

## MODIFIED SCATTERING FOR THE QUADRATIC NONLINEAR KLEIN–GORDON EQUATION IN TWO DIMENSIONS

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ABSTRACT. In this paper, we consider the long time behavior of the solution to the quadratic nonlinear Klein–Gordon equation (NLKG) in two space dimensions:  $(\square + 1)u = \lambda|u|u$ ,  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^2$ , where  $\square = \partial_t^2 - \Delta$  is d’Alembertian. For a given asymptotic profile  $u_{\text{ap}}$ , we construct a solution  $u$  to (NLKG) which converges to  $u_{\text{ap}}$  as  $t \rightarrow \infty$ . Here the asymptotic profile  $u_{\text{ap}}$  is given by the leading term of the solution to the linear Klein–Gordon equation with a logarithmic phase correction. Construction of a suitable approximate solution is based on Fourier series expansion of the nonlinearity.

### 1. INTRODUCTION

We consider the final state problem for the quadratic nonlinear Klein–Gordon equation in two space dimensions:

$$(1.1) \quad \begin{cases} (\square + 1)u = \lambda|u|u & t \in \mathbb{R}, x \in \mathbb{R}^2, \\ u - u_{\text{ap}} \rightarrow 0 & \text{in } L^2 \text{ as } t \rightarrow +\infty, \end{cases}$$

where  $\square = \partial_t^2 - \Delta$  is d’Alembertian,  $u : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is an unknown function,  $u_{\text{ap}} : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is a given function, and  $\lambda$  is a non-zero real constant.

There are many known results on the scattering for the nonlinear Klein–Gordon equation

$$(1.2) \quad (\square + 1)u = \lambda|u|^{p-1}u, \quad t \in \mathbb{R}, x \in \mathbb{R}^n,$$

where  $p > 1$  and  $\lambda \in \mathbb{R} \setminus \{0\}$ . Here, we focus on the results on scattering for small data. For the scattering results for large data, see [1, 2, 7, 26] for instance. For the case  $p_0(n) < p \leq 1 + 4/(n - 2)$  with  $p_0(n) = (n + 2 + \sqrt{n^2 + 12n + 4})/(2n)$ , small data scattering for (1.2) was studied by many authors; see [28, 29, 32] for instance. As for the case  $p \leq p_0(n)$ , Klainerman [17] and Shatah [30] independently proved the global existence of a solution to the Klein–Gordon equation with the quadratic nonlinearity for  $n = 3$  by using the vector field approach and the normal form, respectively. By using the vector field approach, Georgiev and Leconte [5] obtained pointwise decay estimates for solutions to (1.2) for  $p > 1 + 2/n$  with  $n = 1, 2, 3$ . Hayashi and Naumkin have shown in [11] that the nonlinear interaction in (1.2) is a *short range type* for  $p > 1 + 2/n$  with  $n = 1, 2$ ; i.e., solutions to (1.2) scatter to the

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solution to the linear Klein–Gordon equation if  $p > 1 + 2/n$ . On the other hand, Glassey [8] and Matsumura [23] proved that the nonlinear interaction in (1.2) is a *long range type* for  $1 < p \leq 1 + 2/n$  and  $n \geq 2$ ; namely, solutions to (1.2) do not scatter to the solution to the linear Klein–Gordon equation if  $1 < p \leq 1 + 2/n$ . A similar result was obtained in the case where  $p = 3$  and  $n = 1$  by Georgiev and Yordanov [6]. With these results, we see that the exponent  $p = 1 + 2/n$  is the borderline between the short range and long range scattering theories.

We briefly explain why the exponent  $p = 1 + 2/n$  appears as the borderline. Roughly speaking, the nonlinearity is short range if and only if the  $L^2$  norm of the nonlinear term is integrable in  $t \in [1, \infty)$ . Since the pointwise decay of the solution to the linear Klein–Gordon equation is  $O(t^{-n/2})$  as  $t \rightarrow \infty$ , the  $L^2$  norm of the nonlinear term  $|u|^{p-1}u$  has the rate  $O(t^{-n(p-1)/2})$ . Then, we observe that the nonlinearity is short range if and only if the integral  $\int_1^\infty t^{-n(p-1)/2} dt$  is finite. The condition is nothing but  $p > 1 + 2/n$ . The argument also suggests that solutions of the nonlinear equation (1.2) with  $p \leq 1 + 2/n$  may have an asymptotic behavior different from a solution of the linear Klein–Gordon equation. Thus, the threshold is  $p = 1 + 2/n$ .

For the Klein–Gordon equation with the cubic nonlinearity in one dimension, Georgiev and Yordanov [6] studied the pointwise decay of a solution to the initial value problem. Delort [3] obtained an asymptotic profile of a global solution to the equation corresponding to the small initial data with compact support (see also Lindblad and Soffer [19] for an alternative proof). The compact support assumption in [3] was later removed by Hayashi and Naunkin in [9]. We also note that global existence and the asymptotic behavior of a solution to the Klein–Gordon equation with the cubic quasi-linear nonlinearity is studied by Moriyama [25], Katayama [14], and Sunagawa [33] in one space dimension. Concerning the Klein–Gordon equation with the quadratic nonlinearity in two dimensions, Ozawa, Tsutaya, and Tsutsumi [27] proved a global existence result and characterized the asymptotic behavior of a small solution to (1.2) with a smooth, quadratic, semi-linear nonlinearity; i.e., the nonlinear term depends on  $u, \partial_t u, \nabla u$ . Delort, Fang, and Xue [4] extended Ozawa–Tsutaya–Tsutsumi’s result to the case where the nonlinear term is quasi-linear. See also Kawahara and Sunagawa [16] and Katayama, Ozawa, and Sunagawa [15] for related works.

In this paper we consider the scattering problem for (1.2) with the critical nonlinearity  $|u|u$  in two space dimensions. Especially, we consider the final state problem: For a given asymptotic profile  $u_{\text{ap}}$ , we construct a solution to (1.1) which converges to the given asymptotic profile as  $t \rightarrow \infty$ . Notice that the critical nonlinearity  $|u|u$  in two space dimensions was out of the scope of previous works due to the lack of smoothness of the nonlinear term.

Let us introduce the asymptotic profile  $u_{\text{ap}}$  which we work with. To this end, we first recall that the leading term of a solution to the linear Klein–Gordon equation

$$\begin{cases} (\square + 1)v = 0 & t \in \mathbb{R}, x \in \mathbb{R}^2, \\ v(0, x) = \phi_0(x), \quad \partial_t v(0, x) = \phi_1(x), & x \in \mathbb{R}^2, \end{cases}$$

is given by

$$t^{-1} \mathbf{1}_{\{|x| < t\}}(t, x) P_1(\mu) \cos(\langle \mu \rangle^{-1} t) + t^{-1} \mathbf{1}_{\{|x| < t\}}(t, x) Q_1(\mu) \sin(\langle \mu \rangle^{-1} t),$$

where  $\mu = x/\sqrt{t^2 - |x|^2}$ ,  $\mathbf{1}_\Omega(t, x)$  is the characteristic function supported on  $\Omega \subset \mathbb{R}^{1+2}$ , and

$$(1.3) \quad P_1(\mu) = -\langle \mu \rangle^2 \operatorname{Im} \hat{\phi}_0(\mu) - \langle \mu \rangle \operatorname{Re} \hat{\phi}_1(\mu),$$

$$(1.4) \quad Q_1(\mu) = \langle \mu \rangle^2 \operatorname{Re} \hat{\phi}_0(\mu) - \langle \mu \rangle \operatorname{Im} \hat{\phi}_1(\mu);$$

see [13] for instance. For given final state  $(\phi_0, \phi_1)$ , we define the asymptotic profile  $u_{\text{ap}}$  by

$$(1.5) \quad u_{\text{ap}}(t, x) := t^{-1} \mathbf{1}_{\{|x| < t\}}(t, x) P_1(\mu) \cos(\langle \mu \rangle^{-1} t + \Psi(\mu) \log t) + t^{-1} \mathbf{1}_{\{|x| < t\}} Q_1(\mu) \sin(\langle \mu \rangle^{-1} t + \Psi(\mu) \log t),$$

where the phase correction term is given by

$$(1.6) \quad \Psi(\mu) = -\frac{4}{3\pi} \langle \mu \rangle \left| \hat{\phi}_0(\mu) + i \langle \mu \rangle^{-1} \hat{\phi}_1(\mu) \right|.$$

