CONVEX HULLS OF MULTIDIMENSIONAL RANDOM WALKS

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ABSTRACT. Let S_k be a random walk in $\mathbb{R}^d$ such that its distribution of increments does not assign mass to hyperplanes. We study the probability p_n that the convex hull $\mathrm{conv}(S_1, \ldots, S_n)$ of the first n steps of the walk does not include the origin. By providing an explicit formula, we show that for planar symmetrically distributed random walks, p_n does not depend on the distribution of increments. This extends the well-known result by Sparre Andersen (1949) that a one-dimensional random walk satisfying the above continuity and symmetry assumptions stays positive with a distribution-free probability. We also find the asymptotics of p_n as $n \to \infty$ for any planar random walk with zero mean square-integrable increments.

We further developed our approach from the planar case to study a wide class of geometric characteristics of convex hulls of random walks in any dimension $d \geq 2$. In particular, we give formulas for the expected value of the number of faces, the volume, the surface area, and other intrinsic volumes, including the following multidimensional generalization of the Spitzer–Widom formula (1961) on the perimeter of planar walks:

$$\mathbb{E}V_1(\mathrm{conv}(0, S_1, \ldots, S_n)) = \sum_{k=1}^{n} \frac{\mathbb{E}\|S_k\|}{k},$$

where V_1 denotes the first intrinsic volume, which is proportional to the mean width.

These results have applications to geometry and, in particular, imply the formula by Gao and Vitale (2001) for the intrinsic volumes of special path-simplexes, called canonical orthoschemes, which are finite-dimensional approximations of the closed convex hull of a Wiener spiral. Moreover, there is a direct connection between spherical intrinsic volumes of these simplexes and the probabilities p_n.

We also prove similar results for convex hulls of random walk bridges and, more generally, for partial sums of exchangeable random vectors.

1. INTRODUCTION AND RESULTS FOR PLANAR RANDOM WALKS

1.1. **Motivation.** Let $S_n = X_1 + \cdots + X_n$ be a random walk in $\mathbb{R}^d$. This paper was motivated by the following question: What is the probability that $\mathrm{conv}(S_1, \ldots, S_n)$,

the convex hull of the first n steps of the walk, does not include the origin? This is a natural high-dimensional generalization of the classical problem to find the probability that a one-dimensional random walk stays positive (or negative) by time n. In this paper we develop a combinatorial approach that answers the question in some particular cases and, importantly, allows one to obtain further results on expected geometric characteristics of the convex hull, including its expected number of faces, volume, surface area, and other intrinsic volumes. Most of our main results are presented in the form of *exact* non-asymptotic formulas.

Our interest in the probabilities $\mathbb{P}(0 \notin \operatorname{conv}(S_1, \dots, S_n))$ emerged from two different topics. First, we were interested in a multidimensional version of the one-dimensional *persistence problem* of finding the probability that a stochastic process (the random walk, in our case) stays above a certain level. Over the past ten years, such problems have drawn a lot of attention from both mathematical and theoretical physics communities; see the survey papers by Aurzada and Simon [2] and Bray et al. [5].

Second, we were aware of the direct connection to geometry: for random walks with Gaussian increments, $\frac{1}{2}\mathbb{P}(0 \notin \operatorname{conv}(S_1, \dots, S_n))$ equals the d-th *spherical intrinsic volume* of a certain path-simplex in $\mathbb{R}^n$ called the *canonical orthoscheme*. This simplex is defined as the convex hull of n vectors whose Gram matrix coincides with the covariance matrix of a standard Brownian motion sampled at times $1, \dots, n$. Spherical intrinsic volumes are spherical analogues of classical Euclidean intrinsic volumes. The details on this connection of our problem to geometry are explained below in Section 4, where we also discuss the other geometric properties of canonical orthoschemes.

We were also inspired by two famous results. By Sparre Andersen [27, Theorem 2], for any one-dimensional random walk with continuous symmetric distribution of increments,

$$(1) \qquad \qquad \mathbb{P}(S_1 > 0, \dots, S_n > 0) = \frac{(2n-1)!!}{(2n)!!}.$$

That is, the probability to stay positive does not depend on the distribution. The other distribution-free result, which is due to Wendel [36], also concerns symmetric distributions and describes convex hulls of independent identically distributed random vectors. Let $X_1, \dots, X_n$ be such random vectors in $\mathbb{R}^d$ that satisfy two additional assumptions:

$(H_0) \qquad \mathbb{P}(X_1 \in h) = 0$ for any hyperplane $h \subset \mathbb{R}^d$ passing through the origin,

and the distribution of X_1 is centrally symmetric, i.e.,

$$(S) \qquad \qquad \qquad X_1 \stackrel{d}{=} -X_1.$$

Then

$$(2) \qquad \qquad \mathbb{P}(0 \notin \operatorname{conv}(X_1, \dots, X_n)) = \frac{1}{2^{n-1}} \sum_{k=0}^{d-1} \binom{n-1}{k}.$$

Wendel assumed (H_0) to ensure that with probability one, $X_1, \dots, X_n$ are in *general position*; that is, any d of these vectors are a.s. linearly independent. We will need the stronger assumption

$(H) \qquad \qquad \mathbb{P}(X_1 \in h) = 0$ for any affine hyperplane $\quad h \subset \mathbb{R}^d,$

which in particular guarantees that any one-dimensional projection of X_1 has a continuous distribution. We will use this assumption throughout the paper.

1.2. **First results.** There is a similarity between the results of Sparre Andersen and Wendel that stems from the use of combinatorial arguments in their proofs. This motivated our first result, a distribution-free two-dimensional version of (1):

Theorem 1. *Let $d = 2$, and assume that* (H) *and* (S) *hold. Then*

$$(3) \qquad \mathbb{P}(0 \notin \mathrm{conv}(S_1, \ldots, S_n)) = \sum_{k=1}^{n} \frac{(2n - 2k - 1)!!}{k \cdot (2n - 2k)!!}.$$

Let us discuss some corollaries. Here and below we consider the asymptotics as $n \to \infty$. For two positive sequences a_n and b_n, the notation $a_n \sim b_n$ means that $\lim_{n \to \infty} a_n/b_n = 1$.

It is not hard to obtain from (3) (see Section 6 below) that

$$(4) \qquad \mathbb{P}(0 \notin \mathrm{conv}(S_1, \ldots, S_n)) \sim \frac{\log n}{\sqrt{\pi n}}, \qquad d = 2.$$

Note that this probability is of a higher order of asymptotics than its one-dimensional counterpart (1), where

$$(5) \qquad \frac{(2n - 1)!!}{(2n)!!} = \frac{\Gamma(n + 1/2)}{\Gamma(1/2)\Gamma(n + 1)} \sim \frac{1}{\sqrt{\pi n}}.$$

Further, since for symmetric random walks one has

$$\mathbb{P}(0 \notin \mathrm{conv}(S_1, \ldots, S_n)) = \mathbb{P}(-S_n \notin \mathrm{conv}(S_1 - S_n, \ldots, 0))$$
$$= \mathbb{P}(S_n \notin \mathrm{conv}(0, S_1, \ldots, S_{n-1})),$$

the expected *number of updates* of the convex hull is distribution-free and satisfies

$$\sum_{k=1}^{n} \mathbb{P}(S_k \notin \mathrm{conv}(0, S_1, \ldots, S_{k-1})) \sim \frac{\sqrt{n} \log n}{2\sqrt{\pi}}, \qquad d = 2.$$

The other quantity, which is closely related to the probabilities

$$\mathbb{P}(0 \notin \mathrm{conv}(S_1, \ldots, S_n)),$$

is the *opening solid angle*, denoted by Ω_n, of the convex hull observed from the origin. In the planar case we understand Ω_n as the arc angle, and so here $\Omega_n = 2\pi$ if 0 belongs to the interior of the convex hull and $\Omega_n \le \pi$ if otherwise.

It is easy to see[1] that

$$(6) \qquad \mathbb{E}\left(\frac{1}{2} - \frac{\Omega_n}{|\mathbb{S}^{d-1}|}\right)^{+} = \frac{1}{2}\mathbb{P}\left(0 \notin \mathrm{conv}(S_1, \ldots, S_n)|U^{\perp}\right),$$

where: $x^{+} := \max(0, x)$ for any real x; for any direction $u \in \mathbb{S}^{d-1}$, the notation $\cdot|u^{\perp}$ stands for the orthogonal projection onto the hyperplane $u^{\perp}$ passing through the origin that is orthogonal to u; and U is a random vector that is uniformly

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[1]Indeed, consider *any* set $A \subset \mathbb{R}^d$. If $0 \notin \mathrm{Int}(\mathrm{conv}(A))$, then by the definition of solid angle,

$$\Omega(\mathrm{conv}(A)) := \left|\left\{\frac{x}{|x|}, x \in \mathrm{conv}(A)\right\}\right| = \frac{1}{2}\int_{\mathbb{S}^{d-1}} \mathbb{1}\left(0 \in \mathrm{conv}(A)|u^{\perp}\right)\sigma(du),$$

hence $\frac{1}{2} - \frac{\Omega(\mathrm{conv}(A))}{|\mathbb{S}^{d-1}|} = \frac{1}{2}\mathbb{P}(0 \notin \mathrm{conv}(A|U^{\perp}))$. If $0 \in \mathrm{Int}(\mathrm{conv}(A))$, the l.h.s. in the last equality is negative (it equals $-1/2$) and the r.h.s. equals zero, and thus we have $(\frac{1}{2} - \frac{\Omega(\mathrm{conv}(A))}{|\mathbb{S}^{d-1}|})^{+} = \frac{1}{2}\mathbb{P}(0 \notin \mathrm{conv}(A|U^{\perp}))$.

distributed over the unit sphere $\mathbb{S}^{d-1}$ and independent of the random walk S_n. Since for any direction u, $\tilde{S}_n := S_n | u^\perp, n \geq 1$, is a $(d-1)$-dimensional random walk which satisfies assumptions (H) and (S) if the d-dimensional walk S_n does so, (6) combined with (1) and (3) implies the distribution-free relations

$$(7) \qquad \mathbb{E}\,(\pi - \Omega_n)^+ = 2\pi \frac{(2n-1)!!}{(2n)!!}, \qquad d = 2,$$

and

$$(8) \qquad \mathbb{E}\,(2\pi - \Omega_n)^+ = 2\pi \sum_{k=1}^{n} \frac{(2n-2k-1)!!}{k \cdot (2n-2k)!!}, \qquad d = 3.$$

Hence, under the assumptions of Theorem 1, the conditional expected discrepancy between the opening angles of the conic hull of $\mathrm{conv}(S_1, \ldots, S_n)$ and of a full half-plane containing the hull is also distribution-free and satisfies

$$\mathbb{E}\,\big(\pi - \Omega_n \,\big|\, 0 \notin \mathrm{conv}(S_1, \ldots, S_n)\big) \sim \frac{2\pi}{\log n}, \qquad d = 2.$$

The approach of the present paper does not allow one to generalize (3) to higher dimensions where it gives non-sharp upper bounds; see (18) and (16) in the next section. Based on numerical simulations for dimensions $d = 3$ and 4, which were further supported by (8) in dimension three, we suggest the following hypothesis.

Conjecture. *Let $d \geq 3$, and assume that* (H) *and* (S) *hold. Then the probabilities* $\mathbb{P}(0 \notin \mathrm{conv}(S_1, \ldots, S_n))$ *are distribution-free for any $n \geq 1$.*

Since the publication on arXiv of the first version of this paper, this conjecture has been fully resolved in our subsequent paper [12, Theorem 2.3], coauthored with Z. Kabluchko. This new work uses an entirely different method, which, however, applies only for perfectly symmetric distributions of increments.

1.3. **Asymptotic results for general planar random walks.** On the contrary, the approach of the present paper allows us, with an additional effort, to obtain an asymptotic version of Theorem 1 for *asymmetric* planar random walks. It also gives asymptotic upper estimates of the probabilities $\mathbb{P}(0 \notin \mathrm{conv}(S_1, \ldots, S_n))$ in higher dimensions. Surprisingly, for symmetric walks these bounds overestimate the true values merely by a constant factor; cf. (18) and (21) below with [12, Theorems 2.3 and 5.1].

We now present an asymptotic result for general random walks assuming that the increments have zero mean and a finite covariance matrix Σ. This matrix must be non-degenerate by assumption (H). For such walks, we introduce the following definitions. For any non-zero $u \in \mathbb{R}^d$, denote by $T(u) := \inf\{k \geq 1 : S_k \notin H(u)\}$ the exit time from the half-space $H(u) := \{z \in \mathbb{R}^d : \langle z, u \rangle \geq 0\}$, and let

$$(9) \qquad R(u) := -\frac{\mathbb{E}\langle S_{T(u)}, u \rangle}{\sqrt{\langle \Sigma u, u \rangle}}$$

be the expected normalized distance from $H(u)$ to the exit point $S_{T(u)}$. It is easy to see that this function is positive and angular, that is, independent of $|u|$. We will also show that $R(u)$ is bounded. For random walks with centrally symmetric distribution of increments satisfying (H), we have $R(u) \equiv \sqrt{2}/2$. This becomes clear

in the next paragraph if one uses that $T(u)$ is the exit time of the symmetrically distributed random walk $\langle S_n, u \rangle$ from the non-negative half-line.

