UEDA'S PEAK SET THEOREM
FOR GENERAL VON NEUMANN ALGEBRAS

DAVID P. BLECHER AND LOUIS LABUSCHAGNE

Abstract. We extend Ueda's peak set theorem for subdiagonal subalgebras of tracial finite von Neumann algebras to σ-finite von Neumann algebras (that is, von Neumann algebras with a faithful state, which includes those on a separable Hilbert space or with separable predual). To achieve this extension, completely new strategies had to be invented at certain key points, ultimately resulting in a more operator algebraic proof of the result. Ueda showed in the case of finite von Neumann algebras that his peak set theorem is the fountainhead of many other very elegant results, like the uniqueness of the predual of such subalgebras, a highly refined F & M Riesz type theorem, and a Gleason-Whitney theorem. The same is true in our more general setting, and indeed we obtain a quite strong variant of the last mentioned theorem. We also show that set theoretic issues dash hopes for extending the theorem to some other large general classes of von Neumann algebras, for example finite or semi-finite ones. Indeed certain cases of Ueda's peak set theorem for a von Neumann algebra M may be seen as 'set theoretic statements' about M that require the sets to not be 'too large'.

1. Introduction

In a series of papers the authors extended most of the theory of generalized H^p spaces for function algebras from the 1960s to the setting of Arveson's (finite maximal) *subdiagonal algebras*. Most of this is summarized in the survey [8]. We worked in the setting where the subdiagonal algebra A was a unital weak* closed subalgebra of a von Neumann algebra M, and where M possesses a faithful normal tracial state. Ueda followed this work in [41] by removing a hypothesis involving a dimensional restriction on $A \cap A^*$ in four or five of our theorems and also establishing several other beautiful results, such as the fact that such an A has a unique predual, all of which followed from his very impressive noncommutative (Amar-Lederer) peak set type theorem. (We will say more about peak sets and peak projections later in this introduction when we describe notation and technical background. Section 2 of the paper is devoted to general results about peak projections, for example giving some useful characterizations of peak projections in C^*-algebras, von Neumann

algebras, and general operator algebras that do not seem to appear explicitly in the literature.) Ueda's peak set result may be phrased as saying that the support projection in M^{**} of a singular state φ on M is dominated by a peak projection p for A (so $\varphi(p) = 1$) with p in the 'singular part' of M^{**} (that is, p annihilates all normal functionals on M).

With the theory of subdiagonal subalgebras of von Neumann algebras with a faithful normal tracial state reaching a level of maturity, several authors turned their attention to the more general σ-finite von Neumann algebras. Important structural results were obtained by Ji, Ohwada, Saito, Bekjan, and Xu [4, 26–29, 43]. Recently in [34] the second author used Haagerup's reduction theory [21] to make several significant advances in generalizing aspects of the earlier theory to the σ-finite case, most notably the Beurling invariant subspace theory. The present work flowed out of this, being a direct continuation of the line of attack in [34]. Here we extend Ueda's peak set theorem, and its corollaries, to maximal subdiagonal algebras A in more general von Neumann algebras M, thereby demonstrating that such algebras too for example have a unique predual, admit a highly refined F & M Riesz type theorem, have a powerful Gleason-Whitney theorem (in particular, every normal functional on A has a unique Hahn-Banach extension to M, and this extension is also normal), etc. We remark that special cases of two of these results were obtained under an additional semi-finite hypothesis in [42]. The technically difficult extension of Ueda's theorem to the general σ-finite case is found in Section 4, while the applications mentioned a few lines back are discussed in Section 5. In Section 3 we establish some Kaplansky density type results for operator spaces and subdiagonal algebras which we will need.

In Section 6 we discuss the extent to which Ueda's theorem might be generalized beyond the σ-finite case. There is some limited good news: our results will have variants valid for any von Neumann algebra under an appropriate condition on its center, since it is known that any von Neumann algebra is a direct sum of algebras of the form $M_i = R_i \bar{\otimes} B(K_i)$ for σ-finite von Neumann algebras R_i. The central projections e_i corresponding to this direct sum will sometimes allow a decomposition of a maximal subdiagonal algebra A of M as a direct sum of subalgebras A_i of the R_i, and it is easy to see that these are then maximal subdiagonal subalgebras of the σ-finite algebras R_i. The 'bad news' is that there is little hope of proving Ueda's theorem in ZFC for all von Neumann algebras or even for commutative (and hence finite) or semi-finite von Neumann algebras. Indeed we show that the validity of Ueda's theorem for commutative atomic von Neumann algebras is a stronger statement than (it would imply) a ZFC proof of the nonexistence of uncountable measurable cardinals, a famous problem in set theory which nobody today seems to believe is solvable. Indeed certain cases of Ueda's peak set theorem, for a von Neumann algebra M, may be seen as 'set theoretic statements' about M that require the sets to not be 'too large'. These issues are discussed in Section 6, and this also led to a sequel paper with Nik Weaver [13]. Some of the ramifications of [13] are described at the end of the present paper, for example that that work indicates that one cannot generalize Ueda's peak set theorem in ZFC much beyond the σ-finite case (not even to $l^\infty(\mathbb{R})$). Thus the main result of our paper is somewhat sharp.

We now turn to our set-up, background, and notation. We recall that a *σ-finite von Neumann algebra* M is one with the property that every collection of mutually orthogonal projections is at most countable. Equivalently, M has a faithful

normal state (or even just a faithful state) or has a faithful normal representation possessing a cyclic separating vector. We often write $\mathbb{1}$ for the identity of M, which may be viewed as the identity operator on the underlying Hilbert spaces on which M is acting. A projection $p \in M$ is called *finite* if it is not Murray von Neumann equivalent to any proper subprojection; M is said to be finite if $\mathbb{1}$ is finite. Beware: σ-finite von Neumann algebras are not sums of finite ones, nor is every finite von Neumann algebra σ-finite. However a von Neumann algebra M possesses a faithful normal tracial state (the setting of [8] and most of [3]) if and only if it is both finite and σ-finite. (For the difficult direction of this one may compose the center valued trace on a finite von Neumann algebra, with a faithful normal state on the center, which in this case is σ-finite. From this it follows easily from e.g. [35, Proposition 2.2.5]) that any finite von Neumann algebra is a direct sum of algebras with faithful normal tracial states.) Any von Neumann algebra which is separably acting, or equivalently has separable predual M_*, is σ-finite. We will sometimes mention *semi-finite von Neumann algebras*; that is, 1 is a sum of mutually orthogonal finite projections or, equivalently, every nonzero projection has a nonzero finite subprojection.

For a subalgebra A of $C(K)$, the continuous scalar functions on a compact Hausdorff space K, a *peak set* is a set of form $f^{-1}(\{1\})$ for $f \in A, \|f\| = 1$. By replacing f by $(1 + f)/2$ we may assume also that $|f| = 1$ only on E. A noncommutative version of this, called *peak projections*, was considered in [24] and developed there and in a series of papers e.g. [5, 6, 9–12]. There are various useful equivalent definitions of peak projections in the latter papers. They may be defined to be the weak* limits $q = \lim_n a^n$ in the bidual for $a \in \text{Ball}(A)$ in the case such limit exists [11, Lemma 1.3]. We will say much more about peak projections in Section 2 below.

Let M be a σ-finite von Neumann algebra, and let ν be a fixed faithful normal state on M. We write N for the crossed product $M \rtimes_\nu \mathbb{R}$ of M with the modular group (σ_t^ν) induced by ν. If M acts on the Hilbert space $\mathfrak{H}$, this crossed product is constructed by canonically representing the elements a of M as operators on $L^2(\mathbb{R}, \mathfrak{H})$ by means of the prescription $\pi(a)\xi(t) = \sigma_{-t}^\nu(a)(\xi(t))$ and then generating a "larger" von Neumann algebra by means of the elements $\pi(a)$ and the shift operators $\lambda_s(\xi)(t) = \xi(t-s)$. The crossed product is known to admit a dual action of $\mathbb{R}$ in the form of an automorphism group (θ_s) indexed by $\mathbb{R}$ and a normal faithful semi-finite trace τ characterized by the property that $\tau \circ \theta_s = e^{-s}\tau$. (See [40].)

The identification $a \to \pi(a)$ above turns out to be a *-isomorphic embedding of M into N, and we will for the sake of simplicity identify M with $\pi(M)$. For simplicity of notation the canonical Hilbert space on which N acts will be denoted by K rather than $L^2(\mathbb{R}, \mathfrak{H})$. We will work in the space $\tilde{N}$ of all τ-measurable operators on K affiliated to N. We remind the reader that the τ-measurable operators are those closed densely defined affiliated operators f which are "almost" bounded in the sense that for any $\epsilon > 0$ we may find a projection $e \in N$ with $\tau(\mathbb{1}-e) < \epsilon$ and with $fe \in N$. This space turns out to be a very well-behaved complete *-algebra large enough to admit all the noncommutative function spaces of interest. Using the fact that $\tau \circ \theta_s = e^{-s}\tau$, it is a simple matter to show that the group of *-automorphisms (θ_s) above admits an extension to continuous $*$-automorphisms on $\tilde{N}$ (see for example either [19, bottom of p. 42] or [33, Proposition 4.7]). We will retain the notation θ_s for these extensions. Within this framework, the Haagerup L^p-spaces $(0 < p < \infty)$ are defined by $L^p(M) = \{a \in \tilde{N} : \theta_s(a) = e^{-s/p}a, s \in \mathbb{R}\}$. Having thus

been identified as subspaces of $\tilde{N}$, the Haagerup L^p-spaces in a very natural way inherit both a conjugation and an order structure from $\tilde{N}$. In fact even the natural (quasi-)norm topology on $L^p(M)$ is inherited from $\tilde{N}$ – the (quasi-)norm topology on each $L^p(M)$ agrees with the subspace topology inherited from $\tilde{N}$ [40, Proposition II.26]. Via the canonical embedding of $L^1(M)$ in $\tilde{N}$ a functional $\psi \in M_*$ is positive (resp. selfadjoint, i.e., a difference of two positive functionals) if and only if its image in $\tilde{N}$ is positive (resp. selfadjoint) [40].