The final state  $(\phi_0, \phi_1)$  is taken from the function space  $Y$  defined by

$$\begin{aligned} Y &:= \{(\phi_0, \phi_1) \in \mathcal{S}'(\mathbb{R}) \times \mathcal{S}'(\mathbb{R}); \|(\phi_0, \phi_1)\|_Y < \infty\}, \\ \|(\phi_0, \phi_1)\|_Y &:= \|\phi_0\|_{H_x^2} + \|x\phi_0\|_{H_x^3} + \|x^2\phi_0\|_{H_x^4} \\ &\quad + \|\phi_1\|_{H_x^1} + \|x\phi_1\|_{H_x^2} + \|x^2\phi_1\|_{H_x^3}. \end{aligned}$$

The main result in this paper is as follows.

**Theorem 1.1.** *Let  $(\phi_0, \phi_1) \in Y$ . For  $1/2 < d < 1$ , there exist a sufficiently large number  $T \geq e$  and a sufficiently small number  $\varepsilon > 0$  such that if  $\|(\phi_0, \phi_1)\|_Y < \varepsilon$ , then there exists a unique solution  $u(t)$  for the equation (1.1) satisfying*

$$(1.7) \quad \begin{aligned} u &\in C([T, \infty); L_x^2), \\ \sup_{t \geq T} t^d \|u - u_{\text{ap}}\|_{L^\infty((t, \infty); L_x^2)} &< \infty, \end{aligned}$$

where the asymptotic profile  $u_{\text{ap}}$  is defined by (1.5).

*Remark 1.2.* Concerning the final state problem for (1.2) with the cubic nonlinearity in one space dimension, Hayashi and Naumkin [10] constructed modified wave operators for (1.2) for small final data. Furthermore, Lindblad and Soffer [18] showed existence of modified wave operators for (1.2) for large final data in the case where  $\lambda < 0$ .

*Remark 1.3.* The same result holds true for equations with a general quadratic nonlinearity  $F(u) : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $F(\lambda u) = \lambda^2 F(u)$  for all  $\lambda > 0$  and  $u \in \mathbb{R}$ . See Remark 2.1 below for the details.

The rest of the paper is organized as follows. Section 2 is devoted to the exhibition of an outline for the proof of Theorem 1.1. The proof of Theorem 1.1 is based on the contraction principle via the integral equation of Yang-Feldman type associated with (1.1) around a suitable approximate solution. The crucial part of the proof is construction of the suitable approximate solution. We summarize how to do this in this section. In Section 3, we solve an abstract final value problem around an approximate solution. Then, in Section 4, we show that the approximate solution given in Section 2 satisfies the assumptions of the final value problem in Section 3, and we complete the proof of Theorem 1.1.

2. OUTLINE OF THE PROOF OF THEOREM 1.1

In this section, we give an outline of the proof of Theorem 1.1. For  $T > 0$ , we define the function spaces  $X_T$  by

$$\begin{aligned} X_T &:= \{w \in C([T, \infty); L_x^2); \|w\|_{X_T} < \infty\}, \\ \|w\|_{X_T} &:= \sup_{t \geq T} t^d (\|w\|_{L_t^\infty((t, \infty); H_x^{1/2})} + \|w\|_{L^4((t, \infty); L_x^4)}), \end{aligned}$$

where  $1/2 < d < 1$ . Put  $N(u) = \lambda|u|u$ . Let  $A$  be a function satisfying

$$\begin{aligned} (2.1) \quad & \|A(t)\|_{L_x^\infty} \leq \eta t^{-1}, \\ (2.2) \quad & \|(\square + 1)A(t) - N(A)(t)\|_{L_x^2} \leq \eta t^{-1-d}. \end{aligned}$$

We will prove in Section 3 that once we find such a function  $A$ , there exists a unique solution  $u$  to the equation (1.1) satisfying  $u - A \in X_T$ . To prove this assertion, we employ the Strichartz estimate (Lemma 3.2) and the contraction argument. Hence, it suffices to construct a function  $A$  satisfying the conditions (2.1) and (2.2) for a given final state  $(\phi_0, \phi_1) \in Y$ . It will turn out that  $A = u_{\text{ap}}$  does not work well, and so we need further modification.

We now explain how to construct the function  $A = A(t, x)$  satisfying the conditions (2.1) and (2.2). The conclusion is that the choice  $A := u_{\text{ap}} + v_{\text{ap}}$  works, where  $u_{\text{ap}}$  is the *first approximation* given by (1.5) and  $v_{\text{ap}}$  is the *second approximation* which is of the form

$$\begin{aligned} (2.3) \quad v_{\text{ap}} &:= t^{-2} \mathbf{1}_{\{|x| < t\}} \sum_{n=2}^\infty P_n(\mu) \cos(n\langle \mu \rangle^{-1}t + n\Psi(\mu) \log t) \\ &\quad + t^{-2} \mathbf{1}_{\{|x| < t\}} \sum_{n=2}^\infty Q_n(\mu) \sin(n\langle \mu \rangle^{-1}t + n\Psi(\mu) \log t). \end{aligned}$$

Here the phase function  $\Psi$  is the same as (1.6), and the choice of  $P_n$  and  $Q_n$  will be specified later. Remark that  $v_{\text{ap}}(t) = O(t^{-1})$  in  $L_x^2$ . Toward the conclusion, we will observe (i) why the second approximation  $v_{\text{ap}}$  is required, and (ii) what is the appropriate choice of  $P_n$  and  $Q_n$ , by a somewhat heuristic argument. Hereafter, we consider the case  $|x| < t$  only because  $u_{\text{ap}}$  and  $v_{\text{ap}}$  are identically zero in the region  $|x| \geq t$ .