In dimension one, $R(1)$ and $R(-1)$ are the expected values of ascending and descending, respectively, ladder heights of the random walk normalized by standard deviation of its increments. In this case it is well known (see the proof of Lemma 5 in the Appendix) that $R(\pm 1)$ are finite and

$$(10) \qquad \mathbb{P}(S_1 \geq 0, \ldots, S_n \geq 0) \sim \frac{\sqrt{2} R(1)}{\sqrt{\pi n}}, \quad d = 1.$$

This is actually true even without (H), while under (H), all the inequalities above can be taken to be strict. The asymptotic order here is the same in the symmetric case (where $R(\pm 1) = \sqrt{2}/2$; cf. (1) and (5)), but of course the probabilities are now distribution-dependent.

Theorem 2. *Let $d = 2$, and assume that* (H) *holds and that increments of the random walk S_n have zero mean and a finite covariance matrix Σ. Then*

$$\mathbb{P}(0 \notin \mathrm{conv}(S_1, \ldots, S_n)) \sim \sqrt{2} \mathbb{E} R(\Sigma^{-1/2} U) \frac{\log n}{\sqrt{\pi n}},$$

where U is a random vector distributed uniformly over the unit circle $\mathbb{S}^1$ and the expectation above is finite and positive.

1.4. **Further results and references.** We give a further development to the approach used for our initial problem discussed above. This allows us to obtain new results on a very wide class of *geometric characteristics of convex hulls* of general (not necessarily symmetric) multidimensional random walks. In particular, we provide explicit exact formulas for *expected intrinsic volumes* of the convex hull. For details we refer the reader directly to Section 3, with its main result given in Theorem 4, and the applications presented in Section 3.2.

Mean geometric characteristics of convex hulls of planar random walks, for example, the expected number of faces and the expected perimeter, were studied in many papers starting with Spitzer and Widom [28] and followed by a few other works that include Baxter [4], Snyder and Steele [24], and one of the most recent by Wade and Xu [35]. It seems that higher-dimensional versions were first considered by Barndorff-Nielsen and Baxter [3], whose work was overlooked by most of the followers, including us. To the best of our knowledge, the probabilities $\mathbb{P}(0 \notin \mathrm{conv}(S_1, \ldots, S_n))$ were not considered until the very recent works by Eldan [6] and Tikhomirov and Youssef [31], who obtain asymptotic estimates as dimension d increases to infinity for a few special types of random walks.

The paper by Abramson et al. [1] gives an overview and the latest account on the very fine description of the structure of the largest convex minorants of one-dimensional random walks. There are many related papers that consider random walks as the initial step in their studies of convex hulls of continuous time Lévy processes, and of course there is a huge number of works on convex hulls of Brownian motions. These topics are beyond the scope of our paper. A lot of references can be found in Pitman and Uribe Bravo [21]. There is a survey of results on random convex hulls by Majumdar et al. [17].

1.5. **Structure of the paper.** This paper is organized as follows. In the next section we present the main tool of our approach, a somewhat technical Proposition 1, which immediately implies Theorem 1 and its analogue for random walk

bridges, Theorem 3. The proposition also serves as the base for the further studies of geometric properties of convex hulls. We also present an asymptotic version of Proposition 1 for general random walks whose increments have zero mean and finite variance. This result is stated in Proposition 2 of Section 2, and it readily implies Theorem 2.

Section 3 contains Theorem 4, our general result on expected geometric characteristics of convex hulls of multidimensional random walks. This theorem also holds true for random walk bridges and more generally, for partial sums of exchangeable random vectors. As the main application of Theorem 4, we obtain explicit formulas for expected intrinsic volumes of convex hulls of random walks. In Section 4 we consider the special case of random walks with Gaussian increments and present a number of results that explain connections with geometry.

All the proofs are contained in the last two sections. In Section 5 we present our combinatorial results, which are used in Section 6 to prove Proposition 1 and Theorem 4. This last section also contains the proof of Proposition 2, which is based on several rather technical statements concerning uniform convergence. The proofs of these statements are moved to the Appendix since they are very different from the purely combinatorial or semi-combinatorial arguments used throughout the paper. However, we think that a *uniform* version of the *Tauberian theorem* (Theorem 5 of the Appendix) deserves some attention. To our surprise, we did not find a reference to any similar statement.

2. THE MAIN TOOL FOR MULTIDIMENSIONAL RANDOM WALKS

Denote by

$$C_n := \mathrm{conv}(S_0, S_1, \ldots, S_n)$$

the convex hull of the first n steps *including the origin* $S_0 := 0$. In the following consideration we will *always* refer to C_n as the convex hull of the random walk. To avoid trivialities, we assume that $n \geq d$; we also recall our convention that (H) is always satisfied.

With probability one C_n is a convex polytope with boundary of the form

$$(11) \qquad \partial C_n = \bigcup_{f \in \mathcal{F}_n} f,$$

where $\mathcal{F}_n$ is the set of all $(d-1)$-dimensional faces of C_n. Almost surely, each face f is a $(d-1)$-dimensional simplex of the form

$$(12) \qquad f = \mathrm{conv}(S_{i_1(f)}, \ldots, S_{i_d(f)})$$

for some indices $0 \leq i_1(f) < \cdots < i_d(f) \leq n$. It is instructive to think that f is obtained by shifting the simplex with vertices $0, S_{i_2(f)} - S_{i_1(f)}, \ldots, S_{i_d(f)} - S_{i_1(f)}$ by $S_{i_1(f)}$. We say that the ordered $(d-1)$-tuple $(i_2(f) - i_1(f), \ldots, i_d(f) - i_1(f))$ is the *temporal structure* of the face and the ordered d-tuple $(i_1(f), \ldots, i_d(f))$ is the *full temporal structure*.

We shall express the probability that $\mathcal{F}_n$ contains a face of a given temporal or full temporal structure. In order to stress the combinatorial nature of our result, we prove it in a more general setting for the partial sums $S_k = X_1 + \cdots + X_k, 1 \leq k \leq n$, of n-*exchangeable* increments $X_1, \ldots, X_n$. Recalling the definition, this means that for any permutation σ of length n, $(X_{\sigma(1)}, \ldots, X_{\sigma(n)})$ has the same distribution as

$(X_1, \ldots, X_n)$. We will assume that

(G) $\qquad\qquad \mathbb{P}(S_1, \ldots, S_n \text{ are in general position}) = 1$

to ensure that the faces of C_n still are simplexes with probability one. In other words, any d vectors of $S_1, \ldots, S_n$ are linearly independent. This is true, for example, when the exchangeable increments $X_1, \ldots, X_n$ have a joint density or when they are independent (so S_k is a random walk) and satisfy (H).

Our main example of partial sums with dependent n-exchangeable increments is a *random bridge* of length n, where we assume by definition that the n-th partial sum is a.s. zero. We will be interested in *random walk bridges* of two types. For a random walk S_k, the *difference* bridge is the sequence $S_k - (k/n)S_n, 1 \le k \le n$, and the distribution of the *conditional* bridge is given by conditioning on $S_n = 0$. We understand the latter as the well-defined limit of the corresponding conditional distributions $\mathbb{P}(\,\cdot\,\big|\,\|S_n\| \le r)$ as $r \to 0+$. For example, this limit exists if the distribution of increments of the walk has continuous or bounded density and the density of the distribution of S_n is positive at 0. It is easy to see that the first n values of a random walk bridge of length $n + 1$ of either type satisfy (G) if the underlying random walk satisfies (H).

It turns out that the probability that the convex hull C_n contains a face of a given *full* temporal structure is distribution-free for random walk bridges and for random walks with symmetrically distributed increments. Although this probability is not distribution-free for general walks, the probability that C_n contains a face of a given temporal structure is. More precisely, we have the following result.

Proposition 1. *For any $d \ge 1$, let $0 \le i_1 < \cdots < i_d \le n$ be any indices.*

(1) *If the partial sums S_k of n-exchangeable random vectors in $\mathbb{R}^d$ satisfy* (G), *then*

(13)
$$\sum_{i=0}^{n-i_d+i_1} \mathbb{P}(\operatorname{conv}(S_i, S_{i+i_2-i_1}, \ldots, S_{i+i_d-i_1}) \in \mathcal{F}_n)$$
$$= \frac{2}{(i_2 - i_1) \cdot \ldots \cdot (i_d - i_{d-1})}.$$

Moreover, if $S_1, \ldots, S_n, 0$ is a random bridge of length $n + 1$, then

(14) $\quad \mathbb{P}(\operatorname{conv}(S_{i_1}, \ldots, S_{i_d}) \in \mathcal{F}_n) = \dfrac{2}{(i_2 - i_1) \cdot \ldots \cdot (i_d - i_{d-1})(n - i_d + i_1 + 1)}.$

(2) *If S_k is a random walk[2] in $\mathbb{R}^d$ and* (H) *and* (S) *hold, then*

(15) $\quad \mathbb{P}(\operatorname{conv}(S_{i_1}, \ldots, S_{i_d}) \in \mathcal{F}_n) = 2\dfrac{(2i_1 - 1)!!}{(2i_1)!!}\dfrac{(2n - 2i_d - 1)!!}{(2n - 2i_d)!!}\prod_{k=1}^{d-1}\dfrac{1}{i_{k+1} - i_k},$

where by convention $(-1)!! = 1$.

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[2] As explained in Section 6, (15) is true for the partial sums of n-exchangeable random vectors $X_1, \ldots, X_n$ if (G) holds and all the 2^n n-tuples $(\pm X_1, \ldots, \pm X_n)$ have the same distribution; note that Wendel's result (2) also holds true under these relaxed assumptions. Here is an example of such distributions: if $d = 1$ so do the coordinates of any random vector X in $\mathbb{R}^n$ with a rotationally invariant distribution. In this case $X_1, \ldots, X_n$ are not i.i.d. unless X is a multiple of a standard Gaussian vector.

The first application of this result concerns the expected number of faces of C_n that contain the origin as a vertex. Denote the set of such faces by $\mathcal{F}'_n$ and note that (under (G)!)

$$\mathbb{1}(\mathcal{F}'_n \neq \varnothing) \overset{\text{a.s.}}{=} \mathbb{1}(0 \in \partial C_n) \overset{\text{a.s.}}{=} \mathbb{1}(0 \notin \text{conv}(S_1, \dots, S_n)).$$

Then (15) immediately implies that for symmetric random walks,

$$(16) \qquad \mathbb{E}|\mathcal{F}'_n| = 2 \sum_{1 \leq i_2 < \cdots < i_d \leq n} \frac{(2n - 2i_d - 1)!!}{i_2 \cdot (2n - 2i_d)!!} \prod_{k=2}^{d-1} \frac{1}{i_{k+1} - i_k}.$$

This proves Theorem 1 since for $d = 2$, we have

$$(17) \qquad |\mathcal{F}'_n| = \begin{cases} 2, & 0 \notin \text{conv}(S_1, \dots, S_n), \\ 0, & 0 \in \text{conv}(S_1, \dots, S_n). \end{cases}$$

In higher dimensions, (16) gives only an upper bound, as follows by

$$(18) \qquad \mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n)) \leq \mathbb{E}|\mathcal{F}'_n|/d.$$

By the same reasoning, from (14) we obtain the following version of Theorem 1 for random walk bridges.

Theorem 3. *Let $S_1, \dots, S_{n+1}$ be either the difference bridge or a well-defined conditional bridge (both of length $n + 1$) of a random walk in $\mathbb{R}^2$ that satisfies (H). Then*

$$\mathbb{P}(0 \notin \text{conv}(S_1, \dots, S_n)) = \sum_{k=1}^{n} \frac{1}{k(n - k + 1)}.$$

We stress that no additional assumption other than (H) is required.

For the asymptotics, it follows that (see Section 6) for a random walk under (S) and (H),

$$(19) \qquad \mathbb{E}_{RW}|\mathcal{F}'_n| \sim \frac{2(\log n)^{d-1}}{\sqrt{\pi n}},$$

while for a random walk bridge of length $n + 1$ (under (H)),

$$(20) \qquad \mathbb{E}_{Br}|\mathcal{F}'_n| \sim \frac{2d(\log n)^{d-1}}{n}.$$

We conclude this section with an asymptotic version of part (2) of Proposition 1 with $i_1 = 0$ for general (not necessarily symmetric) random walks. Recall that the function $R(u)$ was defined in (9).

Proposition 2. *Let S_k be a random walk in $\mathbb{R}^d, d \geq 2$, with increments that have zero mean, a finite covariance matrix Σ, and satisfy (H). Let U be a random vector distributed uniformly over the unit sphere $\mathbb{S}^{d-1}$. Then for any sequence h_n tending to infinity such that $h_n = o(n)$, we have*

$$\mathbb{P}(\text{conv}(0, S_{i_2}, \dots, S_{i_d}) \in \mathcal{F}_n) = \left(2\sqrt{\frac{2}{\pi}} + o(1)\right) \frac{\mathbb{E}R(\Sigma^{-1/2}U)}{i_2\sqrt{n - i_d + 1}} \prod_{k=2}^{d-1} \frac{1}{i_{k+1} - i_k}$$

uniformly in $1 \leq i_2 < \cdots < i_d \leq n$ such that $\min(i_2, i_3 - i_2, \dots, i_d - i_{d-1}, n - i_d) \geq h_n$, and the $o(1)$ term is uniformly bounded.