We remind the reader that the crossed product admits an operator valued weight from the extended positive part of N to that of M. Using this operator valued weight, any normal weight ω on M may be extended to a dual weight $\tilde{\omega}$ on N by means of the simple prescription $\tilde{\omega} = \omega \circ T$. In our analysis h will denote the Radon-Nikodym derivative of the (faithful normal semi-finite) dual weight $\tilde{\nu}$ of our fixed faithful normal state ν above with respect to τ. It is known that in the σ-finite case, h belongs to the positive cone of the Haagerup space $L^1(M)$. Using this operator it is also known that for each $0 \leq c \leq 1$, $a \to h^{c/2}ah^{(1-c)/p}$ defines a dense embedding of M into $L^p(M)$ $(1 \leq p < \infty)$ [32]. Inspired by this fact, the Hardy spaces $H^p(A)$ $(1 \leq p < \infty)$ are defined to be the closure in $L^p(M)$ of the subspace $h^{c/p}Ah^{(1-c)/p}$ where $0 \leq c \leq 1$. (We remind the reader that the closures for the various values of c all agree [27]).

Given such a von Neumann algebra M and $\mathcal{E}$ a faithful normal conditional expectation from M onto a von Neumann subalgebra D, a *subdiagonal algebra* A in M (with respect to $\mathcal{E}$) is defined to be a weak* closed subalgebra of M containing $1\!\!1$ such that $A + A^*$ is weak* dense in M and for which the action of the conditional expectation $\mathcal{E} : M \to D = A \cap A^*$ is multiplicative on A. We say that A is *maximal* if it is not properly contained in any larger proper subdiagonal algebra in M with respect to $\mathcal{E}$. Maximality of such unital weak* closed subdiagonal algebras satisfying the aforementioned weak* density condition is characterized by the requirement that A be invariant under the modular automorphism group (σ_t^ν) introduced a few paragraphs earlier (ν is as above). (See [34, Theorem 1.1] or equivalently [43, Theorem 1.1] and [28, Theorem 2.4].)

Since we will have occasion to use the Haagerup reduction theorem [21], we pause to explain the essentials of that theory. From the von Neumann algebra M one constructs a larger algebra R by computing the crossed product with the diadic rationals $\mathbb{Q}_D$ (not $\mathbb{R}$). So in this case one uses only the *-automorphisms σ_t^ν with symbols t in $\mathbb{Q}_D$ to similarly construct a copy $\pi_{\mathbb{Q}_D}(M)$ of M inside $B(L^2(\mathbb{Q}_D, \mathfrak{H}))$, with $R = M \rtimes_\nu \mathbb{Q}_D$ then being the algebra generated by the elements belonging to this copy of M and the shift operators λ_s with symbol s in $\mathbb{Q}_D$. The discreteness of the group ensures that in this case the associated operator valued weight from the extended positive part of R to that of M is in fact a faithful normal conditional expectation $\Phi : R \to M$. Inside R one may then construct an increasing net R_n of finite von Neumann algebras and a concomitant net of expectations $\Phi_n : R \to R_n$ for which $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$. (In the present setting this net actually turns out to be a sequence.) Each R_n comes equipped with a faithful state $\tilde{\nu}_n = \nu \circ \Phi|_{R_n}$ and a faithful normal tracial state τ_n.

The vital fact regarding this construction is that it may be adapted to the study of maximal subdiagonal algebras. Following [43], let $\hat{D}$ be the von Neumann subalgebra of R generated by D and the shift operators λ_s ($s \in \mathbb{Q}_D$). (This is in essence just a copy of $D \rtimes_{\sigma^\nu} \mathbb{Q}_D$.) Similarly let $\hat{A}$ be the weak* closed subalgebra generated

by A and the same set of shift operators. Since A is invariant under σ_t^φ in that reference, $\hat{A}$ may be defined as the weak* closure of sums of terms of the form $\lambda(t)x$ with $x \in A$. It is shown in [43] that $\hat{A} \cap \hat{A}^* = \hat{D}$. The canonical expectation $\mathcal{E}: M \to D$ extends to an expectation $\hat{\mathcal{E}}: R \to \hat{D}$, and if indeed A is maximal subdiagonal with respect to $\mathcal{E}$, the algebra $\hat{A}$ will be maximal subdiagonal with respect to $\hat{\mathcal{E}}$. Moreover the expectation $\Phi: R \to M$ maps $\hat{A}$ and $\hat{D}$ onto A and D respectively. By equations (2.5) and (3.2) in [43] and the fact that we mentioned a few lines back regarding the definition of $\hat{A}$, we see that $\mathcal{E} \circ \Phi = \Phi \circ \hat{\mathcal{E}}$ on $\hat{A}$. Hence $\Phi(\hat{A}_0) = A_0$ since if $\hat{\mathcal{E}}(\hat{a}) = 0$, then $\mathcal{E}(\Phi(\hat{a})) = \Phi(\hat{\mathcal{E}}(\hat{a})) = 0$.

Taking this one step further, the subalgebras $\hat{A}_n = \hat{A} \cap R_n$ turn out to be finite maximal subdiagonal subalgebras of the finite von Neumann algebras R_n, with the restriction of $\hat{\mathcal{E}}$ to R_n acting multiplicatively on $\hat{A}_n$ and mapping R_n onto $\hat{D} \cap R_n$. The algebras $\hat{A}_n$ turn out to be an increasing sequence of algebras which are weak* dense in $\hat{A}$.

2. Peak projections

As we said in the introduction, peak projections with respect to an operator algebra A may be defined to be the weak* limits $q = \lim_n a^n$ in the bidual, for $a \in \text{Ball}(A)$ in the case such limit exists. Historically, if A is a C^*-algebra B, then peak projections are very closely related to Edwards and Ruttiman's element $u(a)$ (see e.g. [16]), computed in B^{**}. Certainly they are the same if $a \geq 0$, and in that case they also agree with the B^{**}-valued Borel functional calculus element $\mathbb{1}_{\{1\}}(a)$. Also they are the same; that is, q above is $u(a)$ if $\|1 - 2a\| \leq 1$ (see [9, Corollary 3.3]). Thus we shall sometimes simply write our peak projections as $u(a)$. Indeed every peak projection is of the form $u(x)$ where $\|1 - 2x\| \leq 1$ (if A is unital and $a^n \to q$ in the weak* topology with $a \in \text{Ball}(A)$ set $x = \frac{1}{2}(1 + a)$ [6, Corollary 6.9] or see [9, Theorem 3.4(3)] for the general case). We call $u(x)$ the peak for x. There is an elementary connection with the *support projection* $s(\cdot)$ (computed in B^{**}) which is often useful: if B is a unital C^*-algebra, then

$$u(1 - x) = \mathbb{1}_{\{1\}}(1 - x) = \mathbb{1}_{(0,\infty)}(x)^\perp = s(x)^\perp, \qquad x \in \text{Ball}(B)_+.$$

A similar but more general result holds in a unital nonselfadjoint algebra A: in the notation of Proposition 2.22 in [10] (see also [24, Proposition 5.4]) that result says that if $\|1 - 2x\| \leq 1$, then the peak for x is $u(x) = s(1 - x)^\perp$, where $s(\cdot)$ is the support projection in A^{**} studied in, e.g., [10, Section 2].

The following fact is implicit in the noncommutative peak set theory (see e.g. [5, 9, 10]), but we could not find it stated explicitly (except in the case of two projections; see, e.g., [24]).

Lemma 2.1. *If A is a closed subalgebra of a C^*-algebra B, then the infimum of any countable collection of peak projections for A is a peak projection for A.*

Proof. We may assume that A is unital, for example by [11, Proposition 6.4(1)] (see also [9, Lemma 3.1]). Suppose that q_n is a peak for $a_n \in A$ and that $\|1 - 2a_n\| \leq 1$ (which can always be arranged as we said). Let $q = \bigwedge_n q_n$ and $a = \sum_n \frac{a_n}{2^n}$. We will show that q is the peak for a. By a relation above the lemma we have

$$u(a) = s(1 - a)^\perp = s\left(\sum_n \frac{1 - a_n}{2^n}\right)^\perp = \left(\bigvee_n s(1 - a_n)\right)^\perp = \bigwedge_n s(1 - a_n)^\perp = \bigwedge_n q_n.$$

In the last line we have used e.g. Proposition 2.14 or Theorem 2.16(2) in [10] and the easy and known fact that the support projection of the closure of a sum of right ideals with left contractive approximate identities is the supremum of the individual support projections [10]. □

Remark 2.2. There is also a 'facial' proof of the previous result along the lines of [9, Proposition 1.1]. Another proof follows from an appeal to the next two results.