We first focus on the nonlinear part  $N(u_{\text{ap}}) = \lambda|u_{\text{ap}}|u_{\text{ap}}$ . Since  $N(u) = \lambda|u|u$  is not polynomial in  $(u, \bar{u})$ , it becomes difficult to pick up a *resonant part* from  $N(u_{\text{ap}})$ . Taking a hint from our previous paper [22], we use the Fourier series expansion of  $N(u_{\text{ap}})$  to decompose  $N(u_{\text{ap}})$  into the resonant part and the rest into the *non-resonant part*. This decomposition is done as follows. We rewrite  $u_{\text{ap}}$  as

$$u_{\text{ap}} = \begin{cases} t^{-1} \mathbf{1}_{\{|x| < t\}} \sqrt{P_1(\mu)^2 + Q_1(\mu)^2} \cos(\alpha - \beta) & \text{if } P_1(\mu)^2 + Q_1(\mu)^2 \neq 0, \\ 0 & \text{if } P_1(\mu)^2 + Q_1(\mu)^2 = 0, \end{cases}$$

where  $\alpha = \langle \mu \rangle^{-1}t + \Psi(\mu) \log t$  and  $\beta \in (0, 2\pi]$  is given by

$$(2.4) \quad \cos \beta = \frac{P_1(\mu)}{\sqrt{P_1(\mu)^2 + Q_1(\mu)^2}}, \quad \sin \beta = \frac{Q_1(\mu)}{\sqrt{P_1(\mu)^2 + Q_1(\mu)^2}}.$$

Then, we have

$$\begin{aligned}
 (2.5) \quad N(u_{\text{ap}}) &= \lambda t^{-2}(P_1(\mu)^2 + Q_1(\mu)^2)c_1 \cos(\alpha - \beta) \\
 &\quad + \lambda t^{-2}(P_1(\mu)^2 + Q_1(\mu)^2) \sum_{n \geq 2} c_n \cos(n\alpha - n\beta) \\
 &= \frac{8}{3\pi} \lambda t^{-2} \sqrt{P_1(\mu)^2 + Q_1(\mu)^2} (P_1(\mu) \cos \alpha + Q_1(\mu) \sin \alpha) \\
 &\quad + \lambda t^{-2}(P_1(\mu)^2 + Q_1(\mu)^2) \sum_{n \geq 2} c_n \cos(n\alpha - n\beta) \\
 &=: N_r(u_{\text{ap}}) + N_{\text{nr}}(u_{\text{ap}}),
 \end{aligned}$$

where  $c_n$  are the Fourier coefficients for the function  $|\cos \theta| \cos \theta$ :

$$c_n = \frac{1}{\pi} \int_0^{2\pi} |\cos \theta| \cos \theta \cos n\theta d\theta = \begin{cases} -\frac{8}{\pi} \frac{\sin(\frac{n}{2}\pi)}{n(n^2 - 4)} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

This kind of technique was also used in Sunagawa [33] to pick up the resonant term from the cubic nonlinearity in one space dimension. In that case the Fourier series for  $N(u_{\text{ap}})$  consists of four terms. We would emphasize that, in our setting, the Fourier series consists of *infinitely many terms*, so we need to take care of the convergence of the Fourier series, which seems a new ingredient. Fortunately, it will turn out that the nonlinearity  $|u|u$  has enough smoothness to ensure the convergence of the Fourier series for  $|u|u$ . We mention that a similar but slightly different expansion of a nonlinearity into an infinite Fourier series is used by the first author and Miyazaki [21] in the context of nonlinear Schrödinger equations.

Since both the resonant and non-resonant parts are  $O(t^{-1})$  in  $L_x^2$ , we need to cancel out those terms by the linear part; otherwise (2.2) fails. Thanks to the phase correction  $\Psi$ , we have the desired cancellation of the resonant part. Namely, we have

$$(\square + 1)u_{\text{ap}} = N_r(u_{\text{ap}}) + O(t^{-2}(\log t)^2), \quad \text{in } L^2$$

as  $t \rightarrow \infty$ ; see Lemma 4.2 for the details. We then add a *second approximation*  $v_{\text{ap}}$  of  $u$ , given in (2.3), in order to cancel out the non-resonant term  $N_{\text{nr}}(u_{\text{ap}})$ . This is the reason why we need the second approximation  $v_{\text{ap}}$ .

To obtain the desired cancellation, we will choose suitable  $P_n$  and  $Q_n$ . More precisely, we choose them so that the leading term of the  $n$ th term of  $(\square + 1)v_{\text{ap}}$  and the  $n$ th term of the Fourier expansion of  $N_{\text{nr}}(u_{\text{ap}})$  coincide. By a computation, we have

$$\begin{aligned}
 (\square + 1)v_{\text{ap}} &= t^{-2} \sum_{n=2}^{\infty} (1 - n^2) P_n(\mu) \cos(n\langle \mu \rangle^{-1}t + n\Psi(\mu) \log t) \\
 &\quad + t^{-2} \sum_{n=2}^{\infty} (1 - n^2) Q_n(\mu) \sin(n\langle \mu \rangle^{-1}t + n\Psi(\mu) \log t) \\
 &\quad + O(t^{-2}(\log t)^2), \quad \text{in } L^2
 \end{aligned}$$

as  $t \rightarrow \infty$ ; see Lemma 4.3 for the details. Hence, we obtain the specific choice

(2.6)

$$P_n(\mu) = \begin{cases} \frac{8 \sin(\frac{n}{2}\pi)}{\pi n(n^2-1)(n^2-4)} \lambda(P_1(\mu)^2 + Q_1(\mu)^2) \cos(n\beta) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

(2.7)

$$Q_n(\mu) = \begin{cases} \frac{8 \sin(\frac{n}{2}\pi)}{\pi n(n^2-1)(n^2-4)} \lambda(P_1(\mu)^2 + Q_1(\mu)^2) \sin(n\beta) & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

With this choice, the leading term of the  $n$ th term of  $(\square + 1)v_{\text{ap}}$  and the  $n$ th term of the Fourier expansion for  $N_{\text{nr}}(u_{\text{ap}})$  successfully cancel each other out. Further, it will turn out in Section 4 that the error term can be handled thanks to fast decay of  $P_n$  and  $Q_n$  in  $n$ . Remark that the coefficients of  $P_n$  and  $Q_n$  are of order  $O(|n|^{-5})$  as  $|n| \rightarrow \infty$ . The decay rate of the Fourier coefficients reflects the smoothness of the nonlinearity  $\lambda|u|u$ . Thus, we see that  $A = u_{\text{ap}} + v_{\text{ap}}$  satisfies the conditions (2.1) and (2.2).

This kind of approximation was introduced in Hörmander [13] for the Klein–Gordon equation with *polynomial* nonlinearity in  $(u, \bar{u})$ . See also [24, 31] for the nonlinear Schrödinger equation with polynomial nonlinearity in  $(u, \bar{u})$ .

*Remark 2.1.* Let us consider a generalization of Theorem 1.1. Notice that any real-valued quadratic nonlinearity can be expressed as the linear combination of  $|u|u$  and  $u^2$ . Indeed, if  $F(u) : \mathbb{R} \rightarrow \mathbb{R}$  satisfies  $F(\lambda u) = \lambda^2 F(u)$  for any  $\lambda > 0$  and  $u \in \mathbb{R}$ , then we see that

$$F(u) = \frac{F(1) + F(-1)}{2} u^2 + \frac{F(1) - F(-1)}{2} |u|u.$$

As for the even nonlinearity  $u^2$ , it is easy to pick up the resonant/non-resonant part from  $u_{\text{ap}}^2$  because the Fourier series of  $u_{\text{ap}}^2$  consists of the zeroth and the second terms only. In particular, it contains no resonant part, and so existence of the even part does not change (the leading term of) the asymptotic profile. Thus, we can generalize Theorem 1.1 for general real-valued quadratic nonlinearity in two dimensions. Note that the final state problem for the Klein–Gordon equation with the nonlinearity  $u^2$  in two dimensions was studied by Hayashi and Naumkin [12] by using the normal form method. It is an interesting problem to generalize Theorem 1.1 to equations with *complex-valued* quadratic nonlinearities. As a related problem, we mention that Sunagawa [34] obtained the pointwise decay estimate of the complex valued solution to the initial value problem for the one-dimensional cubic nonlinear Klein–Gordon equation.