Similarly to (19), this gives (see Section 6) the asymptotics

$$\mathbb{E}_{RW}|\mathcal{F}_n'| \sim 2\sqrt{2}\mathbb{E}R(\Sigma^{-1/2}U)\frac{(\log n)^{d-1}}{\sqrt{\pi n}}. \tag{21}$$

Then Theorem 2 readily follows by (17).

3. Geometric properties of convex hulls in $\mathbb{R}^d$

3.1. **Expected additive functionals of faces.** For a further application of Proposition 1, we sum in (13) over all possible indices to obtain that the expected number of faces in the convex hull satisfies

$$\mathbb{E}|\mathcal{F}_n| = 2 \sum_{\substack{j_1 + \cdots + j_{d-1} \leq n \\ j_1, \ldots, j_{d-1} \geq 1}} \frac{1}{j_1 \cdot \ldots \cdot j_{d-1}}. \tag{22}$$

Comparing (22) and (14), we see that

$$\mathbb{E}^{(d)}|\mathcal{F}_n| = \sum_{k \leq n} \mathbb{E}_{Br}^{(d-1)}|\mathcal{F}_{k-1}'|,$$

where the upper indices show the dimension; hence by (20),

$$\mathbb{E}|\mathcal{F}_n| \sim 2(\log n)^{d-1}. \tag{23}$$

We stress that these formulas are valid under (G) only, and (S) is not required.

For $d = 2$, (22) was proved by Baxter [4]. We first generalized his argument to higher dimensions but then found a more direct and intuitive proof for part (1) of Proposition 1 presented below in Section 6. Later we discovered that such a generalization was already done by Barndorff-Nielsen and Baxter [3], who extended the proof of [4].

We followed the steps of Baxter [4] and Snyder and Steele [24] (both papers considered only the planar case) to obtain the following generalization of (22). Let $g : \mathbb{R}^{d \times (d-1)} \to \mathbb{R}$ be any non-negative Borel function. As we noted above, with probability one C_n is a convex polytope with faces of the form (12). Hence we can represent nearly any geometric property of a face f of C_n in terms of

$$g\left(S_{i_2(f)} - S_{i_1(f)}, \ldots, S_{i_d(f)} - S_{i_{d-1}(f)}\right)$$

for some symmetric function g. This quantity has the same expectation for all faces with the same temporal structure, and a conditional version of (13) (see (43) below) readily yields the following result.

Theorem 4. *Let S_k be partial sums of n-exchangeable random vectors in $\mathbb{R}^d$, $d \geq 1$. If* (G) *holds, then for*

$$G_n := \sum_{f \in \mathcal{F}_n} g\left(S_{i_2(f)} - S_{i_1(f)}, \ldots, S_{i_d(f)} - S_{i_{d-1}(f)}\right),$$

we have that

$$\mathbb{E}G_n = 2 \sum_{1 \leq i_1 < \cdots < i_{d-1} \leq n} \frac{\mathbb{E}g(S_{i_1}, S_{i_2} - S_{i_1}, \ldots, S_{i_{d-1}} - S_{i_{d-2}})}{i_1(i_2 - i_1) \cdot \ldots \cdot (i_{d-1} - i_{d-2})}.$$

Notice that if S_n is a random walk that satisfies (H), then the $d-1$ arguments of g in the definition of G_n are independent. In this case $\mathbb{E}G_n$ can be written as

$$(24) \qquad \mathbb{E}G_n = 2 \sum_{\substack{j_1+\cdots+j_{d-1}\leq n \\ j_1,\ldots,j_{d-1}\geq 1}} \frac{\mathbb{E}g\left(S_{j_1}^{(1)}, S_{j_2}^{(2)}, \ldots, S_{j_{d-1}}^{(d-1)}\right)}{j_1 \cdot \ldots \cdot j_{d-1}},$$

where $S_n^{(1)}, \ldots, S_n^{(d-1)}$ are independent copies of the random walk S_n.

We proved Theorem 4 being unaware of the work of Barndorff-Nielsen and Baxter [3], who gave no general statement of this type but did a similar consideration and obtained many of the results discussed in the next section as applications of Theorem 4. Our proof uses both combinatorial and probabilistic reasoning and, in our opinion, is more transparent than that of [3].

The latter proof rests on the smart combinatorial argument proposed by Baxter [4]. It is based on the simple fact that none of the $n!$ permutations of the increments $X_1, \ldots, X_n$ change the distribution of the partial sums. Similarly, Wendel's proof of (2) uses that all the 2^n possible n-tuples $(\pm X_1, \ldots, \pm X_n)$ have the same distribution. If this holds true, his argument works for any random vectors $X_1, \ldots, X_n$ in general position (so our assumption that X_i are i.i.d. is actually superfluous). Both proofs of Baxter and Wendel rely on the corresponding properties of *deterministic* sequences. Sparre Andersen's original proof of (1) does not allow such a nice description, as it combines a simple combinatorial argument with some clever counting which rests on additivity of probability. The widely known proof of this result given by Feller [8, Section XII.6] offers a much clearer combinatorial approach but heavily uses the independence of increments.

3.2. **Applications to intrinsic volumes of convex hulls.** Let us give some corollaries of Theorem 4. In this subsection we always assume that S_n is a random walk that satisfies (H) and impose no other conditions. As in (24), let $S_n^{(1)}, \ldots, S_n^{(d)}$ be independent copies of the walk S_n.

First of all, by considering $g(x_1, \ldots, x_{d-1}) \equiv 1$ in (24), we obtain the formula (22) for the expected number of faces of the convex hull of S_n. Less trivial applications are as follows.

Corollary 1 (Expected surface area)**.** *We have that*

$$\mathbb{E}\,\mathrm{Vol}_{d-1}(\partial C_n) = \frac{2}{(d-1)!} \sum_{\substack{j_1+\cdots+j_{d-1}\leq n \\ j_1,\ldots,j_{d-1}\geq 1}} \frac{\mathbb{E}\,\det^{1/2}\left(\left\langle S_{j_m}^{(m)}, S_{j_\ell}^{(\ell)}\right\rangle\right)_{m,\ell=1}^{d-1}}{j_1 \cdot \ldots \cdot j_{d-1}}.$$

For $d = 2$ this gives a formula by Spitzer and Widom [28] on the average perimeter:

$$(25) \qquad \mathbb{E}\,\mathrm{Vol}_1(\partial C_n) = 2\sum_{j=1}^{n} \frac{\mathbb{E}\|S_j\|}{j}.$$

The three-dimensional version of this result was first obtained by Barndorff-Nielsen and Baxter [3].

Proof. Applying (24) with the Gram determinant formula

$$(26)$$

$$g(x_1, \ldots, x_{d-1}) = \mathrm{Vol}_{d-1}(\mathrm{conv}(0, x_1, \ldots, x_{d-1})) = \frac{1}{(d-1)!} \sqrt{\det\left(\langle x_m, x_l \rangle\right)_{m,l=1}^{d-1}}$$

with $x_1, \ldots, x_{d-1} \in \mathbb{R}^d$, we get

$$\mathbb{E}\,\mathrm{Vol}_{d-1}(\partial C_n) = 2 \sum_{\substack{j_1 + \cdots + j_{d-1} \leq n \\ j_1, \ldots, j_{d-1} \geq 1}} \frac{\mathbb{E}\,\mathrm{Vol}_{d-1}\left(\mathrm{conv}\left(0, S_{j_1}^{(1)}, \ldots, S_{j_{d-1}}^{(d-1)}\right)\right)}{j_1 \cdot \ldots \cdot j_{d-1}}$$

$$= \frac{2}{(d-1)!} \sum_{\substack{j_1 + \cdots + j_{d-1} \leq n \\ j_1, \ldots, j_{d-1} \geq 1}} \frac{\mathbb{E}\,\det^{1/2}\left(\langle S_{j_m}^{(m)}, S_{j_\ell}^{(\ell)} \rangle\right)_{m,\ell=1}^{d-1}}{j_1 \cdot \ldots \cdot j_{d-1}}.$$

$$\square$$

Corollary 2 (Expected volume). *We have that*

$$(27) \qquad \mathbb{E}\,\mathrm{Vol}_d(C_n) = \frac{1}{d!} \sum_{\substack{j_1 + \cdots + j_d \leq n \\ j_1, \ldots, j_d \geq 1}} \frac{\mathbb{E}\left|\det\left[S_{j_1}^{(1)}, \ldots, S_{j_d}^{(d)}\right]\right|}{j_1 \cdot \ldots \cdot j_d}.$$

A version of this result was first obtained by Barndorff-Nielsen and Baxter [3].

Proof. Denote by $' : \mathbb{R}^{d+1} \to \mathbb{R}^d$ the projection onto the first d coordinates. Let $\tilde{S}_n$ be any $(d+1)$-dimensional random walk such that $\tilde{S}_n' = S_n$ and its last coordinate is distributed continuously and independently of S_n. The convex hull $\tilde{C}_n$ of $\tilde{S}_n$ satisfies $(\partial \tilde{C}_n)' = C_n$, and the pre-image under $'$ of any point from $\mathrm{Int}(C_n)$ consists of exactly two points. Together with (11) this gives

$$(28) \qquad 2\,\mathrm{Vol}_d(C_n) = \sum_{f \in \tilde{\mathcal{F}}_n} \mathrm{Vol}_d(f'),$$

where $\tilde{\mathcal{F}}_n$ denotes the set of faces of $\tilde{C}_n$.

Each face $f \in \tilde{\mathcal{F}}_n$ a.s. is a d-dimensional simplex in $\mathbb{R}^{d+1}$ with vertices $\tilde{S}_{i_1(f)}, \ldots, \tilde{S}_{i_{d+1}(f)}$, and so f' a.s. is a d-dimensional simplex in $\mathbb{R}^d$. Its volume is given by

$$\mathrm{Vol}_d(f') = g(\tilde{S}_{i_2(f)} - \tilde{S}_{i_1(f)}, \ldots, \tilde{S}_{i_{d+1}(f)} - \tilde{S}_{i_1(f)}),$$

where

$$g(x_1, \ldots, x_d) = \mathrm{Vol}_d(\mathrm{conv}(0, x_1', \ldots, x_d')) = \frac{1}{d!}\left|\det[x_1', \ldots, x_d']\right|$$

is defined for $x_1, \ldots, x_d \in \mathbb{R}^{d+1}$. The claim then follows by combining (28) with (24) applied for $\tilde{S}_n$ and the above given g. $\square$

The following approach unifies the examples considered above. The volume and the surface area of a convex set are the special cases of so-called *intrinsic volumes* $V_0, \ldots, V_d$, which naturally arise as the coefficients in the *Steiner formula*: for any convex set $K \subset \mathbb{R}^d$,

$$\mathrm{Vol}_d(K + rB_d) = \sum_{k=0}^d \kappa_{d-k} V_k(K) r^{d-k}, \quad r \geq 0,$$

where B_d denotes a d-dimensional unit ball and $\kappa_k := \pi^{k/2}/\Gamma(\frac{k}{2}+1)$ is the volume of B_k. In particular, $V_0(K) = 1, V_d(K) = \text{Vol}_d(K), 2V_{d-1}(K) = \text{Vol}_{d-1}(\partial K)$, and $V_1(K)$ equals the mean width of K divided by the constant $\frac{2\kappa_{d-1}}{d\kappa_d}$, which is the mean width of a unit segment in $\mathbb{R}^d$. The last statement readily follows from the other definition of intrinsic volumes, which sometimes is called the *Crofton formula*:

$$(29) \qquad V_k(K) := \binom{d}{k} \frac{\kappa_d}{\kappa_k \kappa_{d-k}} \int_{\mathcal{L}_k^d} \text{Vol}_k(K|L) \; d\mu_k(L),$$

where $\mathcal{L}_k^d$ is the Grassmannian of all k-dimensional linear subspaces of $\mathbb{R}^d$ equipped with the Haar probability measure μ_k, and $K|L$ is the orthogonal projection of K onto L.

Intuitively, the k-th intrinsic volume of K equals, up to the constant factor, the mean k-dimensional volume of the projection of K onto a uniformly chosen random k-dimensional linear subspace of $\mathbb{R}^d$. The normalization constant $\binom{d}{k}\frac{\kappa_d}{\kappa_k\kappa_{d-k}}$ is chosen so that the intrinsic volumes of K do not depend on whether we consider K as a subset of $\mathbb{R}^d$ or embed it in any higher-dimensional Euclidian space. For an extensive account on integral geometry we refer the reader to the books of Santaló [22] and of Schneider and Weil [23].