For a compact Hausdorff space K, the peak sets for $C(K)$ can be characterized abstractly as the compact G_δ subsets. There is a similar fact for C^*-algebras using Akemann's noncommutative topology (see [2] and references therein): the next result chararacterizes the peak projections for any C^*-algebra B as the 'compact G_δ projections'. A G_δ projection is the infimum in B^{**} of a sequence (p_n) of open projections in B^{**}, where a projection in B^{**} is said to be open if it is a weak* limit of an increasing net from B_+. The orthogonal complement of an open projection is called closed. A compact projection in B^{**} is a projection $q \in B^{**}$ which is closed and satisfies $qa = q$ for some $a \in \mathrm{Ball}(B)_+$ (or equivalently, which is closed with respect to B^1; see, e.g. [2] or 2.47 in [14]). If B is unital, then 'compact' is the same as 'closed'.

We have not been able to find the following result in the literature except for some form (see, e.g., [14]) of parts of the unital case:

Proposition 2.3. *If B is a C^*-algebra and q is a projection in B^{**}, the following are equivalent:*

   (i) *q is a peak projection with respect to B.*
   (ii) *q is a compact G_δ projection.*
   (iii) *q is the weak* limit of a decreasing sequence from B_{sa}.*

Proof. (i) $\Rightarrow$ (ii) If $q = u(a)$ for $a \in \mathrm{Ball}(B)_+$ let p_n be the B^{**}-valued spectral projection of $(1-\frac{1}{n}, 1+\frac{1}{n})$ for a. This gives a decreasing sequence of open projections in B^{**} whose infimum (= weak* limit) equals q by the Borel functional calculus. It is well known that peak projections are compact (e.g. since $q = aq$).

   (i) $\Rightarrow$ (iii) Clearly $a^n \searrow u(a)$ weak* if $a \in \mathrm{Ball}(B)_+$.

   If B is unital, then one may finish the proof using the relation $u(1-x) = s(x)^\perp$ discussed above and known results about the support projection $s(\cdot)$. Thus (ii) implies by, e.g., [14, Corollary 3.34] that $1 - q$ is a support projection, so that q is a peak projection. Similarly if B is unital, then (iii) implies that $1 - q$ is the weak* limit of an increasing sequence (a_n) from B_+. Let $k = \sum_{k=1}^\infty \frac{1}{2^n} a_n$; then $k \leq 1 - q$. A standard argument shows that k is strictly positive in the hereditary subalgebra (HSA) determined by $1 - q$ (any state of that HSA annihilating k also annihilates each a_n, hence also $1 - q$, which is impossible). Thus $1 - q$ is the support projection of k, so that q is the peak projection of $1 - k$.

   If B is nonunital, then (ii) or (iii) implies similar conditions with respect to B^1, so that by the unital case q is a peak for $a + t1$ for some $t \in [0, 1]$ and $a \in \mathrm{Ball}(B)_+$. The norm of $a + t1$ is $\|a\| + t = 1$, and so $0 \leq t = 1 - \|a\| < 1$ (or else $a = 0$, which is impossible). It is then easy to see by, e.g., the functional calculus for a that $q = u(a + t1) = u(a/\|a\|)$, giving (i). □

We now describe general peak projections in terms of the C^*-algebraic peak projections characterized in the last result.

Lemma 2.4. *If A is a closed subalgebra of a C^*-algebra B and $q \in B^{**}$, then q is a peak projection for A if and only if $q \in A^{\perp\perp}$ and q is a peak projection for B.*

Proof. If q is a peak projection for A, the peak for $x \in \mathrm{Ball}(A)$, then q is the weak* limit of (x^n), which is in $A^{\perp\perp}$. It is also the peak for some $a \in \mathrm{Ball}(B)_+$ (e.g. for x^*x or $|x|$, this follows for example from the proof of [9, Lemma 3.1] or from the formula $u(x^*x) = u(x)^*u(x)$).

The converse is Corollary 4.5 of [12], and has even been generalized to Jordan operator algebras in a recent paper of the first author with Neal. A sketch of a proof: Suppose that $q \in A^{\perp\perp}$ and q is a peak projection for B. We may assume that B is unital. Let A^1 be the span of A and 1_B. By Proposition 2.3(ii) and [12, Lemma 4.4], q is a peak projection for A^1 and by [11, Proposition 6.4(1)] (see also [9, Lemma 3.1]), q is a peak projection for A. $\square$

The following result, which characterizes peak projections in subalgebras of von Neumann algebras, will also be used in [13].

Theorem 2.5. *If A is a closed subalgebra (not necessarily with any kind of approximate identity) of a von Neumann algebra M and q is a projection in M^{**}, then q is a peak projection for A if and only if $q \in A^{\perp\perp}$ and $q = \bigwedge_n q_n$, the infimum in M^{**} of a (decreasing, if desired) sequence (q_n) of projections in M.*

Proof. If q is a peak projection for $x \in \mathrm{Ball}(A)$, then by the last lemma q is in $A^{\perp\perp}$, and q is the peak for some $a \in \mathrm{Ball}(M)_+$, so that $q = 1_{\{1\}}(a)$, the M^{**}-valued spectral projection of $\{1\}$. Let q_n be the M-valued spectral projection of $(1 - \frac{1}{n}, 1 + \frac{1}{n})$ for a. We claim that the decreasing sequence (q_n) in M has infimum q in M^{**}. To see this note that as in Proposition 2.3, q is the infimum of (p_n) in M^{**} where p_n is the M^{**}-valued spectral projection of $(1 - \frac{1}{n}, 1 + \frac{1}{n})$ for a. However, $q_n \leq p_n \searrow q$. This may be seen from viewing the M-valued Borel functional calculus as the M^{**}-valued Borel functional calculus multiplied by the canonical central projection z with $zM^{**} \cong M$ (this follows in turn from the uniqueness property of the Borel functional calculus). Also $q \leq q_n$ (as may be seen e.g. by the above functional calculi, using continuous f with $1_{\{1\}} \leq f \leq 1_{(1 - \frac{1}{n}, 1 + \frac{1}{n})}$).

Conversely, suppose that $q = \bigwedge_n q_n$. Note that q_n is clearly a peak projection for M; hence so is q by Lemma 2.1. Now apply the last lemma. $\square$

3. A KAPLANSKY DENSITY TYPE RESULT

The following simple principle will be useful for dealing with Kaplansky density type results in unital operator spaces.

Lemma 3.1. *Let M be a unital operator space or operator system. Let σ be any linear topology on M weaker than the norm topology, e.g. the weak or weak* topology (the latter if M is a dual space too). Let X be a subspace of M for which $\mathrm{Ball}(X)$ is dense in $\mathrm{Ball}(M)$ in the topology σ. Then $\{x \in X : x + x^* \geq 0\}$ is dense in $\{x \in M : x + x^* \geq 0\}$ in the topology σ.*

Proof. Suppose that $x \in M$ with $x + x^* \geq 0$. Then $z = x + \frac{1}{n}$ satisfies $z + z^* \geq 0$ and

$$z + z^* \geq \frac{2}{n} \geq Cz^*z$$

for some constant $C > 0$. This implies that

$$C^2 z^*z - C(z + z^*) + 1 = (1 - Cz)^*(1 - Cz) \leq 1.$$

We may then approximate $1 - Cz$ in the topology σ by a net $x_t \in \mathrm{Ball}(X)$, and so $\frac{1}{C}(1 - x_t) \to z$ with respect to σ. Since $2 - x_t - x_t^* \geq 0$ we have shown that z is in the closure of $\{x \in X : x + x^* \geq 0\}$ in the topology σ. Hence so is x. $\qquad\square$

The following is a Kaplansky density type result generalizing the one in Corollary 4.3 in [7] and in [41, Section 4] (where Ueda points out that the dimensional restriction in [7, Corollary 4.3] can be removed).

Theorem 3.2. *If A is a maximal subdiagonal algebra in a σ-finite von Neumann algebra M, then $\mathrm{Ball}(A + A^*)$ is weak* dense in $\mathrm{Ball}(M)$. Hence $\mathrm{Ball}(A + A^*)_{\mathrm{sa}}$ is weak* dense in $\mathrm{Ball}(M)_{\mathrm{sa}}$. Moreover, $(A + A^*)_+$ is weak* dense in M_+. Also, in all of these statements we can replace 'weak*' by σ-strong*.*

Proof. The first assertion is known in the case that M has a faithful normal tracial state (this is the case discussed immediately before the theorem). Let $x \in \mathrm{Ball}(M)$. As stated in the introduction, one may construct a σ-finite von Neumann superalgebra R of M with M appearing as the image of a faithful normal conditional expectation $\Phi : R \to M$. This R may be constructed so that it appears as the weak* closure of an increasing sequence R_n of finite von Neumann algebras each of which is the image of a faithful normal conditional expectation $\Phi_n : R \to R_n$ for which we have that $\Phi_n \circ \Phi_m = \Phi_m \circ \Phi_n = \Phi_n$ when $n \geq m$. In fact each $x \in R$ is the weak* limit of the sequence $\Phi_n(x)$.

As shown by [43], this construction can be modified in such a way that R admits a maximal subdiagonal subalgebra $\hat{A}$ for which Φ will map $\hat{A}$ onto A and $\hat{A} \cap \hat{A}^*$ onto $A \cap A^*$. Moreover the subalgebras $\hat{A}_n \cap R_n \subset R_n$ are each maximal subdiagonal in R_n, with $\bigcup_{n=1}^\infty \hat{A}_n$ weak*-dense in $\hat{A}$. By known results $\mathrm{Ball}(\hat{A}_n + \hat{A}_n^*)$ is for each n weak* dense in $\mathrm{Ball}(R_n)$. So the subset $\bigcup_{n=1}^\infty \mathrm{Ball}(\hat{A}_n + \hat{A}_n^*)$ of $\mathrm{Ball}(\hat{A} + \hat{A}^*)$ must be weak* dense in the weak* closure of $\bigcup_{n=1}^\infty \mathrm{Ball}(R_n)$, namely $\mathrm{Ball}(R)$. It therefore follows that $\Phi(\mathrm{Ball}(\hat{A} + \hat{A}^*)) = \mathrm{Ball}(A + A^*)$ is weak* dense in $\Phi(\mathrm{Ball}(R)) = \mathrm{Ball}(M)$.

