Publ. Math. Debrecen
82/2 (2013), 341–358
DOI: 10.5486/PMD.2013.5305

Convex solutions of the polynomial-like iterative equation in Banach spaces

By XIAOBING GONG (Chengdu) and WEINIAN ZHANG (Chengdu)

Abstract. Although convex (concave) solutions were investigated for the polynomial-like iterative equation on a compact interval of $\mathbb{R}$, there are much more difficulties in discussion on convexity of solutions in Banach spaces. In this paper we consider a partial order in Banach spaces, which is defined by an order cone, and discuss monotonicity and convexity of operators under iteration in Banach spaces. Then we give the existence of monotone solutions in the ordered real Banach spaces and further obtain conditions under which the solutions are convex or concave in the order. Moreover, the uniqueness and continuous dependence of those solutions are also discussed.

1. Introduction

As indicated in the books [9], [26] and the surveys [2], [34], the polynomial-like iterative equation

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x), \quad x \in S, \tag{1.1}$$

where S is a subset of a linear space over $\mathbb{R}$, $F : S \to S$ is a given function, λ_is $(i = 1, \ldots, n)$ are real constants, $f : S \to S$ is the unknown function and f^i is the ith iterate of f, i.e., $f^i(x) = f(f^{i-1}(x))$ and $f^0(x) = x$ for all $x \in S$, is one of important forms of functional equation since the problem of iterative roots and the problem of invariant curves can be reduced to the kind of equations. For $S \subset \mathbb{R}$, in addition to those works for linear F (see e.g. [6], [7], [16], [17], [18], [24], [30]),

Mathematics Subject Classification: 39B12, 58F08.
Key words and phrases: functional equation, iteration, Banach spaces, order, increasing operator, decreasing operator.
Supported by NSFC and MOE grants.

many results were given to the case of nonlinear F in, for example, [15], [39] for $n = 2$, [35] for general n, [14], [36] for smoothness, and [22] for analyticity. Efforts were also made to the case that $S \subset \mathbb{R}^N$ ($N \geq 2$). Radially monotonic solutions were discussed in $\mathbb{R}^N$ in [37] by properties of orthogonal groups. The existence of Lipschitzian solutions were proved in [11] for $n = \infty$ on a compact convex subset of $\mathbb{R}^N$ by applying the Schauder's fixed point theorem. In 2004, results of [11] were generalized partly to an arbitrary closed (not necessarily convex) subset of a Banach space in [25]. In addition, in [23] J. TABOR investigated the difference equation $\sum_{k=0}^{+\infty} A_k x_{k+n} = 0$ in a Banach space, where $(A_k)_{k\in\mathbb{N}}$ is a sequence of bounded linear operators, and applied the results to iterative equations. In [19] equation (1.1) was discussed for linear operator f on an (infinite dimensional) linear space S and the results were applied to answering the question on annihilating polynomials of linear operators. Along with the deep research of functional equations, generalization to infinite dimensional cases attracts attentions as in [3], [20].

Convexity is an important properties of functions and often used in optimization, mathematical programming and game theory. The study of convexity for iterative equations can be traced to 1968, when KUCZMA and SMAJDOR [10] investigated the convexity of iterative roots. Some recent results can be found from [27], [28], [38]. In [38], convex solutions and concave ones of equation (1.1) were discussed under the normalization condition: $\sum_{j=1}^{n} \lambda_j = 1$ on a compact interval. More concretely, the existence and uniqueness of convex (resp. concave) solutions with uniform non-positiveness of $\lambda_2, \ldots, \lambda_n$ and increasing convex (resp. concave) with uniform non-negativeness of $\lambda_2, \ldots, \lambda_n$ are proved. Later,as a continuation of [38], increasing convex (or concave) solutions and decreasing convex (or concave) solutions of equation (1.1) are investigated in [28], relaxing the normalization condition and the requirement of uniform sign of coefficients. In [27], nondecreasing convex solutions for equation (1.1) on open intervals are discussed. Up to now, there are no further results on convexity of solutions for iterative equations in high dimensional spaces. Actually, in high dimensional spaces there are much more difficulties on convexity as well as monotonicity.

In this paper we study convexity of solutions for equation (1.1) in Banach spaces. Unlike the 1-dimensional case (in which there is a natural order), both monotonicity and convexity depend on an appropriate order in infinite-dimensional cases. We consider a partial order in Banach spaces, which is defined by an order cone, and give existence of increasing solutions and decreasing solutions in the ordered real Banach spaces. Then we further give conditions under which those

solutions are convex or concave. The uniqueness and continuous dependence of those solutions are also discussed.

2. Convexity in ordered Banach spaces

As indicated in the Introduction, we need a partial order when we discuss convexity of solutions in Banach spaces. As in [33], a nonempty subset K of a real vector space X is called a *cone* if $x \in K$ implies that $ax \in K$ for all $a > 0$. A nonempty and nontrivial ($\neq \{\theta\}$, where θ denotes the zero element of X) subset $K \subset X$ is called an *order cone* in X if K is a convex cone and satisfies $K \cap (-K) = \{\theta\}$. Having chosen such an order cone K in X, we can define a partial order $x \leq_K y$ in X, simply called the *K-order*, if

$$y - x \in K.$$

Sometimes, we write $y \geq_K x$ for convenience when $x \leq_K y$. As intuitive examples, the set $\{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \geq 0\}$ is an order cone in $\mathbb{R}^2$; in contrast, the set $\{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ is a cone but not an order cone. As indicated in [1, p. 626], the above defined K-order is linear, i.e., it satisfies

(C1) $x \leq_K y$ in X implies $x + z \leq_K y + z$ for all $z \in X$, and

(C2) $x \leq_K y$ in X implies $\alpha x \leq_K \alpha y$ for all $\alpha \in [0, +\infty)$.

A real vector space X equipped with a K-order is called an *ordered vector space*, abbreviated by OVS and denoted by (X, K). A real Banach space $(X, \|\cdot\|)$ associated with a K-order is called an *ordered real Banach space*, abbreviated by OBS and denoted by $(X, K, \|\cdot\|)$, if K is closed.

Lemma 2.1. *Let $(X, \|\cdot\|)$ be an ordered real Banach space with the order $\leq_K$. Then* (i) *$x \leq_K y$ implies $-y \leq_K -x$, and* (ii) *$x_n \leq_K y_n, x_n \to x$ and $y_n \to y$ imply $x \leq_K y$, where $x, x_n, y, y_n \in X$.*

The results (ii) in Lemma 2.1 can be found from [33, p. 277, Proposition 7.5]. If $x \leq_K y$ then $y - x \in K$, implying $-x - (-y) \in K$, i.e., $-y \leq_K -x$. Hence (i) is proved.