Corollary 3 (Expected intrinsic volumes). *We have*

$$\mathbb{E}V_k(C_n) = \frac{1}{k!} \sum_{\substack{j_1+\cdots+j_k \leq n \\ j_1,\ldots,j_k \geq 1}} \frac{\mathbb{E}\det^{1/2}\left(\langle S_{j_m}^{(m)}, S_{j_\ell}^{(\ell)} \rangle\right)_{m,\ell=1}^{d-1}}{j_1 \cdot \ldots \cdot j_k}, \quad k = 1,\ldots,d.$$

In particular, the Spitzer–Widom formula (25) *naturally extends to any dimension:*

$$\mathbb{E}V_1(C_n) = \sum_{j=1}^n \frac{\mathbb{E}\|S_j\|}{j}.$$

These results were already used by Molchanov and Wespi [19, Theorem 2.3] to compute intrinsic volumes of the closed convex hull of a symmetric α-stable Lévy process in $\mathbb{R}^d$ with $\alpha \in (1,2]$. For a standard Brownian motion, the intrinsic volumes V_1 and V_2 were found in the earlier paper by Kampf et al. [15].

Proof. For any $L \in \mathcal{L}_k^d$, the sequence $\tilde{S}_n := S_n|L, n \geq 0$, is a k-dimensional random walk satisfying (H), and its convex hull is $\tilde{C}_n = C_n|L$. Hence by Corollary 2, one has

$$\begin{aligned}
\mathbb{E}\,\text{Vol}_k(C_n|L) &= \frac{1}{k!} \sum_{\substack{j_1+\cdots+j_k \leq n \\ j_1,\ldots,j_k \geq 1}} \frac{\mathbb{E}\left|\det\left[\tilde{S}_{j_1}^{(1)}, \ldots, \tilde{S}_{j_k}^{(d)}\right]\right|}{j_1 \cdot \ldots \cdot j_d} \\
&= \sum_{\substack{j_1+\cdots+j_k \leq n \\ j_1,\ldots,j_k \geq 1}} \frac{\mathbb{E}\,\text{Vol}_k\left(\text{conv}\left(0, S_{j_1}^{(1)}, \ldots, S_{j_k}^{(k)}\right)|L\right)}{j_1 \cdot \ldots \cdot j_k}.
\end{aligned}$$

Integrate this equation over $\mathcal{L}_k^d$ with respect to μ_k, normalize according to the definition of the intrinsic volume, and apply the Fubini theorem to both sides to

get

$$\mathbb{E}V_k(C_n) = \sum_{\substack{j_1+\cdots+j_k \leq n \\ j_1,\ldots,j_k \geq 1}} \frac{\mathbb{E}V_k\big(\text{conv}\big(0, S_{j_1}^{(1)}, \ldots, S_{j_k}^{(k)}\big)\big)}{j_1 \cdot \ldots \cdot j_k}.$$

The linear dimension of $K = \text{conv}\big(0, S_{j_1}^{(1)}, \ldots, S_{j_k}^{(k)}\big)$, which is a convex hull of $k+1$ points, is k a.s.; hence $V_k(K) = \text{Vol}_k(K)$, and the claim follows if we use the Gram determinant formula (26). $\square$

4. Applications of Gaussian convex hulls to geometry

In this section we always assume that $X_1, \ldots, X_n$ are independent standard Gaussian vectors in $\mathbb{R}^d$.

4.1. **Intrinsic volumes of canonical orthoschemes.** Consider the Gaussian random $d \times n$ matrix A with the columns $X_1, \ldots, X_n$. Its rows $Y_1, \ldots, Y_d$ are standard Gaussian vectors in $\mathbb{R}^n$. It is known that the linear span of $Y_1, \ldots, Y_d$ (which are in general position with probability one) is a random d-dimensional linear subspace of $\mathbb{R}^n$ uniformly distributed on the Grassmannian $\mathcal{L}_d^n$ with respect to the Haar probability measure. Using this fact and the Crofton formula (29), it can be shown that for any convex body $K \subset \mathbb{R}^n$,

$$V_d(K) = \frac{(2\pi)^{d/2}}{d!\kappa_d} \mathbb{E}\,\text{Vol}_d(\text{conv}(\{Ax \, : \, x \in K\})).$$

This equation is a finite-dimensional version of a general result of Sudakov [30] (for $d = 1$) and Tsirelson [32–34] (for general d) on Gaussian measures in infinite-dimensional spaces.

Consider the simplex $T_n \subset \mathbb{R}^n$ with vertices

$$(0, 0, \ldots, 0), (1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, \ldots, 1),$$

which we call the *Schläfli canonical orthoscheme*. Such simplexes are also called *path-simplexes*.

Now

$$\text{conv}(\{Ax \, : \, x \in T_n\}) = C_n,$$

which implies that

$$(30) \qquad V_d(T_n) = \frac{(2\pi)^{d/2}}{d!\kappa_d} \mathbb{E}\,\text{Vol}_d(C_n).$$

Combining this equality with (27), we obtain

$$V_d(T_n) = \frac{(2\pi)^{d/2}}{(d!)^2\kappa_d} \sum_{\substack{j_1+\cdots+j_d \leq n \\ j_1,\ldots,j_d \geq 1}} \frac{\mathbb{E}\left|\det\left[S_{j_1}^{(1)}, \ldots, S_{j_d}^{(d)}\right]\right|}{j_1 \cdot \ldots \cdot j_d},$$

where $S_1^{(1)}, \ldots, S_d^{(d)}$ are independent standard Gaussian random walks in $\mathbb{R}^d$. Let M be a $d \times d$ matrix with independent standard normal entries. Then $\mathbb{E}|\det M| = \mathbb{E}\sqrt{\det(MM^\top)}$, where $MM^\top$ is a Wishart matrix whose determinant has a well-known distribution and moments. Hence (see for example Kabluchko and Zaporozhets [13])

$$\mathbb{E}\left|\det\left[S_{j_1}^{(1)}, \ldots, S_{j_d}^{(d)}\right]\right| = \frac{d!\kappa_d}{(2\pi)^{d/2}}\sqrt{j_1 \cdot \ldots \cdot j_d},$$

which implies that

$$
(31) \qquad V_d(T_n) = \frac{1}{d!} \sum_{\substack{j_1 + \cdots + j_d \leq n \\ j_1, \ldots, j_d \geq 1}} \frac{1}{\sqrt{j_1 \cdot \ldots \cdot j_d}}.
$$

This result was first obtained by Gao and Vitale [10], who considered a direct geometric approach using a formula for intrinsic volumes of convex polytopes. The simplex $T_n/\sqrt{n} \in \mathbb{R}^n$ is a finite-dimensional approximation of the closed convex hull T of a *Wiener spiral*[3] in a Hilbert space, which was introduced by Kolmogorov in 1940; [10] calls T the Brownian motion body. Note that T is isometric to the subset of non-increasing functions of $L_2[0,1]$ that take values in $[0,1]$. Gao and Vitale [10] used (31) to prove that

$$
V_d(T) = \frac{\kappa_d}{d!}.
$$

Due to Tsirelson [32–34], the normalized d-th intrinsic volume of T is equal to the expected volume of the convex hull of a d-dimensional Brownian motion; see Kabluchko and Zaporozhets [14] for details. The latter quantity was calculated by Eldan [7] using direct methods.

4.2. **Spherical intrinsic volumes of canonical orthoschemes.** Let us consider the unit sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. By saying that $K \subset \mathbb{S}^n$ is convex we mean that the conic hull of K in $\mathbb{R}^{n+1}$ is convex and line-free. Following Santaló (see [22, Section IV.4]), for a convex body K in $\mathbb{S}^n$ we can use a spherical counterpart of the Crofton formula (29) to define

$$
U_k(K) := \frac{1}{2} \int_{S_{n-k}^n} \mathbb{1}_{\{K \cap s \neq \varnothing\}} \, d\nu_{n-k}(s),
$$

where S_k^n denotes the space of k-dimensional great subspheres of $\mathbb{S}^n$ equipped with the rotationally invariant probability measure ν_k. The functionals U_k can be considered as *spherical* counterparts of Euclidean intrinsic volumes V_k. However, there are other possible definitions of spherical intrinsic volumes. For basic facts from spherical integral geometry we refer the reader to Gao et al. [9], McCoy and Tropp [18], and Schneider and Weil [23, Section 6.5].

Similarly to (30), it can be shown (see Götze et al. [11] for details) that

$$
U_d(\tilde{T}_n) = \frac{1}{2} \mathbb{P}(0 \in C_n),
$$

where $\tilde{T}_n$ denotes the intersection of the conic hull of T_n with $\mathbb{S}^{n-1}$. To eliminate any misunderstanding, by the spherical intrinsic volumes of the canonical orthoscheme T_n mentioned in the abstract and in Section 1 we mean exactly $U_d(\tilde{T}_n)$. It follows from (1) and Theorem 1 that

$$
U_1(\tilde{T}_n) = \frac{1}{2} - \frac{(2n-1)!!}{(2n)!!}, \quad U_2(\tilde{T}_n) = \frac{1}{2} - \sum_{k=1}^{n} \frac{(2n-2k-1)!!}{2k \cdot (2n-2k)!!}.
$$

As we explained in the introduction, the exact values of the other spherical intrinsic volumes of $\tilde{T}_n$ are not accessible by the method of this paper. They are available

---

[3]This is the deterministic curve $\{W(t), t \in [0,1]\}$, where W is a standard Wiener process, in the Hilbert space of square-integrable zero mean random variables.

in our most recent work [12], coauthored with Z. Kabluchko, where U_k are referred to as half-tail functionals.

5. COMBINATORIAL ARGUMENTS

For any $x_1, \ldots, x_n \in \mathbb{R}^d$, denote by

$$s_0 := 0, \; s_k := x_1 + \cdots + x_k, \quad k = 1, \ldots, n,$$

the sequence of partial sums. For any permutation $\sigma = (\sigma(1), \ldots, \sigma(n))$, denote

$$s_0(\sigma) := 0, s_k(\sigma) := x_{\sigma(1)} + \cdots + x_{\sigma(k)}, \; k = 1, \ldots, n.$$

We first proved a simple combinatorial statement which generalizes two-dimensional Lemma 1 from Baxter [4] to higher dimensions. Later we found this result in the paper by Barndorff-Nielsen and Baxter [3]. The one-dimensional version is known as the "cycle lemma"; for example, see Steele [29, Section 4] and references therein for further combinatorial applications. For the reader's convenience we present the proof here.

Lemma 1. *Let $x_0, x_1, \ldots, x_n \in \mathbb{R}^d$, and let H be a closed half-space such that*

$$x_0, x_0 + s_n \in \partial H \quad and \quad x_0 + s_j - s_i \notin \partial H, 0 \leq i < j \leq n - 1.$$

There exists exactly one cyclic permutation $\sigma = (k+1, \ldots, n, 1, \ldots, k)$ such that

$$x_0, x_0 + s_1(\sigma), \ldots, x_0 + s_n(\sigma) \in H.$$

Proof. By the assumption, there exists exactly one point $x_0 + s_k$ among

$$\{x_0 + s_i\}_{i=0}^{n-1} \cap (\text{Int}(H))^c$$

that is at the maximum distance (possibly zero) from ∂H. Then $\sigma := (k + 1, \ldots, n, 1, \ldots, k)$ is a required permutation, and it is unique by the uniqueness of k. $\square$

Our next goal is to obtain stochastic versions of this result. For any points $x_1, \ldots, x_d \in \mathbb{R}^d$, define

$$H_\pm(x_1, \ldots, x_d) := \{z \in \mathbb{R}^d : \pm \det[x_2 - x_1, \ldots, x_d - x_1, z - x_1] \geq 0\}.$$

If there is a unique hyperplane through these points, then this definition gives a rule to distinguish between the two half-spaces H_+ and H_- lying on different sides of the hyperplane. If such a hyperplane is not unique, then $H_\pm = \mathbb{R}^d$.

Lemma 2. *Assume that the partial sums S_k of n-exchangeable random vectors $X_1, \ldots, X_n$ in $\mathbb{R}^d, d \geq 2$, satisfy (G). For any indices $1 \leq i_1 < \cdots < i_{d-2} \leq n - 1$, we have*

$$\mathbb{P}\big(S_1, \ldots, S_n \in H_\pm(0, S_{i_1}, \ldots, S_{i_{d-2}}, S_n)\big) = \frac{1}{i_1(i_2 - i_1) \cdot \ldots \cdot (n - i_{d-2})}$$

and, moreover,

$$\mathbb{P}\big(S_1, \ldots, S_n \in H_\pm(0, S_{i_1}, \ldots, S_{i_{d-2}}, S_n)\big| S_{i_1}, \ldots, S_{i_{d-2}}, S_n\big)$$
$$= \frac{1}{i_1(i_2 - i_1) \cdot \ldots \cdot (n - i_{d-2})} \; a.s.$$

This is a little generalization of the well-known fact that the trajectory of any continuously distributed one-dimensional random walk S_n lies above the line joining $(0,0)$ and (n, S_n) with probability $1/n$; see Feller [8, Section XII.9]. The fact follows from Lemma 2 if we consider the two-dimensional walk $\tilde{S}_n := (n, S_n)$ with deterministic first component.

Proof. With probability one, there exists a unique half-plane through $0, S_{i_1}, \ldots, S_{i_{d-2}}, S_n$ (otherwise we could add any other point S_k and arrive at a contradiction with (G).) Hence almost surely,

$$H_\pm(S) := H_\pm(0, S_{i_1}, \ldots, S_{i_{d-2}}, S_n)$$

are half-spaces.