### 3. THE FINAL VALUE PROBLEM

In this section, we solve a Cauchy problem at infinite initial time for the equation (1.1) in an abstract framework. Let  $A(t, x)$  be a given asymptotic profile of a solution to (1.1). We show that if  $A(t, x)$  is well chosen, then we obtain a solution which asymptotically behaves like  $A(t, x)$ . We will give one sufficient condition for the choice of  $A(t, x)$ .

Let  $N(u) = \lambda|u|u$ . We introduce the *error function*  $F(t, x)$  by

$$(3.1) \quad F := (\square + 1)A - N(A).$$

For  $T > 0$ , we introduce the function space

$$X_T = \{w \in C([T, \infty); L_x^2); \|w\|_{X_T} < \infty\},$$

where

$$\|w\|_{X_T} = \sup_{t \geq T} t^d (\|w\|_{L^\infty((t, \infty); H_x^{1/2})} + \|w\|_{L^4((t, \infty); L_x^4)})$$

with  $1/2 < d < 1$ . For  $\rho > 0$  and  $T > 0$ , we define

$$\tilde{X}_T(\rho) = \{w \in C([T, \infty); L_x^2); \|w\|_{X_T} \leq \rho\}.$$

The function space  $X_T$  is a Banach space with the norm  $\|\cdot\|_{X_T}$ , and  $\tilde{X}_T(\rho)$  is a complete metric space with the  $\|\cdot\|_{X_T}$ -metric.

**Proposition 3.1.** *Let  $d$  be a constant such that  $1/2 < d < 1$ . Then there exist a sufficiently large  $T > 0$  and a sufficiently small  $\eta > 0$  such that if  $A(t, x)$  satisfies*

$$(3.2) \quad \|A(t)\|_{L_x^\infty} \leq \eta t^{-1},$$

$$(3.3) \quad \|F(t)\|_{L_x^2} \leq \eta t^{-1-d},$$

where  $F(t, x)$  is given by (3.1), then there exists a unique solution  $u$  for the equation (1.1) satisfying

$$u \in C([T, \infty); L_x^2),$$

$$(3.4) \quad \sup_{t \geq T} t^d (\|u - A\|_{L^\infty((t, \infty); H_x^{1/2})} + \|u - A\|_{L^4((t, \infty); L_x^4)}) < \infty.$$

To prove Proposition 3.1, we use the following inhomogeneous Strichartz estimates associated with the Klein-Gordon equation. Let

$$(3.5) \quad \mathcal{G}[g](t) := \int_t^\infty \sin((t - \tau)\sqrt{1 - \Delta})(1 - \Delta)^{-1/2} g(\tau) d\tau.$$

**Lemma 3.2.** *Let  $2 \leq q < \infty$  and  $1/p + 1/q = 1/2$ . Then we have*

$$\|\mathcal{G}[g]\|_{L_t^p([T, \infty), L_x^q)} \leq C \|(1 - \Delta)^{1/2-2/q} g\|_{L_t^{p'}([T, \infty), L_x^{q'})},$$

$$\|\mathcal{G}[g]\|_{L_t^\infty([T, \infty), L_x^2)} \leq C \|(1 - \Delta)^{-1/q} g\|_{L_t^{p'}([T, \infty), L_x^{q'})},$$

$$\|\mathcal{G}[g]\|_{L_t^p([T, \infty), L_x^2)} \leq C \|(1 - \Delta)^{-1/q} g\|_{L_t^1([T, \infty), L_x^2)}.$$

*Proof of Lemma 3.2.* The above inequalities follow from the  $L^p$ - $L^q$  estimate for the solution to the Klein-Gordon equation by [20] and the duality argument by [35]. Since the proof is now standard, we omit the details.  $\square$

*Proof of Proposition 3.1.* We put  $v = u - A$ . Then the equation (1.1) is equivalent to

$$(3.6) \quad (\square + 1)v = N(v + A) - N(A) - F,$$

where  $F$  is defined by (3.1). The associate integral equation to the equation (3.6) is

$$(3.7) \quad v = \mathcal{G}[\{N(v + A) - N(A)\} - F],$$

where  $\mathcal{G}$  is given by (3.5). It suffices to show the existence of a unique solution  $v$  to the equation (3.7) in  $X_T$  for sufficiently large  $T > 0$  and sufficiently small  $\eta > 0$ .

We prove this assertion by the contraction argument. Define the nonlinear operator  $\Phi$  by

$$\Phi v := \mathcal{G}[\{N(v + A) - N(A)\} - F]$$

for  $v \in \tilde{X}_T(\rho)$ . We show that for any  $\rho > 0$ ,  $\Phi$  is a contraction map on  $\tilde{X}_T(\rho)$  if  $T > 0$  is sufficiently large and  $\eta > 0$  is sufficiently small. Let  $\rho > 0$  be arbitrary, and let  $T, \eta > 0$ , which will be determined below. Let  $v \in \tilde{X}_T(\rho)$  and  $t \geq T$ . By the assumptions and Lemma 3.2, we see that

$$\begin{aligned} & \|(\Phi v)(t)\|_{L^\infty((t,\infty);H_x^{1/2})} + \|\Phi v\|_{L^4((t,\infty);L_x^4)} \\ & \leq C(\|v^2\|_{L^{4/3}((t,\infty);L_x^{4/3})} + \|(1 - \Delta)^{-1/4}Av\|_{L^1((t,\infty);L_x^2)}) \\ & \quad + \|(1 - \Delta)^{-1/4}F\|_{L^1((t,\infty);L_x^2)} \\ & \leq C\left\{ \|v\|_{L^4((t,\infty);L_x^4)} \left( \int_t^\infty \|v(\tau)\|_{L_x^2}^2 d\tau \right)^{1/2} + \int_t^\infty \|A(\tau)\|_{L_x^\infty} \|v(\tau)\|_{L_x^2} d\tau \right. \\ & \quad \left. + \int_t^\infty \|F(\tau)\|_{L_x^2} d\tau \right\} \\ & \leq C\left\{ \rho t^{-d} \left( \int_t^\infty \rho^2 \tau^{-2d} d\tau \right)^{1/2} + \int_t^\infty \eta \tau^{-1} \rho \tau^{-d} d\tau + \int_t^\infty \eta \tau^{-1-d} d\tau \right\} \\ & \leq C t^{-d} (\rho^2 t^{-d+1/2} + \rho \eta + \eta). \end{aligned}$$

Therefore we obtain

$$(3.8) \quad \|\Phi v\|_{X_T} \leq C_1(\rho^2 T^{-d+1/2} + \rho \eta + \eta).$$

In the same way as above, for  $v_1, v_2 \in \tilde{X}_T(\rho)$ , we can show that

$$\begin{aligned} (3.9) \quad & \|\Phi v_1 - \Phi v_2\|_{X_T} \\ & \leq C_2((\|v_1\|_{X_T} + \|v_2\|_{X_T})T^{-d+1/2} + \eta)\|v_1 - v_2\|_{X_T} \\ & \leq C_2(\rho T^{-d+1/2} + \eta)\|v_1 - v_2\|_{X_T}. \end{aligned}$$

We note that for  $\rho > 0$ , there exists a sufficiently large  $T > 0$  and a sufficiently small  $\eta > 0$  such that

$$\begin{aligned} C_1(\rho^2 T^{-d+1/2} + \rho \eta + \eta) & \leq \rho, \\ C_2(\rho T^{-d+1/2} + \eta) & \leq \frac{1}{2}, \end{aligned}$$

since  $d > 1/2$ . From this observation, the estimates (3.8) and (3.9) show that the operator  $\Phi$  is a contraction map on  $\tilde{X}_T(\rho)$  for sufficiently large  $T > 0$  and sufficiently small  $\eta > 0$ . Therefore for any  $\rho > 0$ , there exist  $T > 0, \eta > 0$ , and a unique solution to the integral equation (3.7) in  $\tilde{X}_T(\rho)$ . The uniqueness of solutions to the equation (3.7) in  $X_T$  follows from the first inequality of the estimate (3.9) for solutions  $v_1 \in X_T$  and  $v_2 \in X_T$ . Hence the equation (3.7) has a unique solution in  $X_T$ . This completes the proof of Proposition 3.1.  $\square$

#### 4. CONSTRUCTION OF A SUITABLE ASYMPTOTIC PROFILE

In this section, we complete the proof of Theorem 1.1 by showing that the asymptotic profile  $A(t, x)$  introduced in Section 2 satisfies the assumptions in Proposition 3.1.