In an ordered real vector space (X, K) one can define increasing (decreasing) operators and convex (concave) operators as in [1]. An operator $f : D \subset X \to X$ is said to be *increasing (*resp. *decreasing)* in the sense of the K-order if $x \leq_K y$ implies $f(x) \leq_K f(y)$ (resp. $f(x) \geq_K f(y)$). An operator $f : D \to X$, where $D \subset X$ is a convex subset, is said to be *convex (*resp. *concave)* in the sense of the K-order if $f(\lambda x + (1 - \lambda)y) \leq_K \lambda f(x) + (1 - \lambda)f(y)$ (resp. $f(\lambda x + (1 -$

$\lambda)y) \geq_K \lambda f(x) + (1-\lambda)f(y))$ for all $\lambda \in [0,1]$ and for every pair of distinct comparable points $x, y \in D$ (i.e., either $x \leq_K y$ or $x \geq_k y$). When $X = \mathbb{R}$ and $K := [0, +\infty)$, an increasing (resp. decreasing) operator in the sense of K-order is a usual increasing (resp. decreasing) function of single real variable, and a convex (resp. concave) operator in the sense of K-order is a usual convex (resp. concave) function of single real variable.

Let Ω be a compact convex subset of an ordered real Banach space $(X, K, \|\cdot\|)$ with nonempty interior and $C(\Omega, X)$ consist of all continuous functions $f : \Omega \to X$. $C(\Omega, X)$ is a Banach space equipped with the norm $\|f\|_{C(\Omega,X)} := \sup_{x\in\Omega} \|f(x)\|$. For $0 \leq m \leq M < +\infty$, define

$$\begin{aligned}
C^+(\Omega, m, M) := \{&f \in C(\Omega, X) : f(\Omega) \subset \Omega,\\
&m(y-x) \leq_K f(y) - f(x) \leq_K M(y-x) \text{ if } x \leq_K y, \text{ and}\\
&\|f(y) - f(x)\| \leq M\|y-x\| \text{ if } x \text{ and } y \text{ are not comparable}\},\\
C^-(\Omega, m, M) := \{&f \in C(\Omega, X) : f(\Omega) \subset \Omega,\\
&m(y-x) \leq_K f(x) - f(y) \leq_K M(y-x) \text{ if } x \leq_K y, \text{ and}\\
&\|f(y) - f(x)\| \leq M\|y-x\| \text{ if } x \text{ and } y \text{ are not comparable}\},\\
C^+_{cv}(\Omega, m, M) := \{&f \in C^+(\Omega, m, M) : f \text{ is convex on } \Omega \text{ in } K\text{-order}\},\\
C^+_{cc}(\Omega, m, M) := \{&f \in C^+(\Omega, m, M) : f \text{ is concave on } \Omega \text{ in } K\text{-order}\},\\
C^-_{cv}(\Omega, m, M) := \{&f \in C^-(\Omega, m, M) : f \text{ is convex on } \Omega \text{ in } K\text{-order}\},\\
C^-_{cc}(\Omega, m, M) := \{&f \in C^-(\Omega, m, M) : f \text{ is concave on } \Omega \text{ in } K\text{-order}\}.
\end{aligned}$$

In contrast to the classes of functions considered in, e.g., [28], [38], we need an additional condition called *Lipschitzian without K-order*, i.e.,

$$\|f(y) - f(x)\| \leq M\|y-x\| \text{ if } x \text{ and } y \text{ are not comparable},$$

in classes $C^+(\Omega, m, M)$ and $C^-(\Omega, m, M)$ because the space X is not ordered totally by $\leq_K$.

As shown in [1], [33], an order cone K in an ordered real Banach space $(X, \|\cdot\|)$ is said to be *normal* if there exists a constant $N > 0$ such that $\|x\| \leq N\|y\|$ if $\theta \leq_K x \leq_K y$ in X. The smallest constant N, denoted by $N(K)$, is called the normal constant of K. Actually, every real Banach space X has a normal order cone $K := \{\alpha e \mid \alpha \in [0, +\infty)\}$, where $e \neq \theta$ is chosen arbitrarily in X. One can verify that K is both an order cone and a closed subset and satisfies that $\|x\| \leq \|y\|$ if $\theta \leq_K x \leq_K y$.

Lemma 2.2. *Let $(X, K, \|\cdot\|)$ be an ordered real Banach space such that K is normal. Then the above defined $C^{\pm}(\Omega, m, M), C^{+}_{cv}(\Omega, m, M)$ and $C^{+}_{cc}(\Omega, m, M)$ are compact convex subsets of $C(\Omega, X)$.*

PROOF. We only prove that $C^{+}(\Omega, m, M)$ and $C^{+}_{cv}(\Omega, m, M)$ are compact convex subsets of $C(\Omega, X)$. It will be similar for $C^{-}(\Omega, m, M)$ and $C^{+}_{cc}(\Omega, m, M)$. We first consider $C^{+}(\Omega, m, M)$. For every $f \in C^{+}(\Omega, m, M)$ we have

$$\|f(y) - f(x)\| \leq MN(K)\|y - x\|, \quad \forall x, y \in \Omega, \tag{2.1}$$

if either $x \leq_K y$ or $y \leq_K x$. In fact,

$$m(y - x) \leq_K f(y) - f(x) \leq_K M(y - x)$$

if $x \leq_K y$ and a similar inequality holds if $y \leq_K x$. Then (2.1) follows because K is normal. On the other hand, if x and y are not comparable, i.e., $x - y \notin K$ and $y - x \notin K$, then by the definition of $C^{+}(\Omega, m, M)$

$$\|f(y) - f(x)\| \leq M\|y - x\|. \tag{2.2}$$

Summarizing (2.1) and (2.2), we get

$$\|f(y) - f(x)\| \leq M_0\|y - x\|, \ \forall x, y \in \Omega, \tag{2.3}$$

where $M_0 := \max\{M, MN(K)\}$. It implies that $C^{+}(\Omega, m, M)$ is equicontinuous. In addition, for each $x \in \Omega$ the set $\Xi := \{f(x) : f \in C^{+}(\Omega, m, M)\}$ is relatively compact, i.e., its closure is compact, because the fact $f(\Omega) \subset \Omega$ implies that the set Ξ is a subset of the compact set Ω and therefore sequentially compact. It concludes by Ascoli's Theorem (see the Appendix) that $C^{+}(\Omega, m, M)$ is relatively compact.