The second assertion follows from the first by taking the real part. The third follows by applying the previous lemma to the first. The last assertion follows from the previous assertions and [38, Theorem 2.6(iv)]. $\qquad\square$

We give a corollary of this which we will use later. Note that any element in $A + A^*$ has a unique representation $a^* + d + b$ with $a, b \in A_0$ and $d \in D$. This is because if $a^* + d + b = 0$, then applying $\mathcal{E}$ we see that $d = 0$. Also $A_0 \cap A_0^* \subset D \cap A_0 = (0)$. Thus $A + A^* = A_0 \oplus D \oplus A_0^*$. It follows from this that selfadjoint elements x in $A + A^*$ are of the form $a + d + a^*$ for $a \in A_0, d \in D_{\mathrm{sa}}$, and d must be positive if $x \geq 0$ since $d = \mathcal{E}(x)$. We write H for the Hilbert transform on $L^2(M)$ with respect to A as presented in [27]. It is shown there that H is continuous on $L^2(M)$. In this insightful paper Ji shows that for each fixed $1 < p < \infty$ the operators

$$H_\theta(h^{\theta/p}(a + d + b^*)h^{(1-\theta)/p} = ih^{\theta/p}(b^* - a)h^{(1-\theta)/p}, \qquad a, b \in A, d \in D, \theta \in [0, 1],$$

extend to a unique bounded operator on $L^p(M)$ (the Hilbert transform) not dependent on the parameter θ. See [27, Theorem 3.2]. In the case $p = \infty$, the Hilbert transform H is only partially defined on M by means of the formula by $H(a + d + b^*) = i(b^* - a)$, for $a, b \in A, d \in D$. We remind the reader that h is the Radon-Nikodym derivative of the (faithful normal semi-finite) dual weight $\tilde{\nu}$

with respect to the canonical trace on N, and its role in the definition of $H^p(A)$ is discussed in the introduction. Similarly, the selfadjointness and positivity referred to below were discussed there too.

Lemma 3.3. *Let A be as in the previous result, and let H be the Hilbert transform on $L^2(M)$ with respect to A. If $x \in M_{sa}$, then $h^{\frac{1}{2}} H(xh^{\frac{1}{2}})$ is selfadjoint. Moreover $H(xh^{\frac{1}{2}})^* = H(h^{\frac{1}{2}}x)$.*

Proof. It suffices to prove the claim for the case where $x \in M_+$. If $a \in A_0, d \in D_{\mathrm{sa}}$, then $H((a^* + d + a)h^{\frac{1}{2}}) = i(a^* - a)h^{\frac{1}{2}}$ by the definition in [27]. Hence

$$h^{\frac{1}{2}} H((a^* + d + a)h^{\frac{1}{2}}) = ih^{\frac{1}{2}}(a^* - a)h^{\frac{1}{2}}$$

is selfadjoint. Any $x \in M_+$ is the weak* limit of a net $x_\lambda = a_\lambda^* + d_\lambda + a_\lambda$, where $a_\lambda \in A$ and $d_\lambda \in D_+$, by Theorem 3.2 and the comment following it. Hence the net $(x_\lambda h^{\frac{1}{2}})$ converges weakly to $xh^{\frac{1}{2}}$ in L^2. To see this note that for any $b \in L^2$, $h^{\frac{1}{2}} b$ will be in L^1, whence $\mathrm{tr}(x_\lambda h^{\frac{1}{2}} b) \to \mathrm{tr}(xh^{\frac{1}{2}} b)$. Since any norm continuous operator is also weakly continuous, the L^2 continuity of H ensures that $(H(x_\lambda h^{\frac{1}{2}}))$ converges weakly to $H(xh^{\frac{1}{2}})$ in L^2. This in turn ensures that $(h^{\frac{1}{2}} H(x_\lambda h^{\frac{1}{2}}))$ converges weakly to $h^{\frac{1}{2}} H(xh^{\frac{1}{2}})$ in L^1. By the lines at the start of this paragraph, $h^{\frac{1}{2}} H(x_\lambda h^{\frac{1}{2}})$ is selfadjoint. Hence $h^{\frac{1}{2}} H(xh^{\frac{1}{2}})$ is selfadjoint with respect to the conjugation structure inherited from $\tilde{N}$. Similarly in view of the fact that $(x_\lambda h^{\frac{1}{2}})^* = h^{\frac{1}{2}} x_\lambda$ is weakly convergent to $(xh^{\frac{1}{2}})^* = h^{\frac{1}{2}} x$, we again have that $H(h^{\frac{1}{2}} x_\lambda)$ is weakly convergent to $H(h^{\frac{1}{2}} x)$. It is obvious that $H(h^{\frac{1}{2}} x_\lambda)^* = H(x_\lambda h^{\frac{1}{2}})$ for each λ, from which it follows that $H(xh^{\frac{1}{2}})^* = H(h^{\frac{1}{2}} x)$, as required. $\square$

4. Ueda's peak set theorem for σ-finite M

Theorem 4.1. *Let A be a maximal subdiagonal subalgebra of a σ-finite von Neumann algebra M. For a nonzero singular $\varphi \in M^*$, there exist a contraction $a \in A$ and a projection $p \in M^{**}$ with*

(1) $a^n \to p$ *weak* in M^{**},*
(2) $a^n \to 0$ *weak* in M or equivalently $\psi(p) = 0$ for all $\psi \in M_*$,*
(3) $|\varphi|(p) = |\varphi|(1)$, *where $|\varphi|$ is the absolute value of φ regarded as a member of the predual of the W^*-algebra M^{**}.*

Since φ is known to be singular if and only if $|\varphi|$ is singular [38], one may assume that φ is a state if one wishes. In this case as in [41], (1)–(3) may be restated as saying that (1) p is a peak projection, (2) p is dominated by the 'singular part' projection of M^{**}, and (3) $\varphi(p) = 1$.

The present section is devoted to generalizing Ueda's elegant proof of the tracial state case of Theorem 4.1. As in Theorem 1 of [41] we may find a decreasing sequence (p_n) of projections in M with strong limit 0 and $|\varphi|(p_n) = |\varphi|(p_0) = |\varphi|(1) \neq 0$ for all n, where p_0 is the strong limit of (p_n) in M^{**}. We may also assume that $\nu(p_n) < n^{-6}$, where ν is the fixed faithful normal state on M. The formal series $g = \sum_k k p_k$ may then be shown to correspond to a well-defined positive unbounded closed and densely defined operator affiliated to M. Moreover the formal prescription $g \to h^{1/2} g$ yields a well-defined embedding of this operator into $L^2(M)$. These facts are proved in the following lemma.

Lemma 4.2. *Let the projections p_n be as in the previous paragraphs, for $n \in \mathbb{N}$. Then the formal operator $g = \sum_k kp_k$ corresponds to a densely defined positive selfadjoint operator affiliated to M. Moreover $h^{1/2}g$ is a densely defined closable operator for which the closure is a well-defined element of $L^2(M)$ which appears as the L^2-norm limit of the sequence $(h^{1/2}g_n) \subset L^2(M)$, where $g_n = \sum_{k=1}^n kp_k$. (In other words $[h^{1/2}(\sum_{k=1}^\infty kp_k)] = \sum_{k=1}^\infty k[h^{1/2}p_k]$.) Similarly the formal operator $gh^{1/2}$ may be regarded as the sum in $L^2(M)$ of the series $\lim_{n\to\infty} g_n h^{1/2} = \sum_{k=1}^\infty k[p_k h^{1/2}]$.*

Proof. We first prove the claim regarding the affiliation of g. For this we will make use of the well known theory of the extended positive part of a von Neumann algebra. (See [39, §IX.4].) Observe that $g_n = \sum_{k=1}^n kp_k$ may in a canonical way be regarded as an increasing sequence of elements of the extended positive part of M. It is clear from the discussions following [39, Definitions IX.4.4 and IX.4.6] that the supremum of this sequence (which we identify with g) is a well-defined element of the extended positive part of M.

Next recall that by [39, Corollary IX.4.9], the action of the canonical faithful normal state ν extends to the extended positive part of M. In terms of this action, we must have that $\nu(g) = \sup_{n\in\mathbb{N}} \nu(g_n)$ and hence that

$$\nu(g) = \sup_{n\in\mathbb{N}} \nu(g_n) \leq \lim_{n\to\infty} \sum_{k=1}^n k^{-5} < \infty.$$

However by [39, Theorem IX.4.8], g has a spectral decomposition of the form

$$g(\omega) = \int_0^\infty \lambda \, d\omega(e_\lambda) + \infty\omega(p) \quad \omega \in M_*^+.$$

On considering the case where $\omega = \nu$ and comparing the resulting formula to the the fact that $\nu(g) < \infty$, it is clear that we must then have that $\nu(p) = 0$, i.e., $p = 0$. It is now not difficult to conclude from the discussion in [39] that this can only be the case if the "operator part" of g (see [39, Lemma IX.4.7]) is all of g. See e.g. the last paragraph of the proof of Theorem IX.4.8 there. Hence g is a densely defined affiliated operator.