**Proposition 4.1.** *Assume that the final state  $(\phi_0, \phi_1)$  satisfies  $(\phi_0, \phi_1) \in Y$ . Let  $u_{\text{ap}}$  be defined by (1.5), where  $P_1, Q_1$ , and  $\Psi$  are given by (1.3), (1.4), and (1.6), respectively. Let  $v_{\text{ap}}$  be defined by (2.3), where  $P_n$  and  $Q_n$  are given by (2.6) and (2.7). Then for  $A = u_{\text{ap}} + v_{\text{ap}}$ , there exist positive constants  $C$  such that the inequalities*

$$(4.1) \quad \|A(t)\|_{L_x^\infty} \leq Ct^{-1}\|(\phi_0, \phi_1)\|_Y(1 + \|(\phi_0, \phi_1)\|_Y),$$

$$(4.2) \quad \begin{aligned} \|(\square + 1)A(t) - N(A(t))\|_{L_x^2} \\ \leq Ct^{-2}(\log t)^2\|(\phi_0, \phi_1)\|_Y(1 + \|(\phi_0, \phi_1)\|_Y^3) \end{aligned}$$

hold for any  $(\phi_0, \phi_1) \in Y$  and  $t \geq e$ .

To prove Proposition 4.1, we first calculate  $(\square + 1)u_{\text{ap}}$ .

**Lemma 4.2.** *Assume that  $(\phi_0, \phi_1) \in Y$ . Let  $u_{\text{ap}}$  be defined by (1.5), where  $P_1, Q_1$ , and  $\Psi$  are given by (1.3), (1.4), and (1.6), respectively. Let  $N_r(u_{\text{ap}})$  be given by (2.5). Then it holds that*

$$(4.3) \quad \begin{aligned} \|(\square + 1)u_{\text{ap}} - N_r(u_{\text{ap}})\|_{L_x^2} \\ \leq Ct^{-2}(\log t)^2\|(\phi_0, \phi_1)\|_Y(1 + \|(\phi_0, \phi_1)\|_Y^2). \end{aligned}$$

*Proof of Lemma 4.2.* A simple calculation shows that

$$(4.4) \quad \begin{aligned} &(\square + 1)\{t^{-m} \cos(n\langle\mu\rangle^{-1}t)\} \\ &= (1 - n^2)t^{-m} \cos(n\langle\mu\rangle^{-1}t) + 2n(m - 1)t^{-m-1}\langle\mu\rangle \sin(n\langle\mu\rangle^{-1}t) \\ &\quad + m(m + 1)t^{-m-2} \cos(n\langle\mu\rangle^{-1}t) \end{aligned}$$

for  $m, n \in \mathbb{Z}_+$ . In a similar way,

$$\begin{aligned} &(\square + 1)\{t^{-m} \sin(n\langle\mu\rangle^{-1}t)\} \\ &= (1 - n^2)t^{-m} \sin(n\langle\mu\rangle^{-1}t) - 2n(m - 1)t^{-m-1}\langle\mu\rangle \cos(n\langle\mu\rangle^{-1}t) \\ &\quad + m(m + 1)t^{-m-2} \sin(n\langle\mu\rangle^{-1}t). \end{aligned}$$

We now consider a function of the form  $g = g(t, \mu)$ . Changing variables  $(s, \mu) = (t, x/\sqrt{t^2 - |x|^2})$ ,

$$\begin{aligned} \partial_t g &= \partial_s g - s^{-1}\langle\mu\rangle^2 \mu_1 \partial_{\mu_1} g - s^{-1}\langle\mu\rangle^2 \mu_2 \partial_{\mu_2} g, \\ \partial_{x_1} g &= s^{-1}\langle\mu\rangle(1 + \mu_1^2) \partial_{\mu_1} g + s^{-1}\langle\mu\rangle \mu_1 \mu_2 \partial_{\mu_2} g, \\ \partial_{x_2} g &= s^{-1}\langle\mu\rangle \mu_1 \mu_2 \partial_{\mu_1} g + s^{-1}\langle\mu\rangle(1 + \mu_2^2) \partial_{\mu_2} g. \end{aligned}$$

Hence,

$$(4.5) \quad \begin{aligned} \square(g(t, \mu)) &= \partial_s^2 g - s^{-2}\langle\mu\rangle^2(1 + \mu_1^2) \partial_{\mu_1}^2 g - s^{-2}\langle\mu\rangle^2(1 + \mu_2^2) \partial_{\mu_2}^2 g \\ &\quad - 2s^{-1}\langle\mu\rangle^2 \mu_1 \partial_s \partial_{\mu_1} g - 2s^{-1}\langle\mu\rangle^2 \mu_2 \partial_s \partial_{\mu_2} g \\ &\quad - 2s^{-2}\langle\mu\rangle^2 \mu_1 \mu_2 \partial_{\mu_1} \partial_{\mu_2} g \\ &\quad - 2s^{-2}\langle\mu\rangle^2 \mu_1 \partial_{\mu_1} g - 2s^{-2}\langle\mu\rangle^2 \mu_2 \partial_{\mu_2} g. \end{aligned}$$

By using the above identities, we now calculate  $(\square + 1)u_{\text{ap}}$ . To this end, we split it into the following four pieces:

$$\begin{aligned}
 (4.6) \quad (\square + 1)u_{\text{ap}} &= (\square + 1) \{t^{-1}P_1(\mu) \cos(\langle\mu\rangle^{-1}t) \cos(\Psi(\mu) \log t)\} \\
 &\quad - (\square + 1) \{t^{-1}P_1(\mu) \sin(\langle\mu\rangle^{-1}t) \sin(\Psi(\mu) \log t)\} \\
 &\quad + (\square + 1) \{t^{-1}Q_1(\mu) \sin(\langle\mu\rangle^{-1}t) \cos(\Psi(\mu) \log t)\} \\
 &\quad + (\square + 1) \{t^{-1}Q_1(\mu) \cos(\langle\mu\rangle^{-1}t) \sin(\Psi(\mu) \log t)\} \\
 &=: I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

Further, we split  $I_1$  into the following five pieces:

$$\begin{aligned}
 (4.7) \quad I_1 &= (\square + 1) \{t^{-1} \cos(\langle\mu\rangle^{-1}t)\} P_1(\mu) \cos(\Psi(\mu) \log t) \\
 &\quad + t^{-1} \cos(\langle\mu\rangle^{-1}t) \square \{P_1(\mu) \cos(\Psi(\mu) \log t)\} \\
 &\quad + 2\partial_t \{t^{-1} \cos(\langle\mu\rangle^{-1}t)\} \partial_t \{P_1(\mu) \cos(\Psi(\mu) \log t)\} \\
 &\quad - 2\partial_{x_1} \{t^{-1} \cos(\langle\mu\rangle^{-1}t)\} \partial_{x_1} \{P_1(\mu) \cos(\Psi(\mu) \log t)\} \\
 &\quad - 2\partial_{x_2} \{t^{-1} \cos(\langle\mu\rangle^{-1}t)\} \partial_{x_2} \{P_1(\mu) \cos(\Psi(\mu) \log t)\} \\
 &=: J_1 + J_2 + J_3 + J_4 + J_5.
 \end{aligned}$$