Furthermore, we prove that $C^{+}(\Omega, m, M)$ is a closed subset of $C(\Omega, X)$. Let $\{f_n\} \subset C^{+}(\Omega, m, M)$ be such a sequence that $\lim_{n\to\infty} f_n = f$ in $C(\Omega, X)$, i.e.,

$$\lim_{n\to\infty} \|f_n(x) - f(x)\| = 0, \quad \forall x \in \Omega. \tag{2.4}$$

By the condition **(C1)** of the compatibility with the addition, given just before Lemma 2.1, we have

$$f(y) - f(x) \leq_K f(y) - f_n(y) + M(y - x) + f_n(x) - f(x),$$
$$f(y) - f(x) \geq_K f(y) - f_n(y) + m(y - x) + f_n(x) - f(x)$$

for all $x \leq_K y$ in Ω. Hence, by (2.4) and (ii) in Lemma 2.1, we see that

$$m(y-x) \leq_K f(y) - f(x) \leq_K M(y-x), \quad \forall x \leq_K y \text{ in } \Omega.$$

On the other hand, if x and y are not comparable, i.e., $x - y \notin K$ and $y - x \notin K$, then

$$\|f(x) - f(y)\| \leq \|f(x) - f_n(x)\| + M\|x - y\| + \|f_n(y) - f(y)\|.$$

It follows from (2.4) that $\|f(x) - f(y)\| \leq M\|x - y\|$. At last, we can prove that $f(\Omega) \subset \Omega$ because $f_n(\Omega) \subset \Omega$ and Ω is a compact set. Thus, we have proved that $C^+(\Omega, m, M)$ is a closed set and therefore a compact subset of $C(\Omega, X)$.

Finally, we claim the convexity of $C^+(\Omega, m, M)$. Consider $f, g \in C^+(\Omega, m, M)$ and $\lambda \in [0, 1]$. Obviously, the combination $h := \lambda f + (1-\lambda)g$ is a continuous function. The convexity of Ω implies that $h(x) = \lambda f(x) + (1-\lambda)g(x) \in \Omega$ for each $x \in \Omega$. Thus, by the definition of $C^+(\Omega, m, M)$, if $x \leq_K y$ in Ω then

$$h(y) - h(x) = \lambda(f(y) - f(x)) + (1-\lambda)(g(y) - g(x)) \leq_K M(y-x),$$
$$h(y) - h(x) = \lambda(f(y) - f(x)) + (1-\lambda)(g(y) - g(x)) \geq_K m(y-x),$$

where the condition **(C2)** of the compatibility with the scalar product is employed. On the other hand, if x and y are not comparable, i.e., $x - y \notin K$ and $y - x \notin K$, then

$$\begin{aligned}\|h(y) - h(x)\| &\leq \|\lambda(f(y) - f(x))\| + \|(1-\lambda)(g(y) - g(x))\| \\ &\leq \lambda M\|y - x\| + (1-\lambda)M\|y - x\| = M\|y - x\|.\end{aligned}$$

Hence $C^+(\Omega, m, M)$ is convex. Consequently, we have proved that $C^+(\Omega, m, M)$ is a compact convex subset of $C(\Omega, X)$.

The proof for $C^+_{cv}(\Omega, m, M)$ is almost the same but, in addition, we need to claim the following:

(V1) If $\{f_n\} \subset C^+_{cv}(\Omega, m, M)$ is a sequence such that $\lim_{n\to\infty} f_n = f$ in $C(\Omega, X)$ then

$$f(\lambda x + (1-\lambda)y) \leq_K \lambda f(x) + (1-\lambda)f(y)$$

for every pair of distinct comparable points $x, y \in \Omega$ and $\lambda \in [0, 1]$.

(V2) If $f, g \in C^+_{cv}(\Omega, m, M)$ then

$$\begin{aligned}(tf + (1-t)g)(\lambda x + (1-\lambda)y) \leq_K\ & \lambda(tf + (1-t)g)(x) \\ & + (1-\lambda)(tf + (1-t)g)(y)\end{aligned}$$

for every pair of distinct comparable points $x, y \in \Omega$ and $t, \lambda \in [0, 1]$.

By (ii) in Lemma 2.1, the result **(V1)** follows the fact that $f_n(\lambda x+(1-\lambda)y) \leq_K \lambda f_n(x) + (1-\lambda)f_n(y)$. Let $f, g \in C^+_{cv}(\Omega, m, M)$ and $h := tf + (1-t)g$, where $t \in [0,1]$. Then for every pair of distinct comparable points $x, y \in \Omega$, $\lambda \in [0,1]$,

$$\begin{aligned} h(\lambda x + (1-\lambda)y) &\leq_K t\lambda f(x) + t(1-\lambda)f(y) + (1-t)\lambda g(x) + (1-t)(1-\lambda)g(y) \\ &= \lambda h(x) + (1-\lambda)h(y), \end{aligned}$$

i.e., **(V2)** is proved, where the condition **(C1)** is employed. The proof is completed. □

3. Increasing and decreasing solutions

Before discussing convexity, we prove the existence of increasing and decreasing solutions of equation (1.1) in the ordered real Banach space $(X, K, \|\cdot\|)$ such that K is normal. First, we investigate increasing solutions. Consider equation (1.1) with the following hypotheses:

(**H1**) $\lambda_1 > 0$, $\lambda_i \leq 0$, $i = 2, 3, \ldots, n$, and

(**H2**) the normalization condition $\sum_{i=1}^n \lambda_i = 1$.

Theorem 3.1. *Suppose that* **(H1)** *and* **(H2)** *hold and* $F \in C^+(\Omega, 0, M_1)$, *where* $M_1 \in (0, +\infty)$ *is a constant. If*

$$M_1 \leq \lambda_1 M + \lambda_2 M^2 + \cdots + \lambda_n M^n \tag{3.1}$$

for a constant $M \in (0, +\infty)$, *then equation* (1.1) *has a solution* $f \in C^+(\Omega, 0, M)$.

As shown in many papers (e.g. [11], [14], [27], [28], [38]), the nonlinearity of iteration causes so great difficulties that the conditions in those obtained results are very restrictive although many efforts (e.g. [4], [25], [29]) have been made. In our Theorem 3.1 the condition (3.1) also looks so complicated. We will discuss on it in the final section.

In order to prove Theorem 3.1, we need the following lemmas.

Lemma 3.1. *Let* $(X, K, \|\cdot\|)$ *be an ordered real Banach space. Then composition* $f \circ g$ *is convex (resp. concave) if both* f *and* g *are convex (resp. concave) and increasing. In particular, for increasing convex (resp. concave) operator* f, *the iterate* f^k *is also convex (resp. concave).*

Lemma 3.2. *Let $(X, K, \|\cdot\|)$ be an ordered real Banach space such that K is normal and let $f, g \in C^+(\Omega, m, M)$ (resp. $C^-(\Omega, m, M)$, $C^+_{cv}(\Omega, m, M)$ and $C^+_{cc}(\Omega, m, M)$), where $0 \le m \le M \le +\infty$. Then*

$$\|f^k - g^k\|_{C(\Omega,X)} \le \sum_{j=0}^{k-1} M_0^j \|f - g\|_{C(\Omega,X)}, \quad \forall k = 1, 2, \ldots.$$

The above lemmas are the same as Lemmas 2 and 3 given in [38] but the concepts "increasing", "decreasing", "convex" and "concave" are set up on the K-order. Noting (2.3), we can prove them similarly.