We proceed to verify the claim regarding $h^{1/2}g$ (the $gh^{1/2}$ statement following e.g. by duality). Firstly note that by the choice of the p_k's we have that $\sum_k \|kp_k h^{1/2}\|_2 < +\infty$. Indeed

$$\|kp_k h^{1/2}\|_2^2 = k^2 \mathrm{tr}(h^{1/2}p_k h^{1/2}) = k^2 \nu(p_k) < \frac{1}{k^4}.$$

So the formal series $\sum_{k=1}^\infty k[p_k h^{1/2}] = \lim_{n\to\infty} g_n h^{1/2}$ must correspond to a well-defined τ-measurable element G of $L^2(M) \subset \tilde{N}$.

Recall that $p_m p_k = p_k p_m = p_m$ whenever $m \geq k$. This ensures that for any fixed $m \geq 1$ and any $1 \leq n \leq \infty$, we have that

$$g_n(p_m - p_{m+1}) = \sum_{k=1}^n kp_k(p_m - p_{m+1}) = (\sum_{k=1}^m k)(p_m - p_{m+1}).$$

(Here g_∞ is identified with g.) So for each $m \geq 1$ we then have that $h^{1/2}g(p_m - p_{m+1}) = \lim_{n\to\infty} h^{1/2}g_n(p_m - p_{m+1})$. Taking into account that $p_1 = \bigoplus_{m=1}^\infty (p_m - p_{m+1})$, it follows that $h^{1/2}g = h^{1/2}gp_1 = \lim_{n\to\infty} h^{1/2}g_n p_1 = \lim_{n\to\infty} h^{1/2}g_n = G^*$ as required. $\qquad\square$

Remark 4.3. We note that if $g = \sum_n np_n$, viewed as a supremum in the extended positive part $\hat{M}_+$ of M, then $h^{\frac{1}{2}}gh^{\frac{1}{2}} \in L^1(M)$, and the latter can be shown to be the supremum and limit in $L^1(M)$ of $(h^{\frac{1}{2}}g_nh^{\frac{1}{2}})$. We will not use this though.

Let $\tilde{g}$ (resp. $\tilde{g}_n$) be the Hilbert transform of $gh^{\frac{1}{2}}$ (resp. $g_nh^{\frac{1}{2}}$) as in [27, Section 3], and let $f = gh^{\frac{1}{2}} + i\tilde{g}$ (resp. $f_n = g_nh^{\frac{1}{2}} + i\tilde{g}_n$). Then $f_n, f \in H^2(A)$.

In the following result we use the notion of accretive operators (see, e.g., [22, Appendix C.7]). In $L^p(M)$ an operator is accretive if the associated operator $T \in \tilde{N}$ has $T + T^*$ positive in $\tilde{N}$.

Corollary 4.4. *With $g = \sum_k kp_k$ as above and $f = gh^{\frac{1}{2}} + i\tilde{g}$, we have $h^{\frac{1}{2}}\tilde{g}$ is selfadjoint in $L^1(M)$, so that $h^{\frac{1}{2}}f = h^{\frac{1}{2}}gh^{\frac{1}{2}} + ih^{\frac{1}{2}}\tilde{g}$ is accretive in the sense above.*

Proof. If g_n is as defined above, then by Lemma 3.3 we have that $h^{\frac{1}{2}}H(g_nh^{\frac{1}{2}})$ is selfadjoint. By Lemma 4.2 and the continuity of H from [27], it follows that $h^{\frac{1}{2}}\tilde{g}$ is selfadjoint. Thus, and since g is positive, $h^{\frac{1}{2}}f = h^{\frac{1}{2}}gh^{\frac{1}{2}} + ih^{\frac{1}{2}}\tilde{g}$ is accretive.   □

A σ-finite von Neumann algebra M has a convenient 'standard form'. Indeed as we recalled in the introduction, a characterization of σ-finite algebras is the existence of a (normal faithful) Hilbert space representation $\mathfrak{H}$ possessing a fixed cyclic and separating vector Ω. Then $\nu(x) = (\Omega, x\,\Omega)$ is a faithful normal state on M. It is known that in this context

(4.1) $(M, \mathfrak{H}, \mathcal{P}, J, \Omega)$

is a 'standard form' for M, where $\mathcal{P}$ and J respectively denote the naturally associated cone and the modular conjugation. The modular automorphism group σ_t is implemented by $\sigma_t(\cdot) = \Delta^{it} \cdot \Delta^{-it}$, where Δ is the modular operator. By the universality of the standard form (see [1, 20, 40]) and hence also of the natural cone, we may identify the context

$$(M, \mathfrak{H}, \mathcal{P}, J, \Omega)$$

with

$$(M,\ L^2(M),\ L^2_+(\mathfrak{M}),\ {}^*, h^{\frac{1}{2}}).$$

In what follows we choose to work with the copy of M living inside $B(L^2(M))$ as multiplication operators. In view of the above correspondence, we may do so without loss of generality. We view $h^{\frac{1}{2}}$ as the fixed cyclic and separating vector for this action of M.

Lemma 4.5. *For each $k \in \mathbb{N}$, there exist nets $(a(k)_\lambda) \subset A_0$, $(d(k)_\lambda) \subset D_+$ such that $(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda) \in M_+$, with $(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda)$ converging to g_k in the σ-strong* topology. Hence for any $q \in L^2(M)$, the nets $(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda)q$ and $q(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda)$ will respectively converge in L^2-norm to gq and qg.*

Proof. This follows from Theorem 3.2 and the observation following it.   □

The Hilbert transform H in the next result is the map partially defined on M by $H(a + d + b^*) = i(b^* - a)$, for $a, b \in A, d \in D$.

Lemma 4.6. *Given $a \in A_0$, $d \in D_+$ with $a^* + d + a \in M_+$, the element $(a^* + d + a + 1\!\!1) + iH(a^* + d + a)$ has an inverse v belonging to A, with both v and $1 - v$ contractive.*

Proof. Observe that with a and d as in the hypothesis, $H(a^* + d + a) = i(a^* - a)$ is selfadjoint. Thus $x = a^* + d + a + iH(a^* + d + a)$ is accretive. By the basic theory of accretive operators (see, e.g., [22, Appendix C.7]) $1\!1 + x$ has a contractive inverse v. Note that $v \in A$ since the numerical range and hence the spectrum of x in A is in the right half plane. Also $x(1\!1 + x)^{-1} = 1\!1 - (1\!1 + x)^{-1} = 1\!1 - v$ is a contraction in A, being the average of $1\!1$ and the well-known Cayley transform of x. We remark that the map $x \mapsto x(1\!1 + x)^{-1}$ is called the $\mathfrak{F}$-transform in recent papers of Charles Read and the first author. $\qquad\square$

Proposition 4.7. *There exist elements (w_k) and w_g of A for which each of w_k, w_g, $w_k - 1\!1$, and $w_g - 1\!1$ are contractions, with*

$$w_k[(g_k + 1\!1)h^{1/2} + iH(g_k h^{1/2})] = h^{1/2} = [h^{1/2}(g_k + 1\!1) + iH(h^{1/2}g_k)]w_k$$

and

$$w_g[(g + 1\!1)h^{1/2} + iH(gh^{1/2})] = h^{1/2} = [h^{1/2}(g + 1\!1) + iH(h^{1/2}g)]w_g.$$

Moreover there exists a subnet of (w_k) which is weak* convergent to w_g.

Proof. Choose nets $(a(k)_\lambda) \subset A_0$ and $(d(k)_\lambda) \subset D_+$ as in Lemma 4.5. By Lemma 4.6 each $(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + 1\!1) + iH(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda)$ has an inverse $v(k)_\lambda$ belonging to A, with both $v(k)_\lambda$ and $1 - v(k)_\lambda$ contractive. By passing to a subnet if necessary, we may assume that $(v(k)_\lambda)$ is weak* convergent. Let w_k be the weak* limit of $(v(k)_\lambda)$. (Since both $(v(k)_\lambda)$ and $(v(k)_\lambda - 1\!1)$ are contained in the weak* compact set Ball(A), it is clear that both w_k and $w_k - 1\!1$ will also be in this set.) We wish to prove that

$$w_k[(g_k + 1\!1)h^{1/2} + iH(g_k h^{1/2})] = h^{1/2} = [h^{1/2}(g_k + 1\!1) + iH(h^{1/2}g_k)]w_k.$$

In view of the similarity of the proofs, we prove only the first equality. Notice that $(v(k)_\lambda[(g_k + 1\!1)h^{1/2} + iH(g_k h^{1/2})])$ converges weakly in L^2 to $w_k[(g_k + 1\!1)h^{1/2} + iH(gh^{1/2})]$. By Lemma 4.5 we have that $(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda)h^{1/2} \to g_k h^{1/2}$ in L^2-norm, and so also $H((a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda)h^{1/2}) \to H(g_k h^{1/2})$ in L^2-norm by the continuity of H established in [27]. Since the $v(k)_\lambda$'s are contractive, it easily follows that

$$\|v(k)_\lambda[(g_k + 1\!1)h^{1/2} + iH(g_k h^{1/2})] - h^{1/2}\|_2$$
$$= \|v(k)_\lambda[(g_k + 1\!1)h^{1/2} + iH(g_k h^{1/2})]$$
$$\qquad -v(k)_\lambda[((a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + 1\!1) + iH(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda))h^{1/2}]\|_2$$
$$\le \|[(g_k + 1\!1)h^{1/2} + iH(g_k h^{1/2})]$$
$$\qquad -((a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + 1\!1) + iH(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda))h^{1/2}\|_2.$$

Hence $(v(k)_\lambda[(g_k + 1\!1)h^{1/2} + iH(g_k h^{1/2})])$ is norm convergent to $h^{1/2}$. The claim regarding the g_k's now follows from the uniqueness of limits.

