \times B)/\langle a \times b \rangle \times X)$ which is flat over X with pairwise nonisomorphic fibers. One can deduce this from the fact that $\mathcal{F}$ is $\langle a \times b \times f \times g\rangle$-linearizable; compare [KS15a, Sect. 3]. Hence, we can consider X as a moduli space of sheaves on $(A \times B)/\langle a \times b\rangle$ with universal family $\check{\mathcal{F}}$.

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