By (4.4), we see that

$$(4.8) \quad |J_1(\mu)| \leq Ct^{-3}|P_1(\mu)|.$$

By (4.5), we have

$$\begin{aligned}
 (4.9) \quad |J_2(\mu)| &\leq Ct^{-3}(\log t)^2 \{(|P_1(\mu)| |\Psi(\mu)|^2 + |P_1(\mu)| |\Psi(\mu)|) \\
 &\quad + \langle\mu\rangle^3 (|P_1(\mu)| |\Psi(\mu)| |D\Psi(\mu)| + |DP_1(\mu)| |\Psi(\mu)| \\
 &\quad + |P_1(\mu)| |D\Psi(\mu)| + |DP_1(\mu)|) \\
 &\quad + \langle\mu\rangle^4 (|P_1(\mu)| |D\Psi(\mu)|^2 + |DP_1(\mu)| |D\Psi(\mu)| \\
 &\quad + |P_1(\mu)| |D^2\Psi(\mu)| + |D^2P_1(\mu)|)\},
 \end{aligned}$$

where  $|Df(\mu)| = |\partial_{\mu_1}f(\mu)| + |\partial_{\mu_2}f(\mu)|$  and  $|D^2f(\mu)| = |\partial_{\mu_1}^2f(\mu)| + |\partial_{\mu_1}\partial_{\mu_2}f(\mu)| + |\partial_{\mu_2}^2f(\mu)|$ . For  $J_j$ ,  $j = 3, 4, 5$ , an elementary calculation yields

$$\begin{aligned}
 J_3 &= -2t^{-1}\langle\mu\rangle \sin(\langle\mu\rangle^{-1}t) \\
 &\quad \times \{\partial_t - t^{-1}\langle\mu\rangle^2\mu_1\partial_{\mu_1} - t^{-1}\langle\mu\rangle^2\mu_2\partial_{\mu_2}\}P_1(\mu) \cos(\Psi(\mu) \log t) \\
 &\quad - 2t^{-2} \cos(\langle\mu\rangle^{-1}t) \\
 &\quad \times \{\partial_t - t^{-1}\langle\mu\rangle^2\mu_1\partial_{\mu_1} - t^{-1}\langle\mu\rangle^2\mu_2\partial_{\mu_2}\}P_1(\mu) \cos(\Psi(\mu) \log t) \\
 &=: J_{31} + J_{32} + J_{33} + J_{34} + J_{35} + J_{36}, \\
 J_4 &= -2t^{-1}\mu_1 \sin(\langle\mu\rangle^{-1}t) \\
 &\quad \times \{t^{-1}\langle\mu\rangle(1 + \mu_1^2)\partial_{\mu_1} + t^{-1}\langle\mu\rangle\mu_1\mu_2\partial_{\mu_2}\}P_1(\mu) \cos(\Psi(\mu) \log t) \\
 &=: J_{41} + J_{42}, \\
 J_5 &= -2t^{-1}\mu_2 \sin(\langle\mu\rangle^{-1}t) \\
 &\quad \times \{t^{-1}\langle\mu\rangle\mu_1\mu_2\partial_{\mu_1} + t^{-1}\langle\mu\rangle(1 + \mu_2^2)\partial_{\mu_2}\}P_1(\mu) \cos(\Psi(\mu) \log t) \\
 &=: J_{51} + J_{52}.
 \end{aligned}$$

Since  $J_{32} + J_{41} + J_{51} = 0$  and  $J_{33} + J_{42} + J_{52} = 0$ , we see that

$$\begin{aligned}
 (4.10) \quad J_3 + J_4 + J_5 &= J_{31} + J_{34} + J_{35} + J_{36} \\
 &= 2t^{-2}\langle\mu\rangle P_1(\mu)\Psi(\mu) \sin(\langle\mu\rangle^{-1}t) \sin(\Psi(\mu) \log t) \\
 &\quad + J_{34} + J_{35} + J_{36}.
 \end{aligned}$$

Substituting (4.10) into (4.7), we obtain

$$(4.11) \quad I_1 = 2t^{-2}\langle\mu\rangle P_1(\mu)\Psi(\mu) \sin(\langle\mu\rangle^{-1}t) \sin(\Psi(\mu) \log t) + R_{11},$$

where  $R_{11} = J_1 + J_2 + J_{34} + J_{35} + J_{36}$ . Hence by (4.8) and (4.9),

$$\begin{aligned}
 (4.12) \quad |R_{11}(t, \mu)| &\leq Ct^{-3}(\log t)^2 \{(|P_1(\mu)| |\Psi(\mu)|^2 + |P_1(\mu)| |\Psi(\mu)| + |P_1(\mu)|) \\
 &\quad + \langle\mu\rangle^3 (|P_1(\mu)| |\Psi(\mu)| |D\Psi(\mu)| + |DP_1(\mu)| |\Psi(\mu)| \\
 &\quad + |P_1(\mu)| |D\Psi(\mu)| + |DP_1(\mu)|) \\
 &\quad + \langle\mu\rangle^4 (|P_1(\mu)| |D\Psi(\mu)|^2 + |DP_1(\mu)| |D\Psi(\mu)| \\
 &\quad + |P_1(\mu)| |D^2\Psi(\mu)| + |D^2P_1(\mu)|)\}.
 \end{aligned}$$

In a similar way, we have

$$(4.13) \quad I_2 = 2t^{-2}\langle\mu\rangle P_1(\mu)\Psi(\mu) \cos(\langle\mu\rangle^{-1}t) \cos(\Psi(\mu) \log t) + R_{12},$$

$$(4.14) \quad I_3 = -2t^{-2}\langle\mu\rangle Q_1(\mu)\Psi(\mu) \cos(\langle\mu\rangle^{-1}t) \sin(\Psi(\mu) \log t) + R_{13},$$

$$(4.15) \quad I_4 = -2t^{-2}\langle\mu\rangle Q_1(\mu)\Psi(\mu) \sin(\langle\mu\rangle^{-1}t) \cos(\Psi(\mu) \log t) + R_{14},$$

where  $R_{12}$  satisfies (4.12), and  $R_{13}$  and  $R_{14}$  satisfy (4.12) with  $Q_1$  instead of  $P_1$ . Substituting (4.11)–(4.15) into (4.6), we obtain

$$\begin{aligned}
 (4.16) \quad &|(\square + 1)u_{\text{ap}} - N_{\text{r}}(u_{\text{ap}})| \\
 &\leq |R_{11}| + |R_{12}| + |R_{13}| + |R_{14}| \\
 &\leq Ct^{-3}(\log t)^2 \sum_{Z=P,Q} \{(|Z_1(\mu)| |\Psi(\mu)|^2 + |Z_1(\mu)| |\Psi(\mu)| + |Z_1(\mu)|) \\
 &\quad + \langle\mu\rangle^3 (|Z_1(\mu)| |\Psi(\mu)| |D\Psi(\mu)| + |DZ_1(\mu)| |\Psi(\mu)| \\
 &\quad + |Z_1(\mu)| |D\Psi(\mu)| + |DZ_1(\mu)|) \\
 &\quad + \langle\mu\rangle^4 (|Z_1(\mu)| |D\Psi(\mu)|^2 + |DZ_1(\mu)| |D\Psi(\mu)| \\
 &\quad + |Z_1(\mu)| |D^2\Psi(\mu)| + |D^2Z_1(\mu)|)\}.
 \end{aligned}$$