Lemma 3.3. *Let $(X, K, \|\cdot\|)$ be an ordered real Banach space and let $f \in C^-(\Omega, m, M)$, where $0 \le m \le M < +\infty$. Then*

$$-M^{2n-1}(y-x) \le_K f^{2n-1}(y) - f^{2n-1}(x) \le_K -m^{2n-1}(y-x), \quad n = 1, 2, \ldots, \tag{3.2}$$

and

$$m^{2n}(y-x) \le_K f^{2n}(y) - f^{2n}(x) \le_K M^{2n}(y-x), \quad n = 1, 2, \ldots, \tag{3.3}$$

for all $x \le_K y$ in Ω.

Proof. By the definition of $C^-(\Omega, m, M)$, for $f \in C^-(\Omega, m, M)$ we see that

$$-M(y-x) \le_K f(y) - f(x) \le_K -m(y-x), \quad \forall x, y \in \Omega, \tag{3.4}$$

if $x \le_K y$. It further implies that

$$m^2(y-x) \le_K f^2(y) - f^2(x) \le_K M^2(y-x), \quad \forall x, y \in \Omega, \tag{3.5}$$

if $x \le_K y$. In order to prove (3.3), assume that

$$m^{2k}(y-x) \le_K f^{2k}(y) - f^{2k}(x) \le_K M^{2k}(y-x)$$

for some positive integer k, if $x \le_K y$. It implies that

$$\begin{aligned} f^{2k+2}(y) - f^{2k+2}(x) &= f^2(f^{2k}(y)) - f^2(f^{2k}(x)) \le_K M^2(f^{2k}(y) - f^{2k}(x)) \\ &\le_K M^2 M^{2k}(y-x) = M^{2k+2}(y-x) \end{aligned}$$

and

$$\begin{aligned} f^{2k+2}(y) - f^{2k+2}(x) &= f^2(f^{2k}(y)) - f^2(f^{2k}(x)) \ge_K m^2(f^{2k}(y) - f^{2k}(x)) \\ &\ge_K m^2 m^{2k}(y-x) = m^{2k+2}(y-x). \end{aligned}$$

Therefore inequality (3.3) is proved by induction. The proof of inequality (3.2) can be given similarly. □

PROOF OF THEOREM 3.1. Under the hypotheses **(H1)** and **(H2)**, we can rewrite equation (1.1) as

$$f(x) = \frac{1}{\lambda_1}F(x) - \frac{\lambda_2}{\lambda_1}f^2(x) - \cdots - \frac{\lambda_n}{\lambda_1}f^n(x), \quad x \in \Omega, \tag{3.6}$$

where

$$\frac{1}{\lambda_1} - \sum_{i=2}^{n}\frac{\lambda_i}{\lambda_1} = 1. \tag{3.7}$$

Define a mapping $L : C^+(\Omega, 0, M) \to C(\Omega, X)$ by

$$Lf(x) = \frac{1}{\lambda_1}F(x) - \frac{\lambda_2}{\lambda_1}f^2(x) - \cdots - \frac{\lambda_n}{\lambda_1}f^n(x). \tag{3.8}$$

We first claim that L is a self-mapping on $C^+(\Omega, 0, M)$. Obviously $Lf(x) \in C(\Omega, X)$ and $Lf(\Omega) \subset \Omega$ because of the convexity of Ω. Further, when $x, y \in \Omega$ are not comparable, i.e., $x - y \notin K$ and $y - x \notin K$, by the definition of $C^+(\Omega, 0, M)$ we have

$$\begin{aligned}
&\|Lf(x) - Lf(y)\| \\
&= \left\| \frac{1}{\lambda_1}(F(x) - F(y)) - \frac{\lambda_2}{\lambda_1}(f^2(x) - f^2(y)) - \cdots - \frac{\lambda_n}{\lambda_1}(f^n(x) - f^n(y)) \right\| \\
&\le \left\| \frac{1}{\lambda_1}(F(x) - F(y)) \right\| + \left\| -\frac{\lambda_2}{\lambda_1}(f^2(x) - f^2(y)) \right\| + \cdots + \left\| -\frac{\lambda_n}{\lambda_1}(f^n(x) - f^n(y)) \right\| \\
&\le \left(\frac{1}{\lambda_1}M_1 - \frac{\lambda_2}{\lambda_1}M^2 - \cdots - \frac{\lambda_n}{\lambda_1}M^n \right) \|x - y\|,
\end{aligned}$$

which implies that

$$\|Lf(x) - Lf(y)\| \le M\|x - y\| \tag{3.9}$$

because of inequality (3.1). When $x, y \in \Omega$ are comparable, suppose that $x \le_K y$. By the definition of $C^+(\Omega, 0, M)$,

$$\begin{aligned}
\theta &\le_K Lf(y) - Lf(x) \\
&= \frac{1}{\lambda_1}(F(y) - F(x)) - \frac{\lambda_2}{\lambda_1}(f^2(y) - f^2(x)) - \cdots - \frac{\lambda_n}{\lambda_1}(f^n(y) - f^n(x)) \\
&\le_K \frac{1}{\lambda_1}M_1(y - x) - \frac{\lambda_2}{\lambda_1}M^2(y - x) - \cdots - \frac{\lambda_n}{\lambda_1}M^n(y - x) \\
&\le_K \left(\frac{1}{\lambda_1}M_1 - \frac{\lambda_2}{\lambda_1}M^2 - \cdots - \frac{\lambda_n}{\lambda_1}M^n \right)(y - x),
\end{aligned}$$

which implies that

$$\theta \leq_K Lf(y) - Lf(x) \leq_K M(y-x) \tag{3.10}$$

because of inequality (3.1). Thus, (3.9) and (3.10) imply that L is a self-mapping on $C^+(\Omega, 0, M)$.

Next, we prove the continuity of L. In fact, for any $f, g \in C^+(\Omega, 0, M)$, by Lemma 3.2 we have

$$\begin{aligned}
\|Lf - Lg\|_{C(\Omega,X)} &= \sup_{x\in\Omega} \left\| -\frac{\lambda_2}{\lambda_1}(f^2(x) - g^2(x)) - \cdots - \frac{\lambda_n}{\lambda_1}(f^n(x) - g^n(x)) \right\| \\
&\leq \sup_{x\in\Omega} \left\| -\frac{\lambda_2}{\lambda_1}(f^2(x) - g^2(x)) \right\| + \cdots + \sup_{x\in\Omega} \left\| -\frac{\lambda_n}{\lambda_1}(f^n(x) - g^n(x)) \right\| \\
&= -\frac{\lambda_2}{\lambda_1}\|f^2 - g^2\|_{C(\Omega,X)} - \cdots - \frac{\lambda_n}{\lambda_1}\|f^n - g^n\|_{C(\Omega,X)} \\
&\leq M_+\|f - g\|_{C(\Omega,X)},
\end{aligned} \tag{3.11}$$

where $M_+ := \sum_{k=2}^{n} \big(-\lambda_k/\lambda_1 \big) \sum_{j=0}^{k-1} M_0^j > 0$. As Lemma 2.2 guarantees that $C^+(\Omega, 0, M)$ is a compact convex subset, by Schauder's fixed point theorem we see that L has a fixed point $f \in C^+(\Omega, 0, M)$. Thus f is an increasing solution of the equation. The proof is completed. □

The following is devoted to decreasing solutions.