For any permutation $\sigma = (\sigma(1), \ldots, \sigma(n))$, introduce the partial sums

$$S_0(\sigma) := 0, \ S_k(\sigma) := X_{\sigma(1)} + \cdots + X_{\sigma(k)}, \quad k = 1, \ldots, n.$$

Put $i_0 := 0, i_{d-1} := n$, and denote by $\mathcal{S}$ the set of $(i_1 - i_0) \cdot \ldots \cdot (i_{d-1} - i_{d-2})$ permutations of length n that are products over j from 1 to $d-1$ of cyclic permutations of the form

$$(32) \qquad\qquad\qquad (k_j + 1, \ldots, i_j, i_{j-1} + 1, \ldots, k_j),$$

where $i_{j-1} + 1 \leq k_j \leq i_j$. Note that any $\sigma \in \mathcal{S}$ does not change $H_\pm$, i.e., $H_\pm(S) = H_\pm(S(\sigma))$, since for every $k \in \{i_1, \ldots, i_{d-2}, n\}$ one has $S_k = S_k(\sigma)$, and the sequences of partial sums S and $S(\sigma)$ have the same distribution by the exchangeability of the increments.

For any $0 \leq j \leq d-2$, the random vectors $S_{i_j}, X_{i_j+1}, \ldots, X_{i_{j+1}}$ and the half-space $H_\pm(0, S_{i_1}, \ldots, S_{i_{d-2}}, S_n)$ satisfy the assumption of Lemma 1 with probability one. Indeed, if for some $i_j \leq m < \ell < i_{j+1}$, one has $S_{i_j} + S_\ell - S_m \in \partial H_\pm(S)$ with positive probability, then among the partial sums $S_k(\sigma)$ with

$$\sigma = (1, \ldots, i_j, m+1, \ldots, \ell, i_j + 1, i_j + m, \ell + 1, \ldots, n)$$

there are d points $S_{i_1}(\sigma), \ldots, S_{i_{d-1}}(\sigma), S_{i_j+\ell-m}(\sigma)$ that belong to the hyperplane $\partial H_\pm$ passing through 0, which contradicts (G) by the exchangeability of increments.

By Lemma 1, there exists an a.s. unique random permutation $\sigma_\pm = \sigma_\pm(S) \in \mathcal{S}$ such that $S_1(\sigma_\pm), \ldots, S_n(\sigma_\pm) \in H_\pm(S(\sigma_\pm)) = H_\pm(S)$. Hence the sum in the r.h.s. of the equality
(33)

$$\mathbb{P}\big(S_1, \ldots, S_n \in H_\pm(S_{i_0}, S_{i_1}, \ldots, S_{i_{d-1}})\big) = \frac{1}{|\mathcal{S}|}\mathbb{E}\left[\sum_{\sigma \in \mathcal{S}} \mathbb{1}\big(S_1(\sigma), \ldots, S_n(\sigma) \in H_\pm(S)\big)\right]$$

equals one a.s. This proves the first assertion of the lemma. Similarly, for any non-negative Borel function $g : \mathbb{R}^{d \times (d-1)} \to \mathbb{R}$, we have

$$\mathbb{E}\Big[g(S_{i_1}, \ldots, S_{i_{d-2}}, S_n)\mathbb{1}\big(S_1, \ldots, S_n \in H_\pm(S_{i_0}, S_{i_1}, \ldots, S_{i_{d-1}})\big)\Big]$$

$$= \frac{1}{|\mathcal{S}|}\mathbb{E}\Big[g(S_{i_1}, \ldots, S_{i_{d-2}}, S_n) \sum_{\sigma \in \mathcal{S}} \mathbb{1}\big(S_1(\sigma), \ldots, S_n(\sigma) \in H_\pm(S)\big)\Big]$$

$$= \frac{1}{|\mathcal{S}|}\mathbb{E}g(S_{i_1}, \ldots, S_{i_{d-2}}, S_n),$$

and the second claim of the lemma follows by the definition of conditional expectation. $\square$

We conclude this section with a result on random bridges.

Lemma 3. *Let S_k be a random bridge of length $n + 1$ in $\mathbb{R}^d, d \geq 2$, such that $S_1, \ldots, S_n$ satisfy* (G). *For any indices $0 \leq i_1 < \cdots < i_{d-1} \leq n$, we have*

$$\mathbb{P}\big(S_1, \ldots, S_n \in H_\pm(0, S_{i_1}, \ldots, S_{i_{d-1}})\big) = \frac{1}{i_1(i_2 - i_1) \cdot \ldots \cdot (n - i_{d-1} + 1)}.$$

The above also holds true if S_k is either the difference bridge or a well-defined conditional bridge of a random walk in $\mathbb{R}^d$ that satisfies (H).

This is a multidimensional counterpart of the fact that in dimension one, a bridge of length n of a continuously distributed random walk stays positive with probability $1/n$.

Proof. As we already mentioned in Section 2, by our understanding of the conditioning it is clear that a conditional random walk bridge satisfies (G) if the increments of the underlying random walk satisfy (H). It is also easy to see that the difference bridge of such a random walk satisfies (G). Thus the latter assumptions hold true in all cases.

By repeating the argument used in the proof of Lemma 2, we see that (33) holds for $i_0 = 0$ and $\mathcal{S}$ defined to be the set of permutations of length $n + 1$ that are products of d cyclic permutations of the form (32), where $0 \leq j \leq d - 1$ and $i_d = n + 1$. $\qquad\square$

6. PROOFS

Proof of Proposition 1. Recall that $0 \leq i_1 < \cdots < i_d \leq n$. By (11) and (12),

$$(34) \quad \mathbb{P}(\text{conv}(S_{i_1}, \ldots, S_{i_d}) \in \mathcal{F}_n) = \mathbb{P}(0, S_1, \ldots, S_n \in H_+(S_{i_1}, \ldots, S_{i_d}))$$
$$+ \mathbb{P}(0, S_1, \ldots, S_n \in H_-(S_{i_1}, \ldots, S_{i_d})).$$

Denote

$$H_\pm := H_\pm(0, S_{i_2} - S_{i_1}, \ldots, S_{i_d} - S_{i_1}).$$

Then by $S_{i_d} - S_{i_1} \in \partial H_\pm$, we have

$$(35) \qquad H_+(S_{i_1}, \ldots, S_{i_d}) = H_\pm + S_{i_1} = H_\pm + S_{i_d}.$$

Proof of (14). Here S_k, $1 \leq k \leq n + 1$, with $S_{n+1} := 0$, is a random bridge of length $n + 1$. Let us consider the transformation that translates the whole trajectory of the bridge by moving the origin to S_{i_1}. The transformed trajectory corresponds to the random bridge of partial sums $S_k(\sigma)$, $1 \leq k \leq n + 1$, with $\sigma = (i_1 + 1, \ldots, n + 1, 1, \ldots, i_1)$. By the first equality in (35), we have

$$\big\{0, S_1, \ldots, S_n \in H_\pm(S_{i_1}, \ldots, S_{i_d})\big\}$$
$$= \big\{S_{n+1} - S_{i_1}, S_{n+1} - S_{i_1} + S_1, \ldots, 0, S_{i_1+1} - S_{i_1}, \ldots, S_n - S_{i_1} \in H_\pm\big\}$$
$$= \big\{S_{n+1-i_1}(\sigma), S_{n+2-i_1}(\sigma), \ldots, 0, S_1(\sigma), \ldots, S_{n-i_1}(\sigma)$$
$$\in H_\pm(0, S_{i_2-i_1}(\sigma), \ldots, S_{i_d-i_1}(\sigma))\big\};$$

hence

$$\mathbb{P}(0, S_1, \ldots, S_n \in H_\pm(S_{i_1}, \ldots, S_{i_d})) = \mathbb{P}(S_1, \ldots, S_n \in H_\pm(0, S_{i_2-i_1}, \ldots, S_{i_d-i_1})),$$

and then (14) follows by (34) and Lemma 3.

Proof of (15). Let us split the trajectory of S_n into three parts: by (35), we have

$$(36) \quad \{0, S_1, \ldots, S_n \in H_\pm(S_{i_1}, \ldots, S_{i_d})\} = \{0, S_1, \ldots, S_{i_1} \in H_\pm + S_{i_1};$$
$$S_{i_1+1}, \ldots, S_{i_d} \in H_\pm + S_{i_1} ; S_{i_d+1}, \ldots, S_n \in H_\pm + S_{i_d}\}.$$

Since S_n is a random walk, by conditioning on $X_{i_1+1}, \ldots, X_{i_d}$, which define $H_\pm$, and using the independence of increments, we obtain that

$$(37) \qquad \mathbb{P}\big(0, S_1, \ldots, S_n \in H_\pm(S_{i_1}, \ldots, S_{i_d})\big)$$
$$= \mathbb{P}\big(-S'_{i_1}, -S'_{i_1-1}, \ldots, 0 \in H_\pm\big) \mathbb{P}\big(S_{i_1+1} - S_{i_1}, \ldots, S_{i_d} - S_{i_1} \in H_\pm\big)$$
$$\times \mathbb{P}\big(S'_1, \ldots, S'_{n-i_d} \in H_\pm\big)$$
$$= \mathbb{P}\big(S'_1, \ldots, S'_{i_1} \in H_\mp\big) \mathbb{P}\big(S'_1, \ldots, S'_{n-i_d} \in H_\pm\big)$$
$$\times \mathbb{P}\big(S_1, \ldots, S_{i_d-i_1} \in H_\pm(0, S_{i_2-i_1}, \ldots, S_{i_d-i_1})\big),$$

where S'_n is an independent copy of the random walk S_n. Let us stress that we obtained (37) assuming that S_n is a random walk satisfying (H) but not (S). Then (15) holds by (34), Lemma 2, and the following simple result.

Lemma 4. *Let S_n be a random walk in $\mathbb{R}^d$, and let H be a half-space such that $0 \in \partial H$. Assume that* (H) *and* (S) *hold. Then*

$$\mathbb{P}(S_1, \ldots, S_n \in H) = \frac{(2n-1)!!}{(2n)!!}.$$

Proof. Denote by $u = u_H$ the unit vector that is orthogonal to ∂H and belongs to H. The distribution of increments of the one-dimensional random walk $S_k^{(u)} := \langle S_k, u \rangle, k \geq 1$, is continuous and symmetric; hence the result follows by (1) and

$$\mathbb{P}(S_1, \ldots, S_n \in H) = \mathbb{P}(S_1^{(u)} > 0, \ldots, S_n^{(u)} > 0). \qquad \square$$

Proof of (13). If $i_1 \neq 0$, we transform the trajectory by interchanging its part from 1 to i_1 with the part from $i_1 + 1$ to i_d; this does not change the part from $i_d + 1$ to n. See Figure 1, where the parts are denoted by $\mathbf{T_1}$, $\mathbf{T_2}$, and $\mathbf{T_3}$, respectively. The key observation is that for the transformed trajectory, S_{i_d} becomes the most distant point from $H_\pm = H_\pm(0, S_{i_2} - S_{i_1}, \ldots, S_{i_d} - S_{i_1})$. Let us prove this formally.

If $i_1 \neq 0$, we rewrite the event $\{0, S_1, \ldots, S_n \in H_\pm(S_{i_1}, \ldots, S_{i_d})\}$ in terms of the partial sums $S_k(\sigma), 1 \leq k \leq n$, with

$$\sigma = (i_1 + 1, \ldots, i_d, 1, \ldots, i_1, i_d + 1, \ldots, n).$$

For parts $\mathbf{T_1}$ and $\mathbf{T_3}$ of the trajectory, we use (35) to obtain

$$\{0, S_1, \ldots, S_{i_1}, S_{i_d+1}, \ldots, S_n \in H_\pm(S_{i_1}, \ldots, S_{i_d})\}$$
$$= \{0, S_1, \ldots, S_{i_1} \in H_\pm + S_{i_1} ; 0, S_{i_d+1}, \ldots, S_n \in H_\pm + S_{i_d}\}$$
$$= \{S_{i_d} - S_{i_1}, S_{i_d} - S_{i_1} + S_1, \ldots, S_{i_d}, 0, S_{i_d+1}, \ldots, S_n \in H_\pm + S_{i_d}\}$$
$$= \{0, S_{i_d-i_1+1}(\sigma), \ldots, S_n(\sigma) \in H_\pm(0, S_{i_2-i_1}(\sigma), \ldots, S_{i_d-i_1}(\sigma)) + S_{i_d}(\sigma)\}.$$

Note that the event in the second line differs from the corresponding part of (36) since we added 0 to the second group of variables. For part $\mathbf{T_2}$,

$$\{S_{i_1+1}, \ldots, S_{i_d} \in H_\pm(S_{i_1}, \ldots, S_{i_d})\}$$
$$= \{S_1(\sigma), \ldots, S_{i_d-i_1}(\sigma) \in H_\pm(0, S_{i_2-i_1}(\sigma), \ldots, S_{i_d-i_1}(\sigma))\}.$$
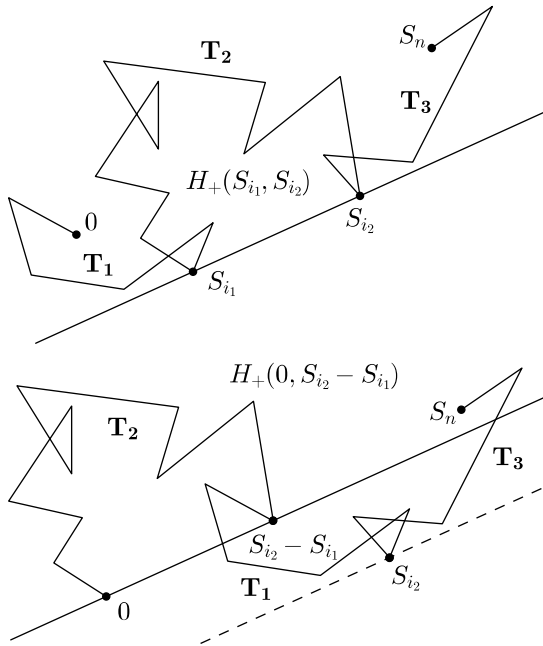
FIGURE 1. The path transform for $d = 2$.