By simple calculations, we see that

$$(4.17) \quad |Z_1(\mu)| \leq C \left( \langle\mu\rangle^2 |\hat{\phi}_0(\mu)| + \langle\mu\rangle |\hat{\phi}_1(\mu)| \right),$$

$$\begin{aligned}
 (4.18) \quad |DZ_1(\mu)| &\leq C \left( \langle\mu\rangle |\hat{\phi}_0(\mu)| + \langle\mu\rangle^2 |D\hat{\phi}_0(\mu)| \right) \\
 &\quad + C \left( |\hat{\phi}_1(\mu)| + \langle\mu\rangle |D\hat{\phi}_1(\mu)| \right),
 \end{aligned}$$

$$\begin{aligned}
 (4.19) \quad |D^2Z_1(\mu)| &\leq C \left( |\hat{\phi}_0(\mu)| + \langle\mu\rangle |D\hat{\phi}_0(\mu)| + \langle\mu\rangle^2 |D^2\hat{\phi}_0(\mu)| \right) \\
 &\quad + C \left( \langle\mu\rangle^{-1} |\hat{\phi}_1(\mu)| + |D\hat{\phi}_1(\mu)| + \langle\mu\rangle |D^2\hat{\phi}_1(\mu)| \right)
 \end{aligned}$$

for  $Z = P, Q$ , and

$$(4.20) \quad |\Psi(\mu)| \leq C \left( \langle \mu \rangle |\hat{\phi}_0(\mu)| + |\hat{\phi}_1(\mu)| \right),$$

$$(4.21) \quad |D\Psi(\mu)| \leq C \left( |\hat{\phi}_0(\mu)| + \langle \mu \rangle |D\hat{\phi}_0(\mu)| \right) \\ + C \left( \langle \mu \rangle^{-1} |\hat{\phi}_1(\mu)| + |D\hat{\phi}_1(\mu)| \right),$$

$$(4.22) \quad |D^2\Psi(\mu)| \leq C \left( \langle \mu \rangle^{-1} |\hat{\phi}_0(\mu)| + |D\hat{\phi}_0(\mu)| + \langle \mu \rangle |D^2\hat{\phi}_0(\mu)| \right) \\ + C \left( \langle \mu \rangle^{-2} |\hat{\phi}_1(\mu)| + \langle \mu \rangle^{-1} |D\hat{\phi}_1(\mu)| + |D^2\hat{\phi}_1(\mu)| \right).$$

Plugging (4.17)–(4.19) and (4.20)–(4.22) into (4.16), we have

$$|(\square + 1)u_{\text{ap}} - N_r(u_{\text{ap}})| \\ \leq Ct^{-3}(\log t)^2 \langle \mu \rangle^2 \\ \times \left\{ \left( 1 + \langle \mu \rangle |\hat{\phi}_0(\mu)| + \langle \mu \rangle^2 |D\hat{\phi}_0(\mu)| \right)^2 \left( \langle \mu \rangle^2 |\hat{\phi}_0(\mu)| + \langle \mu \rangle^3 |D\hat{\phi}_0(\mu)| \right) \right. \\ + \left( 1 + \langle \mu \rangle |\hat{\phi}_0(\mu)| \right) \langle \mu \rangle^4 |D^2\hat{\phi}_0(\mu)| \\ + \left( 1 + |\hat{\phi}_1(\mu)| + \langle \mu \rangle |D\hat{\phi}_1(\mu)| \right)^2 \left( \langle \mu \rangle |\hat{\phi}_1(\mu)| + \langle \mu \rangle^2 |D\hat{\phi}_1(\mu)| \right) \\ \left. + \left( 1 + |\hat{\phi}_1(\mu)| \right) \langle \mu \rangle^3 |D^2\hat{\phi}_1(\mu)| \right\}.$$

Therefore, taking the  $L^2$  norm for the above inequality with respect to the  $x$  variable, we have the estimate (4.3). □

Next we calculate  $(\square + 1)v_{\text{ap}}$ .

**Lemma 4.3.** *Assume that  $(\phi_0, \phi_1) \in Y$ . Let  $u_{\text{ap}}$  be defined by (1.5), where  $P_1, Q_1$ , and  $\Psi$  are given by (1.3), (1.4), and (1.6), respectively. Let  $v_{\text{ap}}$  be defined by (2.3), where  $P_n$  and  $Q_n$  are given by (2.6) and (2.7). Furthermore, let  $N_{\text{nr}}(u_{\text{ap}})$  be given by (2.5). Then, we have*

$$(4.23) \quad \|(\square + 1)v_{\text{ap}} - N_{\text{nr}}(u_{\text{ap}})\|_{L_x^2} \\ \leq Ct^{-2}(\log t)^2 \|(\phi_0, \phi_1)\|_Y^2 (1 + \|(\phi_0, \phi_1)\|_Y^2).$$

*Proof of Lemma 4.3.* In a similar way as in the proof of Lemma 4.2, we have

$$(4.24) \quad (\square + 1)v_{\text{ap}} = t^{-2} \sum_{n=2}^{\infty} (1 - n^2) P_n(\mu) \cos(n\langle \mu \rangle^{-1}t + n\Psi(\mu) \log t) \\ + t^{-2} \sum_{n=2}^{\infty} (1 - n^2) Q_n(\mu) \sin(n\langle \mu \rangle^{-1}t + n\Psi(\mu) \log t) \\ + \sum_{n=2}^{\infty} R_n(t, \mu),$$

where  $R_n$  ( $n \geq 2$ ) satisfies the inequalities

$$\begin{aligned}
 (4.25) \quad |R_n(t, \mu)| &\leq Ct^{-3}(\log t)^2 \\
 &\times \sum_{Z=P,Q} \{ (n^2|Z_n(\mu)||\Psi(\mu)|^2 + n|Z_n(\mu)||\Psi(\mu)| + |Z_n(\mu)|) \\
 &\quad + n\langle \mu \rangle |Z_n(\mu)| \\
 &\quad + \langle \mu \rangle^3 (n^2|Z_n(\mu)||\Psi(\mu)||D\Psi(\mu)| + n|DZ_n(\mu)||\Psi(\mu)| \\
 &\quad \quad + n|Z_n(\mu)||D\Psi(\mu)| + |DZ_n(\mu)|) \\
 &\quad + \langle \mu \rangle^4 (n^2|Z_n(\mu)||D\Psi(\mu)|^2 + n|DZ_n(\mu)||D\Psi(\mu)| \\
 &\quad \quad + n|Z_n(\mu)||D^2\Psi(\mu)| + |D^2Z_n(\mu)|) \}.
 \end{aligned}$$

Then by (2.5) and (4.24), we find that

$$(4.26) \quad \|(\square + 1)v_{\text{ap}} - N_{\text{nr}}(u_{\text{ap}})\|_{L_x^2} \leq \sum_{n=2}^{\infty} \|R_n(t)\|_{L_x^2}.$$