Since (w_k) is bounded, it will admit a weak* convergent subnet (w_γ). Let w_g be the limit of that subnet. The claim regarding w_g can now be verified with an essentially similar proof, but with the roles of $v(k)_\lambda$ and $(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda)$ respectively being played by w_γ and g_γ, and with Lemma 4.2 replacing Lemma 4.5. Thus we begin by noting that

$$w_\gamma[(g + 1\!1)h^{1/2} + iH(gh^{1/2})] \to w_g[(g + 1\!1)h^{1/2} + iH(gh^{1/2})]$$

weakly in L^2. Amending the previous argument as described above now leads to the conclusion that

$$\|w_\gamma[(g + 1)h^{1/2} + iH(gh^{1/2})] - h^{1/2}\|_2 \to 0.$$

So again the claim regarding the g's follows from the uniqueness of limits. □

We proceed to use Proposition 4.7 to analyze the structure of $[(g + 1)h^{1/2} + iH(gh^{1/2})]$.

Theorem 4.8. *For any $n \in \mathbb{N}$, we have $\|p_n w_g\| \leq \sqrt{\frac{2}{n(n+1)}}$.*

Proof. Let g_k and w_k be as in Proposition 4.7; we had there a weak* convergent subnet of the latter sequence with limit w_g. As usual one may replace (w_k) by the subnet. For ease of notation, we will assume that (w_k) is weak* convergent to w_g. It then suffices to show that $\|p_n w_k\| \leq \sqrt{\frac{2}{n(n+1)}}$ for every $k \geq n$. To see this recall that the closed ball of radius $\sqrt{\frac{2}{n(n+1)}}$ is weak* closed. So if each $p_n w_k$ ($k \geq n$) is in this ball, so is $p_n w_g$.

Observe that for a, d, and v as in Lemma 4.6, we have

$$
\begin{aligned}
v^*(a + d + a^* + 1)v &= \frac{1}{2}v^*[((a^* + d + a + 1) + iH(a^* + d + a)) \\
&\qquad + ((a^* + d + a + 1) - iH(a^* + d + a))]v \\
&= \frac{1}{2}[v + v^*].
\end{aligned}
$$

Since v and v^* are both contractive, this means that $v^*(a + d + a^* + 1)v \leq 1$. This in turn ensures that

$$
\begin{aligned}
&(a + d + a^* + 1)vv^*(a + d + a^* + 1) \\
&= (v^{-1})^*v^*(a + d + a^* + 1)vv^*(a + d + a^* + 1)vv^{-1} \\
&\leq (v^{-1})^*v^*(a + d + a^* + 1)vv^{-1} \\
&= (a + d + a^* + 1).
\end{aligned}
$$

Hence

(4.2)
$$
\begin{aligned}
&\|v_\lambda^*(a_\lambda^* + d_\lambda + a_\lambda + 1)w_k h^{1/2}a\|^2 \\
&= \langle w_k^*(a_\lambda^* + d_\lambda + a_\lambda + 1)v_\lambda v_\lambda^*(a_\lambda^* + d_\lambda + a_\lambda + 1)w_k h^{1/2}a , h^{1/2}a\rangle \\
&\leq \langle w_k^*(a_\lambda^* + d_\lambda + a_\lambda + 1)w_k h^{1/2}a , h^{1/2}a\rangle \\
&= \langle(a_\lambda^* + d_\lambda + a_\lambda + 1)w_k h^{1/2}a , w_k h^{1/2}a\rangle.
\end{aligned}
$$

Now let $a \in M$ be given, and let the nets $(a(k)_\lambda) \subset A_0$, $(d(k)_\lambda) \subset D_+$ be as in Lemma 4.5. Then the nets $(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + 1)w_k h^{1/2}a$ converge to $(g_k + 1)w_k h^{1/2}a$ in L^2-norm. As we saw in the proof of Proposition 4.7, on passing to a subnet if necessary, we may assume that the nets $(v(k)_\lambda)$'s described by Lemma 4.6 are weak* convergent to the w_k's.

Since the $v(k)_\lambda$'s are contractive, we have that

$$
\begin{aligned}
&\|[v(k)_\lambda^*(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + 1)w_k h^{1/2}a] - [v(k)_\lambda^*(g_k + 1)w_k h^{1/2}a]\| \\
&\leq \|[(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + 1)w_k h^{1/2}a] - [(g_k + 1)w_k h^{1/2}a]\| \to 0.
\end{aligned}
$$

Thus

$$[v(k)_\lambda^*(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + 1)w_k h^{1/2}a] - [v(k)_\lambda^*(g_k + 1)w_k h^{1/2}a] \to 0$$

in norm. Since also $v(k)_\lambda^*(g_k + \mathbb{1})w_k h^{1/2}a$ is weakly convergent in $L^2(M)$ to $w_k^*(g_k + \mathbb{1})w_k h^{1/2}a$, it follows that $v(k)_\lambda^*(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + \mathbb{1})w_k h^{1/2}a$ is weakly convergent to $w_k^*(g_k + \mathbb{1})w_k h^{1/2}a$.

We proceed to show that $\|(g_k + \mathbb{1})^{1/2}w_k h^{1/2}a\|_2 \le \|h^{1/2}a\|_2$. To see this we first observe that

$$\langle v(k)_\lambda^*(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + \mathbb{1})w_k h^{1/2}a, h^{1/2}a\rangle$$
$$\to \langle w_k^*(g_k + \mathbb{1})w_k h^{1/2}a, h^{1/2}a\rangle = \|(g_k + \mathbb{1})^{1/2}w_k h^{1/2}a\|^2$$

and that

$$\langle(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + \mathbb{1})w_k h^{1/2}a\,,\, w_k h^{1/2}a\rangle$$
$$\to \langle(g_k + \mathbb{1})w_k h^{1/2}a, w_k h^{1/2}a\rangle = \|(g_k + \mathbb{1})^{1/2}w_k h^{1/2}a\|^2.$$

Next observe that by inequality (4.2), we have that

$$\|v(k)_\lambda^*(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + \mathbb{1})w_k h^{1/2}a\|^2$$
$$= \langle(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + \mathbb{1})w_k h^{1/2}a\,,\, w_k h^{1/2}a\rangle.$$

It follows from the above inequality that

$$\langle v(k)_\lambda^*(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + \mathbb{1})w_k h^{1/2}a\,,\, h^{1/2}a\rangle$$
$$\le \|v(k)_\lambda^*(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + \mathbb{1})w_k h^{1/2}a\|.\|h^{1/2}a\|$$
$$\le [\langle(a(k)_\lambda^* + d(k)_\lambda + a(k)_\lambda + \mathbb{1})w_k h^{1/2}a\,,\, w_k h^{1/2}a\rangle]^{1/2}.\|h^{1/2}a\|.$$

On taking limits, we have $\|(g_k + \mathbb{1})^{1/2}w_k h^{1/2}a\|^2 \le \|(g_k + \mathbb{1})^{1/2}w_k h^{1/2}a\|.\|h^{1/2}a\|$ or, equivalently, $\|(g_k + \mathbb{1})^{1/2}w_k h^{1/2}a\| \le \|h^{1/2}a\|$ as claimed.

Finally note that since the p_n's are decreasing, we as before have that

$$\frac{(n+1)n}{2}p_n = \sum_{m=1}^n mp_n \le \sum_{m=1}^n mp_m,$$

which is dominated by $\sum_{m=1}^k mp_m = g_m$. Hence

$$\begin{aligned}
\|p_n w_k h^{1/2}a\|^2 &= \langle w_k^* p_n w_k(h^{\frac{1}{2}}a), (h^{\frac{1}{2}}a)\rangle \\
&\le \frac{2}{n(n+1)}\langle w_k^* g_k w_k(h^{\frac{1}{2}}a), (h^{\frac{1}{2}}a)\rangle \\
&\le \frac{2}{n(n+1)}\langle w_k^*(g_k + \mathbb{1})w_k(h^{\frac{1}{2}}a), (h^{\frac{1}{2}}a)\rangle \\
&= \frac{2}{n(n+1)}\|(g_k + \mathbb{1})^{1/2}w_k h^{1/2}a\|^2 \\
&\le \frac{2}{n(n+1)}\|h^{1/2}a\|^2.
\end{aligned}$$

Since the subspace $\{h^{1/2}a : a \in M\}$ is dense in $L^2(M)$, it follows that the operator of left multiplication by $p_n w_k$ on $L^2(M)$ has norm dominated by $\sqrt{\frac{2}{n(n+1)}}$. This proves the claim. $\qquad\square$

Thus with $b = 1 - w_g$ we deduce that $\|p_k - p_k b\| \le \sqrt{\frac{2}{k(k+1)}}$ as needed for the argument in [41] to proceed. Indeed the rest of that argument is identical. We obtain $p_0 = p_0 b = bp_0$, where p_0 is the strong limit of (p_n) in M^{**}, and if $a = (1 + b)/2$, then (a^n) converges weak* to a peak projection $p \ge p_0$ with

$|\varphi|(p) = |\varphi|(p_0) = |\varphi|(1)$. If $\|a\xi\|_2 = \|\xi\|_2$ for $\xi \in L^2(M)$, then as in [41] we obtain $b\xi = \xi$, so that in our notation above we have $\xi \in \mathrm{Ker}(w_g) = 0$. However $\mathrm{Ker}(w_g) = (0)$. Indeed the projection associated with the kernel is in M, and if $e \in M$ is a projection with $w_g e = 0$, then by the last equality in Proposition 4.7 we obtain $h^{1/2} e = 0$, so that $e = 0$. Hence as in [41] (which relies here on the noncommutative peak theory [24]; see also, e.g., [6,9]) we obtain $a^n \to 0$ weak* in M. This completes the proof of the generalization of Ueda's peak set theorem to σ-finite algebras.