Theorem 3.2. *Suppose that* **(H1)** *and* **(H2)** *hold and all coefficients of even order iterates in equation* (1.1) *are equal to 0. Let* $F \in C^-(\Omega, 0, M_1)$, *where* $M_1 \in (0, +\infty)$ *is a constant. If the condition* (3.1) *holds for a constant* $M \in (0, +\infty)$, *then equation* (1.1) *has a solution* $f \in C^-(\Omega, 0, M)$.

The proof is almost the same as the proof of Theorem 3.1, but a little bit more complicated because of the change of monotonicity. In its proof we note that the function Lf defined in (3.8) is also decreasing because all coefficients of even order iterates in equation (1.1) are equal to 0. Moreover, in order to prove the mapping (3.8) to be a self-mapping on $C^-(\Omega, 0, M)$, we also need a deduction for an inequality like (3.10). For this purpose we will use Lemma 3.3.

4. Convex and concave solutions

On the basis of last section we can discuss on convexity of continuous solutions for equation (1.1) in the ordered real Banach space $(X, K, \|\cdot\|)$ with a normal cone K.

Theorem 4.1. *Suppose that* **(H1)** *and* **(H2)** *hold and* $F \in C_{cv}^{+}(\Omega, 0, M_1)$, *where* $M_1 \in (0, +\infty)$ *is a constant. If*

$$M_1 \leq \lambda_1 M + \lambda_2 M^2 + \cdots + \lambda_n M^n \tag{4.1}$$

for a constant $M \in (0, +\infty)$, *then equation* (1.1) *has a continuous solution* $f \in C_{cv}^{+}(\Omega, 0, M)$.

Proof. Define a mapping $L : C_{cv}^{+}(\Omega, 0, M) \to C(\Omega, X)$ as in Theorem 3.1. In order to prove that L is a self-mapping on $C_{cv}^{+}(\Omega, 0, M)$, it suffices to prove the following inequality

$$Lf(tx + (1-t)y) \leq_K tLf(x) + (1-t)Lf(y), \quad \forall t \in [0, 1], \tag{4.2}$$

for every pair of distinct comparable points $x, y \in \Omega$. In fact, each f^k, $k = 2$, $\ldots, n$, is convex in the sense of K-order because f is increasing and convex by Lemma 3.1. Hence,

$$\begin{aligned}
&Lf(tx + (1-t)y) \\
&= \frac{1}{\lambda_1} F(tx + (1-t)y) - \frac{\lambda_2}{\lambda_1} f^2(tx + (1-t)y) - \cdots - \frac{\lambda_n}{\lambda_1} f^n(tx + (1-t)y) \\
&\leq_K \frac{1}{\lambda_1} \{tF(x) + (1-t)F(y)\} - \frac{\lambda_2}{\lambda_1} \{tf^2(x) + (1-t)f^2(y)\} \\
&\quad - \cdots - \frac{\lambda_n}{\lambda_1} \{tf^n(x) + (1-t)f^n(y)\} \\
&= t \left\{ \frac{1}{\lambda_1} F(x) - \frac{\lambda_2}{\lambda_1} f^2(x) - \cdots - \frac{\lambda_n}{\lambda_1} f^n(x) \right\} \\
&\quad + (1-t) \left\{ \frac{1}{\lambda_1} F(y) - \frac{\lambda_2}{\lambda_1} f^2(y) - \cdots - \frac{\lambda_n}{\lambda_1} f^n(y) \right\} = tLf(x) + (1-t)Lf(y),
\end{aligned}$$

implying (4.2). The continuity of L was proved in the proof of Theorem 3.1. Lemma 2.2 guarantees that $C_{cv}^{+}(\Omega, 0, M)$ is a compact convex subset. Therefore, this proof can be completed by using Schauder's fixed point theorem. □

Similarly, we can prove the following result for concavity of solutions.

Theorem 4.2. *Suppose that* **(H1)** *and* **(H2)** *hold and* $F \in C_{cc}^{+}(\Omega, 0, M_1)$, *where* $M_1 \in (0, +\infty)$ *is a constant. If*

$$M_1 \leq \lambda_1 M + \lambda_2 M^2 + \cdots + \lambda_n M^n \tag{4.3}$$

for a constant $M \in (0, +\infty)$, *then equation* (1.1) *has a continuous solution* $f \in C_{cc}^{+}(\Omega, 0, M)$.

Example 4.1. Consider the equation

$$\frac{6}{5}f(x_1,x_2)-\frac{1}{5}f^2(x_1,x_2)=\left(\frac{1}{2}x_1,\frac{1}{2}x_2^2\right),\quad \forall (x_1,x_2)\in\Omega, \tag{4.4}$$

where $\Omega:=\{(x_1,x_2)\in\mathbb{R}^2 : x_1^2+x_2^2\le 1,\ x_2\ge 0\}$. Clearly, equation (4.4) is of the form (1.1), where $F(x_1,x_2):=\left(\frac{1}{2}x_1,\frac{1}{2}x_2^2\right)$, $\lambda_1=6/5$ and $\lambda_2=-1/5$. Ω is a compact convex subset of the ordered real Banach space $(X,K,\|\cdot\|)$, where $X=\mathbb{R}^2$, $\|x\|=(x_1^2+x_2^2)^{1/2}$ and $K:=\{x=(x_1,x_2): x_1\ge 0,\ x_2\ge 0\}$ is a normal order cone. One can check that **(H1)** and **(H2)** are satisfied. We further claim that $F(x)\in C^+_{cv}(\Omega,0,1)$. In fact,

$$\|F(y)-F(x)\|\le\sqrt{\frac{1}{4}(y_1-x_1)^2+(y_2-x_2)^2}\le\|y-x\|,\quad \forall x,y\in\Omega, \tag{4.5}$$

implying that $F(x)\in C(\Omega,X)$, and

$$\left(\frac{1}{2}x_1\right)^2+\left(\frac{1}{2}x_2^2\right)^2=\frac{1}{4}x_1^2+\frac{1}{4}x_2^4\le 1,\quad \forall x\in\Omega,$$