Differentiating (2.4), we see that  $\beta$  satisfies

$$(4.27) \quad |D\beta(\mu)| \leq C \frac{|DP_1(\mu)| + |DQ_1(\mu)|}{\sqrt{P_1(\mu)^2 + Q_1(\mu)^2}},$$

$$(4.28) \quad |D^2\beta(\mu)| \leq C \frac{|D^2P_1(\mu)| + |D^2Q_1(\mu)|}{\sqrt{P_1(\mu)^2 + Q_1(\mu)^2}} + C \frac{|DP_1(\mu)|^2 + |DQ_1(\mu)|^2}{P_1(\mu)^2 + Q_1(\mu)^2}.$$

Plugging (4.17)–(4.19) and (4.27)–(4.28) into (2.6) and (2.7), we have

$$(4.29) \quad |Z_n(\mu)| \leq Cn^{-5} \left( \langle \mu \rangle^4 |\hat{\phi}_0(\mu)|^2 + \langle \mu \rangle^2 |\hat{\phi}_1(\mu)|^2 \right),$$

$$\begin{aligned}
 (4.30) \quad |DZ_n(\mu)| &\leq Cn^{-4} |\hat{\phi}_0(\mu)| \left( \langle \mu \rangle^3 |\hat{\phi}_0(\mu)| + \langle \mu \rangle^4 |D\hat{\phi}_0(\mu)| \right) \\
 &\quad + Cn^{-4} \langle \mu \rangle^{-1} |\hat{\phi}_1(\mu)| \left( \langle \mu \rangle^2 |\hat{\phi}_1(\mu)| + \langle \mu \rangle^3 |D\hat{\phi}_1(\mu)| \right),
 \end{aligned}$$

$$\begin{aligned}
 (4.31) \quad |D^2Z_n(\mu)| &\leq Cn^{-3} \left\{ \left( |\hat{\phi}_0(\mu)| + \langle \mu \rangle |D\hat{\phi}_0(\mu)| \right) \left( \langle \mu \rangle^2 |\hat{\phi}_0(\mu)| + \langle \mu \rangle^3 |D\hat{\phi}_0(\mu)| \right) \right. \\
 &\quad \left. + |\hat{\phi}_0(\mu)| \langle \mu \rangle^4 |D^2\hat{\phi}_0(\mu)| \right\} \\
 &\quad + Cn^{-3} \left\{ \left( \langle \mu \rangle^{-1} |\hat{\phi}_1(\mu)| + |D\hat{\phi}_1(\mu)| \right) \left( \langle \mu \rangle |\hat{\phi}_1(\mu)| + \langle \mu \rangle^2 |D\hat{\phi}_1(\mu)| \right) \right. \\
 &\quad \left. + \langle \mu \rangle^{-1} |\hat{\phi}_1(\mu)| \langle \mu \rangle^3 |D^2\hat{\phi}_1(\mu)| \right\}
 \end{aligned}$$

for  $Z = P, Q$ . Substituting (4.20)–(4.22) and (4.29)–(4.31) into (4.25), we obtain

$$\begin{aligned}
 |R_n(t, \mu)| &\leq Cn^{-3}t^{-3}(\log t)^2\langle \mu \rangle^2 \\
 &\quad \times \left\{ \left(1 + \langle \mu \rangle |\hat{\phi}_0(\mu)|\right)^2 \left(\langle \mu \rangle^2 |\hat{\phi}_0(\mu)| + \langle \mu \rangle^3 |D\hat{\phi}_0(\mu)|\right)^2 \right. \\
 &\quad + (1 + \langle \mu \rangle |\hat{\phi}_0(\mu)|) \langle \mu \rangle^2 |\hat{\phi}_0(\mu)| \langle \mu \rangle^4 |D^2\hat{\phi}_0(\mu)| \\
 &\quad + \left(1 + |\hat{\phi}_1(\mu)|\right)^2 \left(\langle \mu \rangle |\hat{\phi}_1(\mu)| + \langle \mu \rangle^2 |D\hat{\phi}_1(\mu)|\right)^2 \\
 &\quad \left. + (1 + |\hat{\phi}_1(\mu)|) \langle \mu \rangle |\hat{\phi}_1(\mu)| \langle \mu \rangle^3 |D^2\hat{\phi}_1(\mu)| \right\}.
 \end{aligned}$$

Therefore taking the  $L^2$  norm for  $R_n$  with respect to the  $x$  variable, we have that

$$(4.32) \quad \|R_n(t)\|_{L_x^2} \leq Cn^{-3}t^{-2}(\log t)^2 \|(\phi_0, \phi_1)\|_Y^2 (1 + \|(\phi_0, \phi_1)\|_Y^2).$$

The inequalities (4.26) and (4.32) yield that

$$\begin{aligned}
 \|(\square + 1)v_{\text{ap}} - N_{\text{nr}}(u_{\text{ap}})\|_{L_x^2} &\leq Ct^{-2}(\log t)^2 \sum_{n=2}^{\infty} n^{-3} \|(\phi_0, \phi_1)\|_Y^2 (1 + \|(\phi_0, \phi_1)\|_Y^2) \\
 &\leq Ct^{-2}(\log t)^2 \|(\phi_0, \phi_1)\|_Y^2 (1 + \|(\phi_0, \phi_1)\|_Y^2).
 \end{aligned}$$

This completes the proof of Lemma 4.3. □

*Proof of Proposition 4.1.* The inequality (4.1) follows from the definition of  $A$  immediately. To show (4.2), we first confirm that addition by  $v_{\text{ap}}$  does not change the main part of the nonlinear part. However, it is obvious because  $v_{\text{ap}}$  decays faster than  $u_{\text{ap}}$  in time. Indeed, it can be observed by the elementary inequality

$$\begin{aligned}
 (4.33) \quad \|N(u_{\text{ap}} + v_{\text{ap}}) - N(u_{\text{ap}})\|_{L_x^2} &\leq C(\|u_{\text{ap}}\|_{L_x^\infty} + \|v_{\text{ap}}\|_{L_x^\infty}) \|v_{\text{ap}}\|_{L_x^2} \\
 &\leq Ct^{-2} \|(\phi_0, \phi_1)\|_Y^3 (1 + \|(\phi_0, \phi_1)\|_Y).
 \end{aligned}$$

By Lemma 4.2 (4.3), Lemma 4.3 (4.23) and (4.33), we see that

$$\begin{aligned}
 \|(\square + 1)A(t) - N(A(t))\|_{L_x^2} &\leq \|(\square + 1)u_{\text{ap}} - N_{\text{r}}(u_{\text{ap}})\|_{L_x^2} + \|(\square + 1)v_{\text{ap}} - N_{\text{nr}}(u_{\text{ap}})\|_{L_x^2} \\
 &\quad + \|N(u_{\text{ap}} + v_{\text{ap}}) - N(u_{\text{ap}})\|_{L_x^2} \\
 &\leq Ct^{-2}(\log t)^2 \|(\phi_0, \phi_1)\|_Y (1 + \|(\phi_0, \phi_1)\|_Y^3).
 \end{aligned}$$

Hence, we have the inequality (4.2). This completes the proof of Proposition 4.1. □

*Proof of Theorem 1.1.* By Proposition 4.1, we can apply Proposition 3.1 for  $A = u_{\text{ap}} + v_{\text{ap}}$ . Then there exists a solution  $u$  to (1.1) satisfying (3.4). Hence

$$\begin{aligned}
 \|u - u_{\text{ap}}\|_{L_x^2} &\leq \|u - u_{\text{ap}} - v_{\text{ap}}\|_{L_x^2} + \|v_{\text{ap}}\|_{L_x^2} \\
 &\leq Ct^{-d} + Ct^{-1} \\
 &\leq Ct^{-d},
 \end{aligned}$$

where  $1/2 < d < 1$ . This completes the proof of Theorem 1.1. □

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