Combining the above gives

(38)
$$\mathbb{P}\big(0, S_1, \ldots, S_n \in H_\pm(S_{i_1}, \ldots, S_{i_d})\big) = \mathbb{P}\Big(S_1, \ldots, S_{i_d - i_1} \in H_\pm(0, S_{i_2 - i_1}, \ldots, S_{i_d - i_1});$$

$$0, S_{i_d - i_1 + 1}, \ldots, S_n \in H_\pm(0, S_{i_2 - i_1}, \ldots, S_{i_d - i_1}) + S_{i_d}\Big).$$

By projecting on the orthogonal compliment, we see that S_{i_d} is a most distant point from $H_\pm(\ldots)$ among $0, S_{i_d - i_1 + 1}, \ldots, S_n$. Such a point is a.s. unique by assumption (G). Note that (38) is also valid for $i_1 = 0$ since in this case $H_\pm + S_{i_d} = H_\pm$. Therefore, (38) can be written for all $0 \le i_1 < i_2$ as

(39)
$$\mathbb{P}\big(0, S_1, \ldots, S_n \in H_\pm(S_{i_1}, \ldots, S_{i_d})\big) = \mathbb{P}\Big(S_1, \ldots, S_{i_d - i_1} \in H_\pm(0, S_{i_2 - i_1}, \ldots, S_{i_d - i_1});$$

$$\operatorname*{argmin}_{0 \le k \le n - (i_d - i_1)} \det[S_{i_2 - i_1}, \ldots, S_{i_d - i_1}, S_{i_d - i_1 + k}] = \{i_1\}\Big),$$

For a fixed temporal structure, i.e., the tuple $(i_2 - i_1, \ldots, i_d - i_1)$, it remains to sum in (39) over i_1 from 0 to $n - (i_d - i_1)$. The argmin disappears, and (13) follows by (34) and Lemma 2. Proposition 1 is now proved.

$\square$

Remark on footnote 2. Note that under assumptions made, (1) does hold true in the one-dimensional case; see Sparre Andersen [25, Theorem 1] or [26, Theorem 4]. Strictly speaking, both theorems are stated under slightly stronger assumptions which actually can be weakened to fit our requirements. The latter

theorem yields (15) when applied for the partial sums of $(n - i_d + i_1)$-exchangeable one-dimensional increments $X'_k := \pm \det[S_{i_2-i_1}, \ldots, S_{i_d-i_1}, X_{i_d-i_1+k}]$ (where $1 \leq k \leq n - i_d + i_1$) with $n - i_d + i_1$ substituted for n and the first event in the r.h.s. of (39) substituted for the C_n of [26, Theorem 4].

Proof of Proposition 2. The main ingredient is the following asymptotic version of Lemma 4. Recall that for any half-space H of $\mathbb{R}^d$, by u_H we denote the unit vector that is orthogonal to ∂H and belongs to H and, conversely, for any non-zero $u \in \mathbb{R}^d$, we put $H(u) = \{z \in \mathbb{R}^d : \langle z, u \rangle \geq 0\}$.

Lemma 5. *Let S_k be a random walk in $\mathbb{R}^d, d \geq 1$, with increments that have zero mean, a finite covariance matrix Σ, and satisfy* (H). *Then*

$$(40) \qquad \lim_{n \to \infty} \sqrt{n} \mathbb{P}(S_1, \ldots, S_n \in H) = \sqrt{\frac{2}{\pi}} R(u_H)$$

uniformly over all half-spaces H of $\mathbb{R}^d$ such that $0 \in \partial H$. The limit function $R(u)$ is continuous and positive on $\mathbb{S}^{d-1}$.

We will see that the pointwise convergence in (40) holds by a simple reduction to the well-known one-dimensional result of fluctuation theory. The difficulty is in showing that the convergence is uniform. Since this is quite a technical statement and the main message of our paper is in combinatorial methods, we postpone the proof of Lemma 5 until the Appendix.

Let us conclude the proof of Proposition 2. We first recall that the cross product of $x_1, \ldots, x_{d-1} \in \mathbb{R}^d$ is given by $x_1 \times \cdots \times x_{d-1} = \sum_{k=1}^{d} \det[x_1, \ldots, x_{d-1}, e_k] e_k$, where $e_1, \ldots, e_d$ is the standard basis of $\mathbb{R}^d$. Now consider (37) with $i_1 = 0$. The first probability in the last equation in (37) does not appear. Conditioning on $S_{i_2}, S_{i_3}, \ldots, S_{i_d}$, which determine

$$u_{\pm} := \pm S_{i_2} \times \cdots \times S_{i_d} = \pm (S_{i_2} - S_{i_1}) \times \cdots \times (S_{i_d} - S_{i_{d-1}})$$

and thus fix $H(u_{\pm}) = H_{\pm}(0, S_{i_2}, \ldots, S_{i_d})$, and using Lemma 2 for the third probability in (37), we get

$$\frac{\mathbb{P}(0, S_1, \ldots, S_n \in H_{\pm}(0, S_{i_2}, \ldots, S_{i_d}))}{i_2(i_3 - i_2) \cdot \ldots \cdot (i_d - i_{d-1})} = \mathbb{E}\big[\mathbb{P}\big(S'_1, \ldots, S'_{n-i_d} \in H(u_{\pm}) \big| u_{\pm}\big)\big].$$

The r.h.s. is $O(\frac{1}{\sqrt{n-i_d+1}})$ by Lemma 5, implying the required uniform boundedness of the $o(1)$ term in Proposition 2. Applying Lemma 5 one more time gives that

$$(41)$$
$$\frac{\mathbb{P}(0, S_1, \ldots, S_n \in H_{\pm}(0, S_{i_2}, \ldots, S_{i_d}))}{i_2(i_3 - i_2) \cdot \ldots \cdot (i_d - i_{d-1})} = \Big(\sqrt{\frac{2}{\pi}} + o(1)\Big) \frac{\mathbb{E} R(\pm S_{i_2} \times \cdots \times S_{i_d-i_{d-1}})}{\sqrt{n - i_d}}$$

uniformly in $1 \leq i_2 < i_3 < \cdots < i_d \leq n - h_n$, since by its definition, $R(u)$ is an angular function and $u_{\pm}$ is a.s. non-zero by assumption (H). Then

$$\mathbb{E} R\big(\pm S_{i_2} \times \cdots \times S_{i_d-i_{d-1}}\big) = \mathbb{E} R\Big(\pm \frac{S_{i_2}^{(1)}}{\sqrt{i_2}} \times \frac{S_{i_3-i_2}^{(2)}}{\sqrt{i_3 - i_2}} \times \cdots \times \frac{S_{i_d-i_{d-1}}^{(d-1)}}{\sqrt{i_d - i_{d-1}}}\Big),$$

where recall that $S_k^{(1)}, \ldots, S_k^{(d-1)}$ are independent copies of the random walk S_k.

Let $N_1, \ldots, N_{d-1}$ be independent standard Gaussian random vectors in $\mathbb{R}^d$. Since $R(u)$ is a continuous bounded function on $\mathbb{R}^d \setminus \{0\}$ and the cross product

$x_1 \times \cdots \times x_{d-1}$ is continuous on $\mathbb{R}^{d \times (d-1)}$, the central limit theorem combined with the continuous mapping theorem and the fact that $N_1 \overset{d}{=} -N_1$ implies that

$$(42) \qquad \lim_{n \to \infty} \mathbb{E} R(\pm S_{i_2} \times \cdots \times S_{i_d - i_{d-1}}) = \mathbb{E} R(\Sigma^{1/2} N_1 \times \cdots \times \Sigma^{1/2} N_{d-1})$$

uniformly in $1 \le i_2 < i_3 < \cdots < i_d \le n$ such that $\min(i_2, i_3 - i_2, \ldots, i_d - i_{d-1}) \ge h_n$.

Finally, for any non-degenerate $d \times d$ matrix A one has

$$A N_1 \times \cdots \times A N_{d-1} = \sum_{k=1}^{d} \det[A N_1, \ldots, A N_{d-1}, e_k] e_k$$

$$= \sum_{k=1}^{d} \det(A \cdot [N_1, \ldots, N_{d-1}, A^{-1} e_k]) e_k = \det A \cdot (A^{-1})^T (N_1 \times \cdots \times N_{d-1}).$$

In particular, this shows that the distribution of $N_1 \times \cdots \times N_{d-1}$ is invariant under orthogonal transformations since the standard Gaussian distribution is so. Hence the angular component of this distribution is uniform on $\mathbb{S}^{d-1}$ by the uniqueness of Haar measure (on the special orthogonal group $SO(n)$). Then, since the covariance matrix Σ is symmetric and R is an angular function, we have

$$R(\Sigma^{1/2} N_1 \times \cdots \times \Sigma^{1/2} N_{d-1}) = R(\Sigma^{-1/2}(N_1 \times \cdots \times N_{d-1})) \overset{d}{=} R(\Sigma^{-1/2} U).$$

Combining this fact with (34), (41), and (42) yields the main assertion of Proposition 2.

Proof of Theorem 4. A straightforward extension of the path-transform argument in the proof of (13) in Proposition 1 gives a little strengthening of (39): for any non-negative Borel function $g : \mathbb{R}^{d \times (d-1)} \to \mathbb{R}$,

$$\mathbb{E}\Big[g(S_{i_2} - S_{i_1}, \ldots, S_{i_d} - S_{i_{d-1}}) \mathbb{1}(0, S_1, \ldots, S_n \in H_\pm(S_{i_1}, \ldots, S_{i_d})) \Big]$$

$$= \mathbb{E}\Big[g(S_{i_2 - i_1}, S_{i_3 - i_1} - S_{i_2 - i_1}, \ldots, S_{i_d - i_1} - S_{i_{d-1} - i_1})$$

$$\times \mathbb{1}\big(S_1, \ldots, S_{i_d - i_1} \in H_\pm(0, S_{i_2 - i_1}, \ldots, S_{i_d - i_1})\big)$$

$$\times \mathbb{1}\Big(\underset{0 \le k \le n - (i_d - i_1)}{\arg\min} \det[S_{i_2 - i_1}, \ldots, S_{i_d - i_1}, S_{i_d - i_1 + k}] = \{i_1\} \Big) \Big].$$

For any fixed tuple $(i_2 - i_1, \ldots, i_d - i_1) =: (i'_1, \ldots, i'_{d-1})$, we sum over $i_1 =: i$ from 0 to $n - i'_{d-1}$ to obtain a conditional version of (13):

$$\sum_{i=0}^{n - i'_{d-1}} \mathbb{E}\big(\text{conv}(S_i, S_{i+i'_1}, \ldots, S_{i+i'_{d-1}}) \in \mathcal{F}_n \big| S_{i+i'_1} - S_i, S_{i+i'_2} - S_{i+i'_1},$$

$$\ldots, S_{i+i'_{d-1}} - S_{i+i'_{d-2}}\big)$$

$$(43) \qquad \qquad \qquad \qquad = \frac{2}{i'_1 (i'_2 - i'_1) \cdot \ldots \cdot (i'_{d-1} - i'_{d-2})} \text{ a.s.}$$

Theorem 4 then follows immediately by summation over all temporal structures $(i'_1, \ldots, i'_{d-1})$.

Computation of the asymptotics. (1) We claim that for any sequence a_n such that $a_n \sim (\log n)^a n^{-1/2}$ for some $a \geq 0$, it holds that

$$(44) \qquad \sum_{k=1}^{n} \frac{a_{n-k}}{k} \sim \frac{(\log n)^{a+1}}{\sqrt{n}}.$$

In particular, by (5) this implies (4) if we take $a = 0$.