because $x_1^2+x_2^2\le 1$ and $\frac{1}{2}x_2^2\ge 0$, implying that $F(\Omega)\subset\Omega$. Note that $x\le_K y$ if and only if $y_1-x_1\ge 0$ and $y_2-x_2\ge 0$. Thus, when $x,y\in\Omega$ are not comparable (i.e., $y-x\notin K$ and $x-y\notin K$), we have (4.5); when $x,y\in\Omega$ are comparable, we calculate

$$F(y)-F(x)=\left(\frac{1}{2}(y_1-x_1),\frac{1}{2}(y_2-x_2)(y_2+x_2)\right),\quad \forall x,y\in\Omega,$$

implying that

$$\theta\le_K F(y)-F(x)\le_K (y-x)$$

if $x\le_K y$ because $0\le x_2\le y_2\le 1$. We can also check

$$\begin{aligned}F(\lambda x+(1-\lambda)y)&=\left(\frac{1}{2}(\lambda x_1+(1-\lambda)y_1),\frac{1}{2}(\lambda x_2+(1-\lambda)y_2)^2\right)\\&\le_K\left(\frac{1}{2}\lambda x_1+\frac{1}{2}(1-\lambda)y_1,\frac{1}{2}(\lambda x_2^2+(1-\lambda)y_2^2)\right)\\&=\lambda F(x)+(1-\lambda)F(y)\end{aligned}$$

for all $\lambda\in[0,1]$ and every pair of distinct comparable points $x,y\in\Omega$, implying that $F(x)$ is convex in K-order on Ω. Thus the claim is proved. Since

$$-\frac{1}{5}M^2+\frac{6}{5}M-1=-\frac{1}{5}(M-3)^2+\frac{4}{5}\ge 0$$

for all $M\in[1,5]$, i.e., inequality (4.1) holds for any $M\in[1,5]$, by Theorem 4.1 we see that equation 4.4 has a convex solution $f\in C^+_{cv}(\Omega,0,M)$.

Opposite to the above example, let us consider an example in the infinite-dimensional setting.

Example 4.2. Let $X = C([0,1],\mathbb{R})$ equipped with the norm $\|x\| = \sup\limits_{t\in[0,1]} |x(t)|$ for $x \in X$. Let $\Omega := \{x \in C([0,1],[0,1]) : |x(t_1)-x(t_2)| \le |t_1-t_2|,\ t_1,t_2 \in [0,1]\}$, a subset of X. Then, the equation

$$\frac{9}{8}f(x(t)) - \frac{1}{8}f^2(x(t)) = \sin x(t), \quad \forall x \in \Omega, \tag{4.6}$$

is an iterative equation of the form (1.1) in the infinite-dimensional setting, where $\lambda_1 = 9/8$, $\lambda_2 = -1/8$ and $F(x) := \sin x$. Note that $K := \mathbb{R}_+^{[0,1]} := \{x \in C([0,1],\ \mathbb{R}) \mid x(t) \ge 0\}$ is a normal order cone in X. Then, Ω is a compact convex subset of the ordered real Banach space $(X, K, \|\cdot\|)$. Clearly, **(H1)** and **(H2)** are satisfied. Note that $F \in C(\Omega, X)$ because

$$\|F(y) - F(x)\| = \sup_{t\in[0,1]} |\sin y(t) - \sin x(t)| \le \|y - x\|, \quad \forall x, y \in \Omega. \tag{4.7}$$

Moreover, $F(\Omega) \subset \Omega$ because $|\sin x(t_1) - \sin x(t_2)| \le |x(t_1) - x(t_2)|$. Furthermore, when $x, y \in \Omega$ are comparable, we have

$$0 \le_K F(y) - F(x) \le_K (y - x) \quad \text{if } x \le_K y;$$

otherwise, (4.7) remains valid. This implies that $F \in C^+(\Omega, 0, 1)$. For every $\lambda \in [0,1]$ and every pair of distinct comparable points $x, y \in \Omega$, we can also check that

$$F(\lambda x + (1-\lambda)y) \ge_K \lambda F(x) + (1-\lambda)F(y)$$

implying that $F \in C_{cc}^+(\Omega, 0, 1)$. In this case

$$-\frac{1}{8}M^2 + \frac{9}{8}M - 1 = -\frac{1}{8}\left(M - \frac{9}{2}\right)^2 + \frac{49}{32} \ge 0$$

for all $M \in [1,8]$, i.e., inequality (4.3) holds for any $M \in [1,8]$. By Theorem 4.2, equation (4.6) has a concave solution $f \in C_{cc}^+(\Omega, 0, 1)$.

5. Some remarks

As mentioned just after the statement of Theorem 3.1, the condition (3.1) is complicated but demanded in all of our theorems. It leads to an interesting question: *Given a constant $M_1 > 0$, does the polynomial*

$$p(x) := \lambda_n x^n + \cdots + \lambda_2 x^2 + \lambda_1 x - M_1 \tag{5.1}$$

have a positive root under the hypotheses **(H1)** *and* **(H2)***?*There have been given many methods ([5], [12], [21], [31], [32]) to determine the existence of positive roots for polynomials. The Descartes' Rule of Signs (see [5] or [8, Theorem 2]) indicates that for any nonzero real polynomial the number of coefficient sign variations exceeds the number of positive real roots—counting multiplicities—by a non-negative even integer. In our case the number of coefficient sign variations of the polynomial p defined in (5.1) is 2. The Descartes' Rule of Signs cannot give a definite answer. The criterion given in [12, Theorem 1] requires that all coefficients are positive but our p is not the case. In [21] all roots are required to be real. In addition, Sturm's sequence ([32]) can also be employed for this purpose. Another method is to use the discriminants ([31]) to determine the number of real roots of a polynomial which is modified from p by replacing x with x^2. However, both of them are not efficient for general n and symbolic/literal coefficients.

Concerning the existence of positive roots, let us investigate p on $(0, +\infty)$. Since $p(0) = -M_1 < 0$ and $p(x) \to -\infty$ as $x \to +\infty$, a necessary and sufficient condition for the existence of positive roots is that the maximum of p is greater than or equal to 0. Note that $p''(x) = n(n-1)\lambda_n x^{n-2} + \cdots + 2\lambda_2 < 0$, implying that the derivative $p'(x) = n\lambda_n x^{n-1} + \cdots + 2\lambda_2 x + \lambda_1$ has a unique zero ζ. It is equivalent to say that p has a positive zero if and only if $p(\zeta) \geq 0$. Numerical computation will be helpful to finding the values of ζ and $p(\zeta)$.