5. Consequences of Ueda's peak set theorem for σ-finite M

All the other consequences from [41] of Ueda's peak set theorem now go through with unaltered proofs for maximal subdiagonal subalgebras A of a σ-finite von Neumann algebra M. Indeed this is true rather generally. If A is a weak* closed subalgebra of a von Neumann algebra M, then we say that A is an *Ueda algebra* in M if Ueda's peak set theorem holds for A, that is, if for every singular state on M there is a peak projection q for A with $\varphi(q) = 1$ and q is dominated by the 'singular part' projection of M^{**}, as in the restatement after Theorem 4.1. The ideas in [13, Lemma 9.1] give the following restatement:

Corollary 5.1. *Suppose that A is a weak* closed subalgebra of a von Neumann algebra M. Then A is an Ueda algebra in M if and only if for every singular state φ on M, there exists a (increasing, if desired) sequence (q_n) of projections in $\mathrm{Ker}(\varphi)$ with supremum 1 in M and supremum in M^{**} lying in $A^{\perp\perp}$. If (q_n) is increasing then the last condition means that $\psi(q_n) \to 0$ for any $\psi \in A^\perp$.*

Proof. By Theorem 2.5, the information about q in the lines above the present corollary is equivalent to: there is a (decreasing, if desired) sequence (q_n) of projections in M with infimum q in M^{**} lying in $A^{\perp\perp}$ satisfying $\varphi(q) = 1$ and $\psi(q) = 0$ for all normal states ψ of M. As in [13, Lemma 9.1] the last condition is equivalent to the infimum in M of (q_n) being 0, and $\varphi(q) = 1$ if and only if $\varphi(q_n) = 0$ for all n. Finally set $p = q^\perp$ and replace q_n by $q_n^\perp$. $\square$

We remark that if A is an Ueda algebra, then it is easy to see that so is $A^* = \{x^* : x \in A\}$.

Corollary 5.2. *Suppose that a weak* closed subalgebra A of a von Neumann algebra M is an Ueda algebra. If $\varphi \in M^*$ has nonzero singular part φ_s, then there exist a contraction $a \in A$ and a projection $p \in M^{**}$ with $a^n \to p$ weak* in M^{**}, $a^n \to 0$ weak* in M, and $\varphi_s = \varphi \cdot p$.*

Theorem 5.3. *Suppose that a weak* closed subalgebra A of a von Neumann algebra M is an Ueda algebra. Write A_s^* and A_n^* for the set of restrictions to A of singular and normal functionals on M. Each $\varphi \in A^*$ has a unique Lebesgue decomposition relative to M: $\varphi = \varphi_n + \varphi_s$ with $\varphi_n \in A_n^*$ and $\varphi_s \in A_s^*$. Moreover, $\|\varphi\| = \|\varphi_n\| + \|\varphi_s\|$.*

Corollary 5.4. *Suppose that a weak* closed subalgebra A of a von Neumann algebra M is an Ueda algebra. Then the predual A_* of A is unique and is an L-summand in A^*. Also, A_* has property (V^*) and is weakly sequentially complete.*

(See also, e.g., [31] for recent similar results for a completely different class of dual operator algebras.)

Theorem 5.5 (F & M Riesz type theorem). *Suppose that a weak* closed subalgebra A of a von Neumann algebra M is an Ueda algebra. If $\varphi \in M^*$ annihilates A (that is, $\varphi \in A^\perp$), then the normal and singular parts, φ_n and φ_s, also annihilate A.*

Our proofs from [7] then give the following results (suitably modified by an appeal to Theorem 5.5 instead of to the F & M type theorem in [7]), as noted in [41] and suggested by the referee of that paper. One may define an *F & M Riesz algebra* to be a weak* closed subalgebra A of a von Neumann algebra M such that if $\varphi \in A^\perp$, then the normal and singular parts, φ_n and φ_s, also annihilate A. Theorem 5.5 then says that any Ueda algebra is an F & M Riesz algebra. Again, it is easy to argue (by considering $\psi^*(x) = \overline{\psi(x^*)}$) that if A is an F & M Riesz algebra, then so is $A^* = \{x^* : x \in A\}$. By proofs in [7] we then have:

Corollary 5.6. *Suppose that A is an F & M Riesz algebra in a von Neumann algebra M such that $A + A^*$ is weak* dense in M. If $\varphi \in M^*$ annihilates $A + A^*$, then φ is singular. Any normal functional on M is the unique Hahn-Banach extension of its restriction to $A + A^*$ and in particular is normed by $A + A^*$. In addition, any Hahn-Banach extension to M of a weak* continuous functional on A is normal.*

Corollary 5.7. *If A is an F & M Riesz algebra in a von Neumann algebra M such that $A + A^*$ is weak* dense in M, then $\mathrm{Ball}(A + A^*)$ is weak* dense in $\mathrm{Ball}(M)$.*

Moreover in this case we obtain all the assertions of Theorem 3.2 too.

The last assertion of Corollary 5.6 is related to the well known Gleason-Whitney theorem in function theory. A special case of the following appears in [7, Theorem 4.1] and [34, Theorem 3.4]. We can express some of the ideas in those results more abstractly and generally as follows:

Lemma 5.8. *Suppose A is a weak* closed subalgebra of a von Neumann algebra M. Then $A + A^*$ is weak* dense in M if and only if there is at most one normal Hahn-Banach extension to M of any normal weak* continuous functional on A.*

Proof. ($\Rightarrow$) Choose $a \in \mathrm{Ball}(A)$ such that $\varphi(a) = 1$, and let e be the left support of a, which is the support of aa^*. We may suppose that $\varphi \in M_*$ and that ψ is another normal Hahn-Banach extension of $\varphi_{|A}$. We have

$$1 = \varphi(a) \leq |\varphi|(aa^*) \leq |\varphi|(e),$$

so that $|\varphi|(e^\perp) = 0$. Hence $|\varphi|e^\perp = 0$ and $\varphi e^\perp = 0$. Similarly $\psi e^\perp = 0$ and $(\varphi - \psi)e^\perp = 0$. Next note that φa is contractive and unital, so positive. Similarly for ψ, and so $(\varphi - \psi)a$ is a selfadjoint normal functional. It vanishes on A, hence also on $A + A^*$ and on M. From this it is easy to see that $(\varphi - \psi)e = 0$. So $\varphi - \psi = (\varphi - \psi)e + (\varphi - \psi)e^\perp = 0$.

($\Leftarrow$) It is enough to show that if normal ψ annihilates $A + A^*$, then $\psi = 0$. By taking real and imaginary parts we may assume that $\psi = \psi^*$. Suppose $\psi = \psi_1 - \psi_2$ for positive normal ψ_k. Then $\psi_1 = \psi_2 + \psi$ and ψ_2 agree on A and are normal Hahn-Banach extensions since the norm of a positive functional is its value at 1. So $\psi_1 = \psi_2$ and $\psi = 0$. $\square$

Corollary 5.9 (Gleason-Whitney type theorem). *Suppose that A is an F & M Riesz algebra in a von Neumann algebra M. Then $A + A^*$ is weak* dense in M if and only if every normal functional on A has a unique Hahn-Banach extension to M and if and only if every normal functional on A has a unique normal Hahn-Banach extension to M.*

Of course all of these hold when A is a maximal subdiagonal subalgebra of a σ-finite von Neumann algebra M. Conversely these properties characterize maximal subdiagonal subalgebras. The following is a partial strengthening of [34, Theorem 3.4] (the equivalence of (i) and (iv) there).

Corollary 5.10 (Gleason-Whitney type theorem). *Let A be a weak* closed unital subalgebra of a σ-finite von Neumann algebra M, for which*

- *$\sigma_t^\nu(A) = A$ for each $t \in \mathbb{R}$ (where σ_t^ν is the modular automorphism group for M described in our Introduction), and*
- *the canonical expectation $\mathcal{E} : M \to A \cap A^* = D$ is multiplicative on A.*

Then $A + A^*$ is weak* dense in M (that is, A is maximal subdiagonal with respect to D) if and only if every normal functional on A has a unique normal Hahn-Banach extension to M.

6. The case of semi-finite and general von Neumann algebras

We first briefly discuss the results of our paper in the setting of general von Neumann algebras. We recall from, e.g., [35, Proposition 2.2.5] that any von Neumann algebra M is a direct sum of algebras M_i of the form $R_i \bar{\otimes} B(\mathfrak{H}_i)$ for a σ-finite von Neumann algebra R_i. If A is a maximal subdiagonal algebra in M and if the center of M is contained in the center of $A \cap A^*$, then the central projections corresponding to the direct sum will allow a decomposition of a maximal subdiagonal algebra A of M as a direct sum of algebras $A_i \subset M_i$, and it is easy to see that these are maximal subdiagonal subalgebras of M_i. Assuming that the $B(\mathfrak{H}_i)$'s appearing in the form of M_i above correspond to separable Hilbert spaces $\mathfrak{H}_i$, then $R_i \bar{\otimes} B(\mathfrak{H}_i)$ is σ-finite, and we get Ueda's theorem in this case (Theorem 4.1 but with the σ-finite M replaced by our M above). We immediately deduce that all the results in the last section (Section 5) are valid for this A and M.

We now turn to investigating when Ueda's peak set theorem fails. Of course if Ueda's peak set theorem fails for a von Neumann algebra M, then it also fails for every weak* closed unital subalgebra A of M. Thus henceforth in this section we shall assume that $A = M$.