Let us check that the main contribution to the asymptotics in (44) comes from the terms $k = o(n)$. Since the sum of a_n diverges, for any $\varepsilon \in (0, 1)$,

$$\sum_{k=\varepsilon n}^{n} \frac{a_{n-k}}{k} \leq \frac{1}{\varepsilon n} \sum_{k=\varepsilon n}^{n} a_{n-k} \sim \frac{1}{\varepsilon n} \sum_{k=1}^{(1-\varepsilon)n} \frac{(\log k)^a}{\sqrt{k}} \leq \frac{2(\log n)^a}{\varepsilon \sqrt{n}}.$$

The last expression is of a smaller order of asymptotics than

$$\sum_{k=1}^{\varepsilon n - 1} \frac{a_{n-k}}{k} \sim \sum_{k=1}^{\varepsilon n - 1} \frac{(\log(n-k))^a}{k\sqrt{(n-k)}},$$

since

$$\frac{(\log n)^{a+1}}{\sqrt{n}} \sim \frac{(\log(1-\varepsilon)n)^a}{\sqrt{n}} \sum_{k=1}^{\varepsilon n - 1} \frac{1}{k} \leq \sum_{k=1}^{\varepsilon n - 1} \frac{(\log(n-k))^a}{k\sqrt{(n-k)}}$$

$$\leq \frac{(\log n)^a}{\sqrt{(1-\varepsilon)n}} \sum_{k=1}^{\varepsilon n - 1} \frac{1}{k} \sim \frac{(\log n)^{a+1}}{\sqrt{(1-\varepsilon)n}}.$$

These inequalities clearly imply (44).

(2) We prove (19) by induction in d. The base $d = 2$ holds by (4), which we proved above. Since

$$\sum_{1 \leq i_2 < \cdots < i_d \leq n} \frac{(2n - 2i_d - 1)!!}{i_2 \cdot (2n - 2i_d)!!} \prod_{k=2}^{d-1} \frac{1}{i_{k+1} - i_k}$$

$$= \sum_{i_2=1}^{n-d+2} \frac{1}{i_2} \left[\sum_{1 \leq i'_2 < \cdots < i'_{d-1} \leq n - i_2} \frac{(2(n - i_2) - 2i'_{d-1} - 1)!!}{i'_2 \cdot (2(n - i_2) - 2i'_{d-1})!!} \prod_{k=2}^{d-2} \frac{1}{i'_{k+1} - i'_k} \right],$$

(16) (or (15)) implies that

$$(45) \qquad \mathbb{E}_{RW}^{(d)} |\mathcal{F}'_n| = \sum_{k=1}^{n} \frac{1}{k} \mathbb{E}_{RW}^{(d-1)} |\mathcal{F}'_{n-k}|,$$

where the upper indices show dimension and by definition, $\mathbb{E}_{RW}^{(d)} |\mathcal{F}'_n| := 0$ for $n \leq d - 1$. It remains to use (44) to obtain (19).

(3) Arguing as above and using (14) instead of (15), one can easily show that (45) also holds for a random walk bridge of length $n + 1$. For any sequence b_n such that $b_n \sim (\log n)^b n^{-1}$ for some $b \geq 0$, one has

$$(46) \qquad \sum_{k=1}^{n} \frac{b_{n-k}}{k} \sim \frac{(b+2)(\log n)^{b+1}}{(b+1)n}.$$

The difference with (44) is due to the fact that the main contribution to the asymptotics comes from the indices k that are either $k = o(n)$ or $k = n - o(n)$. The

asymptotics for the base $d = 2$ is then different, namely,

$$\mathbb{E}_{Br}^{(2)}|\mathcal{F}_n'| = \sum_{k=1}^{n} \frac{2}{k(n-k+1)} \sim \frac{4\log n}{n},$$

but the rest is analogous and (20) follows easily.

(4) The assertion (21) immediately follows from (19) once we check that in both summations resulting in these asymptotics, for any slowly varying sequence c_n tending to infinity, the contributions of the indices $1 \le i_2 < i_3 < \cdots < i_d \le n$ with $\min(i_2, i_3 - i_2, \ldots, i_d - i_{d-1}, n - i_d) \le c_n$ are of a smaller order of asymptotics.

We already saw that the main contribution to the asymptotics of the sum in (44) comes from the indices $k = o(n)$. Consequently, the indices with $n - i_d \le c_n$ do not contribute to the asymptotics in (19) and (21). On the other hand,

$$\sum_{k=1}^{c_n} \frac{a_{n-k}}{k} \sim \sum_{k=1}^{c_n} \frac{(\log(n-k))^a}{k\sqrt{n-k}} \sim \frac{(\log n)^a}{\sqrt{n}} \sum_{k=1}^{c_n} \frac{1}{k} \sim \log c_n \frac{(\log n)^a}{\sqrt{n}} = o\Big(\frac{(\log n)^{a+1}}{\sqrt{n}}\Big);$$

hence the indices $k \le c_n$ do not contribute as well to the sum in (44). Consequently, neither do any of the indices satisfying $\min(i_2, i_3 - i_2, \ldots, i_d - i_{d-1}) \le c_n$. $\quad\square$

APPENDIX

Proof of Lemma 5. For any direction $u \in \mathbb{S}^{d-1}$, the one-dimensional random walk $S_k^{(u)} := \langle S_k, u \rangle, k \ge 1$, has increments $\langle X_k, u \rangle$ with zero mean and strictly positive variance $\langle \Sigma u, u \rangle$; recall that Σ is non-degenerate as follows by assumption (H). The random variable $T(u)$, which is the exit time of the random walk S_k from the half-space $H(u)$, coincides with the exit time of the walk $S_k^{(u)}$ from the non-negative half-line. Then

$$R(u) = -\frac{\mathbb{E}\langle S_{T(u)}, u\rangle}{\sqrt{\langle \Sigma u, u\rangle}} = -\frac{\mathbb{E} S_{T(u)}^{(u)}}{\sqrt{\mathrm{Var}(\langle X_1, u\rangle)}}.$$

The last expression admits (Feller [8, Section XVIII.5]) representation in terms of the so-called Spitzer series:

$$(47) \qquad R(u) = \frac{1}{\sqrt{2}} \exp\Big(\sum_{k=1}^{\infty} \frac{1}{n}\big[\mathbb{P}(S_n^{(u)} > 0) - 1/2\big]\Big).$$

The series is known to converge under the zero mean and finite variance assumption on the increments so $R(u)$ is positive and finite on $\mathbb{S}^{d-1}$.

The convergence in (40) holds pointwise (cf. (47) and Feller [8, Section XII.8]) for every fixed $H = H(u_H)$. We will show that the standard proof of this statement can be strengthened to obtain the required uniform version. Let us recall this proof.

For a fixed direction $u \in \mathbb{S}^{d-1}$, we are interested in the asymptotics of the tail probabilities

$$\mathbb{P}(S_1, \ldots, S_n \in H(u)) = \mathbb{P}(S_1^{(u)} > 0, \ldots, S_n^{(u)} > 0) = \mathbb{P}(T(u) > n).$$

The moment-generating function of $T(u)$ is given by the Spitzer identity

$$1 - \mathbb{E}s^{T(u)} = \exp\Big(\sum_{n=1}^{\infty} \frac{s^n}{n}\mathbb{P}(S_n^{(u)} \ge 0)\Big), \qquad 0 \le s < 1,$$

which is valid for any random walk. Since the Spitzer series converges under the zero mean and finite variance assumption on the increments of $S_k^{(u)}$, we have

$$1 - \mathbb{E}s^{T(u)} = \sqrt{1-s}\exp\Big(\sum_{n=1}^{\infty}\frac{s^n}{n}\big[\mathbb{P}(S_n^{(u)} \geq 0) - 1/2\big]\Big).$$

Then

$$\sum_{n=0}^{\infty}\mathbb{P}(T(u) > n)s^n = \frac{1}{\sqrt{1-s}}\exp\Big(\sum_{n=1}^{\infty}\frac{s^n}{n}\big[\mathbb{P}(S_n^{(u)} \geq 0) - 1/2\big]\Big),$$

which can be verified by summation by parts in the l.h.s.

Since the Spitzer series converges, by Abel's theorem and (47) we have

(48)
$$\lim_{s\to 1-}\exp\Big(\sum_{n=1}^{\infty}\frac{s^n}{n}\big[\mathbb{P}(S_n^{(u)} \geq 0) - 1/2\big]\Big) = \sqrt{2}R(u).$$

Hence

(49)
$$\sum_{n=0}^{\infty}\mathbb{P}(T(u) > n)s^n \sim \frac{\sqrt{2}R(u)}{\sqrt{1-s}}, \qquad s \to 1-,$$

and the pointwise version of (40) follows by a Tauberian theorem for power series of sequences with monotone differences. In fact, $U(n) := \sum_{k=0}^{n}\mathbb{P}(T(u) > k)$ has monotone differences $U(n) - U(n+1) = \mathbb{P}(T(u) > n)$.

Now we explain how to modify the above argument to obtain the uniform asymptotics. The key ingredient is that the Spitzer series converges absolutely[4] uniformly in $u \in \mathbb{S}^{d-1}$. This is true by Lemma 6 below applied to the family of random variables $\frac{\langle X_1, u\rangle}{\sqrt{\mathrm{Var}(\langle X_1, u\rangle)}}, u \in \mathbb{S}^{d-1}$, which is uniformly square integrable by the inequality $\frac{\langle X_1, u\rangle^2}{\mathrm{Var}(\langle X_1, u\rangle)} \leq \sigma_1^{-1}\|X_1\|_2^2$, where σ_1 denotes the smallest eigenvalue of Σ.

Since the Spitzer series converges absolutely uniformly in u and it dominates termwise the absolute values of the series in (48), the convergence in (48) is uniform. Then the equivalence in (49) is also uniform in $u \in \mathbb{S}^{d-1}$, and by the second assertion of the uniform Tauberian Theorem 5 below, this implies (40).

Finally, note that each term of the Spitzer series, namely $n^{-1}\big[\mathbb{P}(S_n^{(u)} > 0) - 1/2\big]$, depends continuously on $u \in \mathbb{S}^{d-1}$. This is readily seen from the continuity of probability measures and the fact that the distribution of S_n does not put mass on hyperplanes due to assumption (H). Then $R(u)$ is continuous on $\mathbb{S}^{d-1}$ as a uniform limit of continuous functions. □

Uniform absolute convergence of the Spitzer series. We present a statement stronger than needed for use in the current paper.

Lemma 6. *Let $\{Y_\alpha\}_{\alpha\in I}$, where I is some index set, be random variables with zero mean and unit variance. Let $S_n^{(\alpha)}, n \geq 1$, be a random walk with increments*

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[4]For our proof, it actually suffices to use uniform convergence rather than the uniform absolute convergence. Indeed, it can be shown using Abel's uniform convergence test that the convergence in (48) is uniform as required.

distributed as $Y_\alpha, \alpha \in I$. *If the family* $\{Y_\alpha\}_{\alpha \in I}$ *is uniformly square integrable, then the series*

$$\sum_{n=1}^{\infty} \frac{1}{n} \big| \mathbb{P}(S_n^{(\alpha)} \geq 0) - 1/2 \big|$$

converges uniformly in $\alpha \in I$.

This statement fully rests on the series remainder estimate by Nagaev [20].

Proof. As in [20], for any $\alpha \in I$ denote $n_1(\alpha) = \min(k \geq 1 : \mathbb{E}Y_\alpha^2 \mathbb{1}_{\{Y_\alpha^2 < k\}} > 3/4)$ and $n_0(\alpha) = \max(8, n_1(\alpha))$. Putting together equations (6), (9), and (10) from [20] that estimate the terms of the main bound equation (2) gives that for any $k \geq n_0(\alpha)$,

$$\sum_{n=k}^{\infty} \frac{1}{n} \big| \mathbb{P}(S_n^{(\alpha)} \geq 0) - 1/2 \big| \leq \frac{19}{4\sqrt{k}} + \frac{3}{\sqrt{2}} \mathbb{E}Y_\alpha^2 \mathbb{1}_{\{Y_\alpha^2 \geq k\}} + \frac{2}{\sqrt{k}} \mathbb{E}|Y_\alpha|^3 \mathbb{1}_{\{|Y_\alpha| \leq \sqrt{k}\}}$$
$$+ 4\mathbb{E}|Y_\alpha| \mathbb{1}_{\{|Y_\alpha| > \sqrt{k}\}}.$$

The only difference with Nagaev's estimates is that this inequality is obtained by summation in (2) over $n \geq k$ rather than $n \geq n_0(\alpha)$ as in [20]. We also introduced a minor correction to (9).

Since $n_0 := \sup_\alpha n_0(\alpha)$ is finite by the uniform square-integrability, the remainder estimate applies to all $\alpha \in I$ if k is large enough. The first term vanishes as $k \to \infty$, and by the uniform square-integrability, so does the second one uniformly in $\alpha \in I$. For the fourth term, use the Cauchy–Bunyakovsky–Schwarz inequality. For the remaining third term, for any $\varepsilon > 0$, we have

$$\frac{1}{\sqrt{k}} \mathbb{E}|Y_\alpha|^3 \mathbb{1}_{\{|Y_\alpha| \leq \sqrt{k}\}} \leq \varepsilon \mathbb{E}Y_\alpha^2 \mathbb{1}_{\{|Y_\alpha| \leq \varepsilon\sqrt{k}\}} + \mathbb{E}Y_\alpha^2 \mathbb{1}_{\{\varepsilon\sqrt{k} \leq |Y_\alpha| \leq \sqrt{k}\}}$$
$$\leq \varepsilon + \mathbb{E}Y_\alpha^2 \mathbb{1}_{\{\varepsilon\sqrt{k} \leq |Y_\alpha|\}},$$

where the last term again vanishes uniformly. $\square$

A uniform Tauberian theorem. Although Tauberian theory is a very well studied subject and there are many results on the remainder terms in asymptotics, to our surprise we did not find any reference on uniform convergence. The next result is presented in greater generality than needed for use in the current paper.