Without giving concrete values of coefficients of p, it is hard to obtain an approximation of ζ and $p(\zeta)$. However, we know that

$$(-n\lambda_n - \cdots - 2\lambda_2)x - \lambda_1 \leq -p'(x) \leq (-n\lambda_n - \cdots - 2\lambda_2)x^{n-1} - \lambda_1$$

for all $x \geq 1$ and

$$(-n\lambda_n - \cdots - 2\lambda_2)x^{n-1} - \lambda_1 \leq -p'(x) \leq (-n\lambda_n - \cdots - 2\lambda_2)x - \lambda_1$$

for all $0 < x < 1$.
It follows that ζ lies between $\Gamma(\lambda_1, \ldots, \lambda_n) := \frac{\lambda_1}{-n\lambda_n - (n-1)\lambda_{n-1} - \cdots - 2\lambda_2}$ and its $(n-1)$-th root. Their geometric mean $\tilde{\Gamma}(\lambda_1, \ldots, \lambda_n) := \Gamma(\lambda_1, \ldots, \lambda_n)^{\frac{n}{2(n-1)}}$ is closer to ζ and can be considered as a rough but simple approximation of ζ. Clearly, $p(\tilde{\Gamma}(\lambda_1, \ldots, \lambda_n)) \leq p(\zeta)$. This concludes the following sufficient condition.

Proposition 5.1. *Under the hypotheses* **(H1)** *and* **(H2)**, *the polynomial p defined in* (5.1) *has a positive root if $p(\tilde{\Gamma}(\lambda_1, \ldots, \lambda_n)) \geq 0$.*

The condition obtained in Proposition 5.1 can be demonstrated by Example 4.1 because $\tilde{\Gamma}(\lambda_1, \lambda_2) = \lambda_1/(-2\lambda_2) = 3$ and $\lambda_1\tilde{\Gamma}(\lambda_1, \lambda_2) + \lambda_2\tilde{\Gamma}(\lambda_1, \lambda_2)^2 := 9/5 > M_1 := 1$.

Another remark is that the uniqueness and continuous dependence for solutions with monotonicity and convexity in the sense of K-order can be given by using the same arguments as in [28], [38]. We only show the results but omit their proofs.

Proposition 5.2. *Suppose that* **(H1)** *and* **(H2)** *hold and* $F \in C^+(\Omega, 0, M_1)$ *(resp.* $C^+_{cv}(\Omega, 0, M_1)$ *and* $C^+_{cc}(\Omega, 0, M_1)$*), where* $M_1 \in (0, +\infty)$ *is a constant. If in addition to* (3.1) *the following inequality*

$$\lambda_1 + \sum_{k=2}^{n} \lambda_k \sum_{j=0}^{k-1} M_0^j > 0 \tag{5.2}$$

also holds for a constant $M \in (0, +\infty)$*, where* $M_0 = \max\{M, N(K)M\}$*, then equation* (1.1) *has a unique solution* $f \in C^+(\Omega, 0, M)$ *(resp.* $C^+_{cv}(\Omega, 0, M)$ *and* $C^+_{cc}(\Omega, 0, M)$*). Moreover, the solution depends upon* F *continuously, i.e.,*

$$\|f_1 - f_2\|_{C(\Omega,X)} \le \|F_1 - F_2\|_{C(\Omega,X)} \Big/ \left(\lambda_1 + \sum_{k=2}^{n} \lambda_k \sum_{j=0}^{k-1} M_0^j\right)$$

if f_1, f_2 *are the solutions of equation* (1.1) *with respect to the given functions* F_1*,* F_2 *respectively.*

Proposition 5.3. *Suppose that* **(H1)** *and* **(H2)** *hold and all coefficients of even order iterates in equation* (1.1) *are equal to* 0*. Let* $F \in C^-(\Omega, 0, M_1)$*, where* $M_1 \in (0, +\infty)$ *is a constant. If* (3.1) *and* (5.2) *hold, then equation* (1.1) *has a unique solution* $f \in C^-(\Omega, 0, M)$*. Moreover, the solution depends upon* F *continuously.*

We remark on difficulties encountered in some cases. First, without requiring $\lambda_i \le 0$ for all $i = 2, \dots, n$, we hardly give the existence of increasing (resp. decreasing) solutions because we cannot prove that the mapping defined in (3.8) is a self-mapping, i.e., $Lf(\Omega) \subset \Omega$. On the other hand, we do not consider the mapping $\mathcal{T}$ in the same form $\mathcal{T}f := L_f^{-1} \circ F$ as in [28] because of difficulties in discussing monotonicity of the function $\mathcal{T}f(x)$ in infinite-dimensional spaces. In contrast to [28], we did not discuss decreasing convex (resp. concave) solutions for equation (1.1) because in the infinite-dimensional setting no arguments of "divided difference" are known for us to discuss the convexity of the composition of two decreasing convex (resp. concave) operators and the convexity of the linear combination of iterates of decreasing convex (resp. concave) operators. Besides, the existence of decreasing (resp. increasing) solutions for given $F \in$

$C^+(\Omega, 0, M_1)$ (resp. $C^-(\Omega, 0, M_1)$) and the existence of increasing convex (resp. concave) solution for given $F \in C^+_{cc}(\Omega, 0, M_1)$ (resp. $C^+_{cv}(\Omega, 0, M_1)$) are not given yet for equation (1.1) because of difficulties for the mapping defined in (3.8) to be a self-mapping. Those cases will be subjects of our next work.

As indicated in the Introduction, a Banach space does not have such a natural order as $\mathbb{R}$ has, but monotonicity and convexity of functions require an order. Therefore, in this paper we need to consider an order in the Banach space. The order considered in section 2 is the so-called *linear order*, which is compatible with the linear operations as shown in **(C1)** and **(C2)**. Indicated in [1], for every vector space there is a one-to-one correspondence between the family of linear orders and the family of order cones. Therefore, the structure of order is decided by the order cone. The natural order in $\mathbb{R}$ can be defined by the cone $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$. The natural order in $\mathbb{R}^2$ can be defined by the cone $\mathbb{R}_+ \times \mathbb{R}_+$. The natural order in $C(X, \mathbb{R})$ is defined by $K := \{f \in C(X, \mathbb{R}) : f(x) \geq 0,\ \forall x \in X\}$. Clearly, those spaces can be ordered by another cones, for example, $\mathbb{R}_- := \{x \in \mathbb{R} : x \leq 0\}$ for the one-dimensional $\mathbb{R}$ and $\{(x_1, x_2) : x_1 \leq 0,\ x_2 \geq 0\}$ for the two-dimensional $\mathbb{R}^2$. Different cones define different orders. Different orders define different concepts of monotonicity and convexity. In $\mathbb{R}$ a convex function in the sense of the cone $\mathbb{R}_-$ is a concave function in the sense of the cone $\mathbb{R}_+$. Therefore, our results depend on the choice of the order cone seriously.

Appendix: Ascoli's Theorem

Ascoli's Theorem ([13, Theorem 3.1, p. 55])**.** *Let X be a compact subset of a metric space, and let F be a Banach space. Let Φ be a subset of the space of continuous maps $C(X, F)$ with sup norm. Then Φ is relatively compact in $C(X, F)$ if and only if the following two conditions are satisfied:*

(ASC1) *Φ is equicontinuous.*

(ASC2) *For each $x \in X$, the set $\Phi(x)$ consisting of all values $f(x)$ for $f \in \Phi$ is relatively compact.*

Acknowledgment. The authors are grateful to the referee for his valuable comments and encouragement.