An *Ulam measurable cardinal* is one such that if I is a set of this cardinality, then there exists a free ultrafilter p on I such that every sequence $A_1 \supseteq A_2 \supseteq \cdots$ of nonempty sets in p has nonempty intersection [15, 25]. (Remark: It is a pleasant exercise that an ultrafilter allows no empty countable intersections of members if and only if it is closed under countable intersections. Also, one may always make countable intersections 'decreasing'.) The concept of *measurable cardinal* used in the next result will be explained a little more at the start of its proof. This result shows that there is little chance of generalizing Ueda's peak set theorem to semi-finite von Neumann algebras in the usual set theoretic universe used in most of functional analysis, since this would imply a solution to one of the famous open "problems" in mathematics. We use quotes because nowadays this problem is not believed to be solvable.

The strategy of our proof is simple: it is known that a bound on the size of a set I in relation to being of measurable cardinality is equivalent to being 'realcompact'. Also, I not being realcompact is known to imply that $\beta I \setminus I$ contains points not contained in closed G_δ subsets of βI of a certain type. Finally, for $C(K)$ spaces the closed G_δ sets are exactly the peak sets, by the strict form of the Urysohn

lemma or as in Proposition 2.3. However since we lack a good reference (besides scattered pieces found in an internet search for 'realcompact discrete spaces'; see, e.g., [15, p. 402 ff.]), we will include short arguments for several of these points for the reader's convenience.

Theorem 6.1. *If Ueda's peak set theorem held for all finite von Neumann algebras then there exist no* (*uncountable*) *measurable cardinals.*

Proof. The existence of (uncountable) measurable cardinals is known to be equivalent to the existence of Ulam measurable cardinals [15, 25]. Suppose that I was an uncountable set of Ulam measurable cardinality. Clearly $M = \ell^\infty(I)$ is a finite (and semi-finite) von Neumann algebra, and $A = M$ is a maximal subdiagonal algebra. We may view the free ultrafilter p in the definition of Ulam measurable cardinality as a (singleton) closed set in $\beta I \setminus I$. It is the support of a Dirac probability measure, which can be viewed as a pure state on $M = C(\beta I)$ (evaluation at p). This state is singular (we leave this as an exercise since there are many ways to see this). Moreover via the well known correspondences between minimal projections and pure states and their supports, the support of this state in $C(\beta I)^{**}$ is the minimal projection which is the image e of the characteristic function of $F = \{p\}$ in $C(\beta I)^{**}$ (that is, it is the image of the functional $\mu \mapsto \mu(F)$ on $C(\beta I)^*$, viewing the latter as a space of measures). Indeed here we are just invoking aspects of the well known noncommutative dictionary between the basic theory of probability measures and that of states.

If Ueda's theorem held for M, then there would exist a peak projection $q \in C(\beta I)^{**}$ with $e \leq q \leq z$, where z is the orthogonal complement of the canonical projection in M^{**} corresponding to M_*. These three projections e, z, q correspond to closed sets in βI, namely to sets $F = \{p\}, \beta I \setminus I$, and E say, respectively, and the latter is a classical peak set by the 'peaking' theory [5, 6, 9–11, 24]. (That z corresponds to $\beta I \setminus I$ is well known and was sketched in an earlier version of the present paper, available on arXiv.)

By Theorem 2.5, the characteristic function of any peak set E for M is an intersection of a decreasing sequence of projections in $M = C(\beta I) = l^\infty(I)$. Thus by the theory of the Stone-Čech compactification, $E = \bigcap_{n=1}^\infty [A_n]$, where $[A_n]$ is the (clopen) closure in βI of (open) $A_n \subset I$, where $A_1 \supseteq A_2 \supseteq \cdots$. Also, $\bigcap_{n=1}^\infty [A_n] \subset \beta I \setminus I$ if and only if $\bigcap_{n=1}^\infty A_n = \emptyset$. To see the latter, note that

$$\bigcap_{n=1}^\infty A_n \subset (\bigcap_{n=1}^\infty [A_n]) \cap I \subset (\beta I \setminus I) \cap I = \emptyset$$

if $\bigcap_{n=1}^\infty [A_n] \subset \beta I \setminus I$. The converse follows from the inclusion

$$I \cap (\bigcap_{n=1}^\infty [A_n]) \subset I \cap [A_n] = A_n, \qquad n \in \mathbb{N}.$$

Thus for any closed subset F of a peak set E for $C(\beta I)$, with $E \subset \beta I \setminus I$, we have $F \subset \bigcap_{n=1}^\infty [A_n]$ for sets $A_1 \supseteq A_2 \supseteq \cdots$ in I with empty intersection. In our special case where $F = \{p\}$, the fact that $p \in [A_n]$ implies that $A_n \in p$ for all $n \in \mathbb{N}$, with p regarded as an ultrafilter. This contradicts the property p has in the definition of Ulam measurable cardinality above. So there is no Ulam measurable cardinal. $\square$

Remark 6.2. By the last proof Ueda's peak set theorem holding for $M = A = \ell^\infty(I)$ is equivalent to saying that every closed set F in $\beta I \setminus I$ which is the support of a

Borel probability measure is contained in $\bigcap_{n=1}^{\infty} [A_n]$ for sets $A_1 \supseteq A_2 \supseteq \cdots$ in I with empty intersection. Closed sets in $\beta I \setminus I$ have a nice characterization in the basic literature of the Stone-Čech compactification.

It turns out that Ueda's peak set theorem also fails when $M = A = B(H)$ with H of dimension an Ulam measurable cardinal, or a real-valued measurable cardinal, as is discussed with Nik Weaver in [13]. Indeed in that paper (which was written after the first distributed version of the present paper) it is shown that if M is a von Neumann algebra, then Ueda's peak set theorem fails when $M = A$ if and only if M possesses a singular state φ which is *regular*; that is, $\varphi(\bigvee_n q_n) = 0$ for every sequence of projections (q_n) in $\operatorname{Ker}(\varphi)$. (See [23] for other characterizations and facts about regular states; hence Ueda's peak set theorem is strongly tied to 'quantum measure theory' in the sense of that reference.) This is also equivalent to saying that there is a collection of mutually orthogonal projections in M of cardinality $\geq$ a fixed cardinal κ, namely the first cardinal on which there is a 'regular' singular finitely additive probability measure. (Here and below measures are assumed to be defined on all subsets of the cardinal.) The existence of regular singular states or such regular measures is generally believed to be consistent with ZFC set theory. Indeed as explained in [13] it is believed to be consistent with ZFC set theory that the latter 'first cardinal' is $\leq$ the cardinality of the real numbers. On the other hand, since any cardinal on which there is a singular probability measure dominates the 'first cardinal' above, it follows that if $M \subset B(H)$ where $\dim(H)$ is smaller than any real-valued measurable cardinal (or if measurable cardinals do not exist), then Ueda's peak set theorem holds for M (and taking $A = M$).

From the assertion in the last paragraph about the cardinality of the real numbers, it follows that one should not hope to be able to prove Ueda's peak set theorem for $A = M = l^\infty(\mathbb{R})$ in ZFC. Indeed Ueda's theorem in this case implies by the assertion in the last paragraph about regular states a negative solution to the famous 'Banach measure problem': Is there a probability measure defined on all subsets of $[0, 1]$ which is zero on singletons? (It is well known that if there is, then one can find another that extends Lebesgue measure.) Banach showed that you cannot prove an affirmative answer to this in ZFC. The existence of a negative answer is equivalent to the nonexistence of measurable cardinals in ZFC. However, as we have stated earlier, it is generally believed by set theorists that the existence of measurable cardinals is consistent with ZFC.

This shows that one cannot hope to be able to prove Ueda's peak set theorem in ZFC for von Neumann algebras that are much 'bigger' than σ-finite (the case of the main theorem of our paper). And indeed experts in von Neumann algebras are usually happy to consider only σ-finite von Neumann algebras in their results, because 'bigger' algebras are often pathological. On the other hand, it is shown in [13] that Ueda's peak set theorem holds in ZFC for $A = M = l^\infty(\aleph_1)$, where $\aleph_1$ is the first uncountable cardinal, and this von Neumann algebra is not σ-finite. Hence if we assume the continuum hypothesis, then Ueda's peak set theorem does hold for $A = M = l^\infty(\mathbb{R})$. Assuming the negation of the continuum hypothesis, a remaining question seems to be, for what cardinals κ between the infinite countable cardinal and the cardinality of the reals can one prove Ueda's peak set theorem in ZFC for $A = M = l^\infty(\kappa)$? Nik Weaver has sketched for us a proof in the case of $\aleph_2$, and this trick seems to extend to $\aleph_n$ for $n \in \mathbb{N}$.

Thinking about the last paragraph in conjunction with the proof of our main theorem suggests to us that it may possibly be interesting to study Haagerup's reduction theory, the standard form, and related topics for von Neumann algebras possessing uncountable collections of mutually orthogonal projections of cardinality smaller than the cardinality of the reals (assuming of course the negation of the continuum hypothesis).

Acknowledgments

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DAVID P. BLECHER AND LOUIS LABUSCHAGNE

Department of Mathematics, University of Houston, Houston, Texas 77204-3008
Email address: `dblecher@math.uh.edu`

DST-NRF CoE in Mathematical and Statistical Sciences, Unit for BMI, Internal Box 209, School of Computer, Statistical, and Mathematical Sciences, North-West University, PVT. BAG X6001, 2520 Potchefstroom, South Africa
Email address: `louis.labuschagne@nwu.ac.za`