Theorem 5 (Uniform Tauberian theorem). *Let* $\{U_\alpha\}_{\alpha \in I}$*, where* I *is some index set, be non-decreasing right-continuous functions on* $\mathbb{R}$ *with* $U_\alpha(0-) = 0$ *for every* $\alpha \in I$*, and let* $\{L_\alpha\}_{\alpha \in I}$ *be slowly varying functions. Assume that for some* $\rho \geq 0$*,*

$$\hat{U}_\alpha(s) := \int_0^\infty e^{-sx} dU_\alpha(x) \sim s^{-\rho} L_\alpha(1/s), \quad s \to 0+ \quad \text{uniformly in } \alpha \in I.$$

Then

$$U_\alpha(x) \sim \frac{x^\rho L_\alpha(x)}{\Gamma(1+\rho)}, \quad x \to \infty \quad \text{uniformly in } \alpha \in I.$$

If, in addition, U_α *is absolutely continuous with a monotone density* u_α *and* $L_\alpha(x) \equiv c_\alpha$ *is a positive constant for every* $\alpha \in I$ *and* $\rho > 0$*, then*

$$(50) \qquad u_\alpha(x) \sim \frac{x^{\rho-1} L_\alpha(x)}{\Gamma(\rho)}, \quad x \to \infty \quad \text{uniformly in } \alpha \in I.$$

Remark. It is possible to show that (50) holds under the less strenuous (than $L_\alpha(x) \equiv c_\alpha$) assumption of uniform slow variation for $\{L_\alpha\}_{\alpha \in I}$:

$$\lim_{x \to \infty} \sup_{1 \le s \le 2} \left| \frac{L_\alpha(sx)}{L_\alpha(x)} - 1 \right| = 0 \quad \text{uniformly in } \alpha \in I.$$

Our proof fully follows the one of Korevaar [16, Theorem I.15.3], which is based on explicit estimates of $U_\alpha(x)$ as opposed to more elegant standard proofs (as in Feller [8, Theorem XIII.5.2]) relying on the continuity theorem for Laplace transform.

Proof. For any positive integer m,

$$(51) \quad \hat{U}_\alpha(ks) \sim k^{-\rho} s^{-\rho} L_\alpha(1/s), \quad s \to 0+ \quad \text{uniformly in } \alpha \in I, k \in \{1, \dots, m\}.$$

Then, since for any positive integer k, one has

$$\int_0^\infty e^{-kx} d(x^\rho) = k^{-\rho} \Gamma(1 + \rho),$$

we see from (51) that for any polynomial $P(z) = \sum_{k=1}^m a_k z^k$,

$$(52)$$
$$\int_0^\infty P(e^{-sx}) dU_\alpha(x) \sim \frac{s^{-\rho} L_\alpha(1/s)}{\Gamma(1+\rho)} \int_0^\infty P(e^{-x}) d(x^\rho), \quad s \to 0+ \quad \text{uniformly in } \alpha \in I.$$

As in [16, Theorem I.15.3], denote $g(z) := \mathbb{1}_{[e^{-1}, 1]}(z)$ and, for any $\varepsilon > 0$, consider a polynomial $P_\varepsilon(z)$ approximating the indicator function $g(z)$ on $[0, 1]$ such that

$$P_\varepsilon(z) \ge g(z), \quad z \in [0, 1], \quad \text{and} \quad \int_0^1 (P_\varepsilon(z) - g(z)) \rho(-\log z)^{\rho-1} z^{-1} dz \le \varepsilon.$$

The latter condition ensures that

$$\int_0^\infty P_\varepsilon(e^{-x}) d(x^\rho) \le \int_0^\infty g(e^{-x}) d(x^\rho) + \varepsilon = \int_0^1 d(x^\rho) + \varepsilon = 1 + \varepsilon.$$

Finally, since by the choice of P_ε,

$$\int_0^\infty P_\varepsilon(e^{-sx}) dU_\alpha(x) \ge \int_0^\infty g(e^{-sx}) dU_\alpha(x) = U_\alpha(1/s),$$

from (52) we see that there exists an $s_\varepsilon > 0$ such that

$$U_\alpha(1/s) \le (1 + \varepsilon) \frac{s^{-\rho} L_\alpha(1/s)}{\Gamma(1+\rho)}, \quad \alpha \in I, s \in (0, s_\varepsilon).$$

Similarly, we obtain an analogous lower bound. Both inequalities imply the first assertion of the theorem.

The second assertion (50) that the uniformity is preserved under "differentiation" of the asymptotics can be checked by repeating the elementary proof of Lemma 17.1 in [16]. We omit the details. The assertion of the remark follows along the same lines. $\qquad \square$

References

[1] Josh Abramson, Jim Pitman, Nathan Ross, and Gerónimo Uribe Bravo, *Convex minorants of random walks and Lévy processes*, Electron. Commun. Probab. **16** (2011), 423–434, DOI 10.1214/ECP.v16-1648. MR2831081

[2] Frank Aurzada and Thomas Simon, *Persistence probabilities and exponents*, Lévy matters. V, Lecture Notes in Math., vol. 2149, Springer, Cham, 2015, pp. 183–224, DOI 10.1007/978-3-319-23138-9_3. MR3468226

[3] O. Barndorff-Nielsen and Glen Baxter, *Combinatorial lemmas in higher dimensions*, Trans. Amer. Math. Soc. **108** (1963), 313–325, DOI 10.2307/1993610. MR0156261

[4] Glen Baxter, *A combinatorial lemma for complex numbers*, Ann. Math. Statist. **32** (1961), 901–904, DOI 10.1214/aoms/1177704985. MR0126290

[5] A. J. Bray, S. N. Majumdar, and G. Schehr, *Persistence and first-passage properties in non-equilibrium systems*, Advances in Physics **62** (2013), 225–361.

[6] Ronen Eldan, *Extremal points of high-dimensional random walks and mixing times of a Brownian motion on the sphere* (English, with English and French summaries), Ann. Inst. Henri Poincaré Probab. Stat. **50** (2014), no. 1, 95–110, DOI 10.1214/12-AIHP515. MR3161524

[7] Ronen Eldan, *Volumetric properties of the convex hull of an n-dimensional Brownian motion*, Electron. J. Probab. **19** (2014), no. 45, 34, DOI 10.1214/EJP.v19-2571. MR3210546

[8] William Feller, *An introduction to probability theory and its applications. Vol. II*, Second edition, John Wiley & Sons, Inc., New York-London-Sydney, 1971. MR0270403

[9] Fuchang Gao, Daniel Hug, and Rolf Schneider, *Intrinsic volumes and polar sets in spherical space*, Homage to Luis Santaló. Vol. 1 (Spanish), Math. Notae **41** (2001/02), 159–176 (2003). MR2049442

[10] F. Gao and R. A. Vitale, *Intrinsic volumes of the Brownian motion body*, Discrete Comput. Geom. **26** (2001), no. 1, 41–50, DOI 10.1007/s00454-001-0023-1. MR1832728

[11] F. Götze, Z. Kabluchko, and D. Zaporozhets, *Conic analogues of Sudakov–Tsirelson theorem*, in preparation (2016).

[12] Zakhar Kabluchko, Vladislav Vysotsky, and Dmitry Zaporozhets, *Convex hulls of random walks, hyperplane arrangements, and Weyl chambers*, Geom. Funct. Anal. **27** (2017), no. 4, 880–918, DOI 10.1007/s00039-017-0415-x. MR3678504

[13] D. N. Zaporozhets and Z. Kabluchko, *Random determinants, mixed volumes of ellipsoids, and zeros of Gaussian random fields* (Russian, with English and Russian summaries), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **408** (2012), Veroyatnost′ i Statistika. 18, 187–196, 327; English transl., J. Math. Sci. (N.Y.) **199** (2014), no. 2, 168–173. MR3032216

[14] Zakhar Kabluchko and Dmitry Zaporozhets, *Intrinsic volumes of Sobolev balls with applications to Brownian convex hulls*, Trans. Amer. Math. Soc. **368** (2016), no. 12, 8873–8899, DOI 10.1090/tran/6628. MR3551592

[15] Jürgen Kampf, Günter Last, and Ilya Molchanov, *On the convex hull of symmetric stable processes*, Proc. Amer. Math. Soc. **140** (2012), no. 7, 2527–2535, DOI 10.1090/S0002-9939-2012-11128-1. MR2898714

[16] Jacob Korevaar, *Tauberian theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 329, Springer-Verlag, Berlin, 2004. A century of developments. MR2073637

[17] Satya N. Majumdar, Alain Comtet, and Julien Randon-Furling, *Random convex hulls and extreme value statistics*, J. Stat. Phys. **138** (2010), no. 6, 955–1009, DOI 10.1007/s10955-009-9905-z. MR2601420

[18] Michael B. McCoy and Joel A. Tropp, *From Steiner formulas for cones to concentration of intrinsic volumes*, Discrete Comput. Geom. **51** (2014), no. 4, 926–963, DOI 10.1007/s00454-014-9595-4. MR3216671

[19] Ilya Molchanov and Florian Wespi, *Convex hulls of Lévy processes*, Electron. Commun. Probab. **21** (2016), Paper No. 69, 11 pp., DOI 10.1214/16-ECP19. MR3564216

[20] S. V. Nagaev, *A new proof of the absolute convergence of the Spitzer series* (Russian, with Russian summary), Teor. Veroyatn. Primen. **54** (2009), no. 1, 149–152, DOI 10.1137/S0040585X97984024; English transl., Theory Probab. Appl. **54** (2009), no. 1, 151–154. MR2766651

[21] Jim Pitman and Gerónimo Uribe Bravo, *The convex minorant of a Lévy process*, Ann. Probab. **40** (2012), no. 4, 1636–1674, DOI 10.1214/11-AOP658. MR2978134

[22] Luis A. Santaló, *Integral geometry and geometric probability*, with a foreword by Mark Kac, Encyclopedia of Mathematics and its Applications, Vol. 1, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. MR0433364

[23] Rolf Schneider and Wolfgang Weil, *Stochastic and integral geometry*, Probability and its Applications (New York), Springer-Verlag, Berlin, 2008. MR2455326

[24] Timothy Law Snyder and J. Michael Steele, *Convex hulls of random walks*, Proc. Amer. Math. Soc. **117** (1993), no. 4, 1165–1173, DOI 10.2307/2159548. MR1169048

[25] Erik Sparre Andersen, *On the number of positive sums of random variables*, Skand. Aktuarietidskr. **32** (1949), 27–36. MR0032115

[26] Erik Sparre Andersen, *On sums of symmetrically dependent random variables*, Skand. Aktuarietidskr. **36** (1953), 123–138. MR0060173

[27] Erik Sparre Andersen, *On the fluctuations of sums of random variables. II*, Math. Scand. **2** (1954), 195–223. MR0068154

[28] F. Spitzer and H. Widom, *The circumference of a convex polygon*, Proc. Amer. Math. Soc. **12** (1961), 506–509, DOI 10.2307/2034228. MR0130616

[29] J. Michael Steele, *The Bohnenblust-Spitzer algorithm and its applications*, Probabilistic methods in combinatorics and combinatorial optimization, J. Comput. Appl. Math. **142** (2002), no. 1, 235–249, DOI 10.1016/S0377-0427(01)00472-1. MR1910531

[30] V. N. Sudakov, *Geometric problems of the theory of infinite-dimensional probability distributions* (Russian), Trudy Mat. Inst. Steklov. **141** (1976), 191. MR0431359

[31] Konstantin Tikhomirov and Pierre Youssef, *When does a discrete-time random walk in $\mathbb{R}^n$ absorb the origin into its convex hull?*, Ann. Probab. **45** (2017), no. 2, 965–1002, DOI 10.1214/15-AOP1079. MR3630291

[32] B. S. Tsirel′son, *A geometric approach to maximum likelihood estimation for an infinite-dimensional Gaussian location. I* (Russian, with English summary), Teor. Veroyatnost. i Primenen. **27** (1982), no. 2, 388–395. MR657940

[33] B. S. Tsirelson, *A geometric approach to maximum likelihood estimation for an infinite-dimensional Gaussian location. II* (Russian), Teor. Veroyatnost. i Primenen. **30** (1985), no. 4, 772–779. MR816291

[34] B. S. Tsirelson, *A geometric approach to maximum likelihood estimation for an infinite-dimensional Gaussian location. III* (Russian), Teor. Veroyatnost. i Primenen. **31** (1986), no. 3, 537–549. MR866873

[35] Andrew R. Wade and Chang Xu, *Convex hulls of planar random walks with drift*, Proc. Amer. Math. Soc. **143** (2015), no. 1, 433–445, DOI 10.1090/S0002-9939-2014-12239-8. MR3272767

[36] J. G. Wendel, *A problem in geometric probability*, Math. Scand. **11** (1962), 109–111, DOI 10.7146/math.scand.a-10655. MR0146858

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