References

[1] H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM Rev.* **18** (1976), 620–709.

[2] K. Baron and W. Jarczyk, Recent results on functional equations in a single variable, perspectives and open problems, *Aequationes Math.* **61** (2001), 1–48.

[3] N. Brillouët-Belluot, On regular solutions of some simple iterative functional equations, *Aequationes Math.* **67** (2004), 117–131.

[4] J. Chen and W. Zhang, Leading coefficient problem for polynomial-like iterative equations, *J. Math. Anal. Appl.* **349** (2009), 413–419.

[5] R. Descartes, The Geometry of René Descartes, transl. D. E. Smith and M. L. Lantham, *Dover Publications Inc., New York*, 1954.

[6] J. G. Dhombres, Itèration linèaire d'ordre deux, *Publ. Math. Debrecen* **24** (1977), 177–187.

[7] W. Jarczyk, On an equation of linear iteration, *Aequationes Math.* **51** (1996), 303–310.

[8] W. Krandick and K. Mehlhorn, New bounds for the Descartes method, *J. Symbolic Comput.* **41** (2006), 49–66.

[9] M. Kuczma, B. Choczewski and R. Ger, Iterative functional equations, Encyclopedia Math. Appl., vol. 32, *Cambridge Univ. Press, Cambridge*, 1990.

[10] M. Kuczma and A. Smajdor, Fractional iteration in the class of convex functions, *Bull. Acad. Pol. Sci. Sèr. Sci. Math. Astron. Phys.* **16** (1968), 717–720.

[11] M. Kulczycki and J. Tabor, Iterative functional equations in the class of Lipschitz functions, *Aequationes Math.* **64** (2002), 24–33.

[12] D. C. Kurtz, A sufficient condition for all the roots of a polynomial to be real, *Amer. Math. Monthly* **99** (1992), 259–263.

[13] S. Lang, Real Analysis, (second edition), *Addison-Wesley, New York*, 1983.

[14] J. Mai and X. Liu, Existence, uniqueness and stability of C^m solutions of iterative functional equations, *Sci. China Ser.* **A 43** (2000), 897–913.

[15] M. Malenica, On the solutions of the functional equation $\phi(x) + \phi^2(x) = F(x)$, *Mat. Vesnik.* **6** (1982), 301–305.

[16] J. Matkowski and W. Zhang, On linear dependence of iterates, *J. Appl. Anal.* **6** (2000), 149–157.

[17] A. Mukherjea and J. S. Ratti, On a functional equation involving iterates of a bijection on the unit interval, *Nonlinear Anal.* **7** (1983), 899–908.

[18] A. Mukherjea and J. S. Ratti, On a functional equation involving iterates of a bijection on the unit interval II, *Nonlinear Anal.* **31** (1998), 459–464.

[19] C. T. Ng and W. Zhang, An algebraic equation for linear operators, *Linear Algebra Appl.* **412** (2006), 303–325.

[20] M. Sablik, Differentiable solutions of functional equations in Banach spaces, *Ann. Math. Sil.* **7** (1993), 17–55.

[21] B. I. Sauber, When a polynomial has exactly one positive root and no roots in $(0, 1)$, *Linear Algebra Appl.* **128** (1990), 107–115.

[22] J. Si, Existence of locally analytic solutions of the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$, *Acta Math. Sinica.* **37** (1994), 590–600 (in *Chinese*).

[23] J. Tabor, Difference and iterative equations, *Funct. Diff. Equ.* **11** (2004), 185–193.

[24] J. Tabor and J. Tabor, On a linear iterative equation, *Results Math.* **27** (1995), 412–421.

[25] J. Tabor and M. Żołdak, Iterative equations in Banach spaces, *J. Math. Anal. Appl.* **299** (2004), 651–662.

[26] G. Targonski, Topics in Iteration Theory, *Vandenhoeck & Ruprecht, Göttingen*, 1981.

[27] T. Trif, Convex solutions to polynomial-like iterative equations on open intervals, *Aequationes Math.* **79** (2010), 315–325.

[28] B. Xu and W. Zhang, Decreasing solutions and convex solutions of the polynomial-like iterative equation, *J. Math. Anal. Appl.* **329** (2007), 483–497.

[29] B. Xu and W. Zhang, Construction of continuous solutions and stability for the polynomial-like iterative equation, *J. Math. Anal. Appl.* **325** (2007), 1160–1170.

[30] D. Yang and W. Zhang, Characteristic solutions of polynomial-like iterative equations, *Aequationes Math.* **67** (2004), 80–105.

[31] L. Yang, X. Hou and Z. Zeng, A complete discrimination system for polynomial, *Sci. China (Ser. E)* **39** (1996), 628–646.

[32] C. K. Yap, Fundamental Problems in Algorithmic Algebra, *Oxford Univ. Press, Oxford*, 2000.

[33] E. Zeidler, Nonlinear Functional Analysis and Its Applications, Vol. I: Fixed-Point Theorems, transl. P. R. Wadsack, *Springer-Verlag, New York*, 1986.

[34] J. Zhang, L. Yang and W. Zhang, Some advances on functional equations, *Adv. Math. (China)* **24** (1995), 385–405.

[35] W. Zhang, Discussion on the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$, *Chinese Sci. Bull.* **32** (1987), 1444–1451.

[36] W. Zhang, Discussion on the differentiable solutions of the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$, *Nonlinear Anal.* **15** (1990), 387–398.

[37] W. Zhang, Solutions of equivariance for a polynomial-like iterative equation, *Proc. Roy. Soc. Edinburgh Sect. A* **130** (2000), 1153–1163.

[38] W. Zhang, K. Nikodem and B. Xu, Convex solutions of polynomial-like iterative equations, *J. Math. Anal. Appl.* **315** (2006), 29–40.

[39] L. Zhao, A theorem concerning the existence and uniqueness of solutions of the functional equation $\lambda_1 f(x) + \lambda_2 f^2(x) = F(x)$, *J. Univ. Sci. Tech.* **32** (1983), 21–27 (in *Chinese*).

XIAOBING GONG
DEPARTMENT OF MATHEMATICS
SICHUAN UNIVERSITY
CHENGDU, SICHUAN 610064
P.R. CHINA

E-mail: xbgong@163.com

WEINIAN ZHANG
DEPARTMENT OF MATHEMATICS
SICHUAN UNIVERSITY
CHENGDU, SICHUAN 610064
P.R. CHINA

E-mail: matzwn@126.com, matwnzhang@yahoo.com.cn

(Received July 26, 2011; revised May 2, 2012)