Project: Vaccination and COVID-19.

Introduction At the time of this writing, the FDA has just approved the Pfizer/BioNTech COVID-19 vaccine for use in the United States via an Emergency Use Authorization, and similar authorization for the Moderna vaccine is expected imminently. Initial vaccination of healthcare workers and nursing home residents has begun. In this project we will model vaccination programs for COVID-19 and look at factors that may increase their success.

COVID-19 vaccines might work in several different ways: they can be *disease-modifying*, acting to prevent severe disease and death and limiting the number of infected people who require hospitalization, speeding recovery, and decreasing infectiousness. Vaccines may also reduce susceptibility among uninfected people who have been vaccinated; such vaccines are termed "preventative". A vaccine may have both disease-modifying and preventative attributes, and in this project we will consider such vaccines. While initial Phase 3 studies of the Pfizer and Moderna vaccines have shown high "efficacy" (more than 90%), precise information about the specifics of their preventative and disease-modifying attributes are not yet well understood. The FDA initially established a minimum efficacy threshold of 50% for COVID-19 vaccines, comparable to influenza vaccines but much lower than the efficacy of nearly every other widely used vaccine.

PART I: Model without vaccination. We will begin with an expanded SEIR (susceptible-exposed-infective-removed) compartment model for COVID-19, where the compartment of infective individuals is expanded into four distinct compartments—asymptomatic, mild/moderate, severe, and critical. Initially our model will not include vaccination, so as to have a baseline against which to compare vaccination strategies. The compartment diagram is as shown below, with 8 compartments as follows:

(1) Compartment U of susceptible individuals.
(2) Compartment E of exposed people who are not yet infectious but will progress to at least asymptomatic infection.
(3) Compartment A of people with asymptomatic infection; we will assume that individuals in compartment A are infectious, and some will eventually progress to a symptomatic infection.
(4) Compartment M of people with mild or moderate illness.
(5) Compartment S of people with severe illness; these people are all assumed to be hospitalized.
(6) Compartment C of people with critical illness; these individuals are all assumed to be in an intensive care unit in a hospital.
(7) Compartment D of people who have died of COVID-19.
(8) Compartment R of people who have recovered from COVID-19; we will assume that individuals in R are no longer infectious.

The number of people in each of these compartments at time t will be denoted $u(t)$, $e(t)$, $a(t)$, and so on, and we will measure time in days. Some of the metrics we will be particularly interested in include total deaths (for the period of time $0 \leq t \leq T$ under study), peak hospitalization (the maximum value of $s(t) + c(t)$ for $0 \leq t \leq T$), peak ICU use (the maximum of $c(t)$ for $0 \leq t \leq T$), the total number of infections $u(0) - u(T)$ in the time period under study. We assume the sum $u(t) + e(t) + a(t) + m(t) + s(t) + c(t) + d(t) + r(t)$ is constant, equaling N, the total number of individuals in the population being modeled.
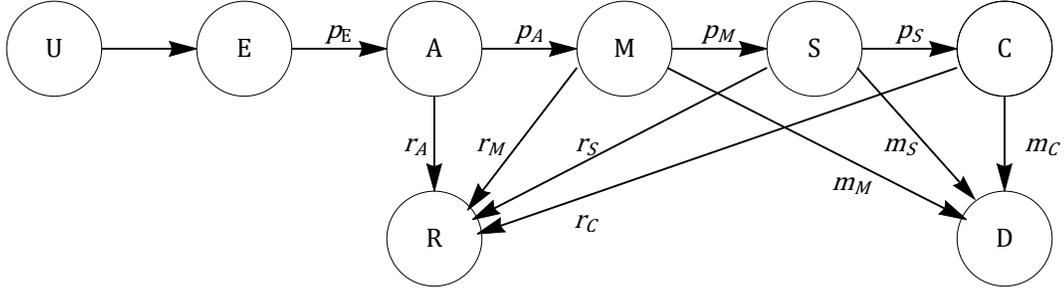
FIGURE 1. Compartment diagram for No Vaccine Model.

Some important assumptions we will make include:

(A1) Only individuals in compartments A and M are capable of passing the infection on to other contacts. While we will generally assume that people in compartments A and M are equally infectious, our set-up will allow for the possibility that individuals in compartment M are more infectious than individuals in compartment A (as some studies have suggested). Since people in compartments S and C are hospitalized, we assume that hospital policies (e.g. no visitors, adequate PPE for healthcare workers) prevent these individuals from infecting others.

(A2) The only deaths that occur are due to COVID.

(A3) Some people in each of the compartments M, S, and C will die of COVID, and exit their compartment directly to compartment D. No asymptomatic person though will die. People in compartments M, S, and C die at a rate proportional to the size of the compartment, with proportionality constants m_M, m_S, and m_C, respectively. These constants are called the **mortality rates** (hence the choice of the letter m).

(A4) Individuals in compartments A, M, S, and C may recover (directly from those compartments), and they do so at a rate proportional to the size of the compartment, with proportionality constants r_A, r_M, r_S, and r_C, respectively. These constants are called the **recovery rates**.

(A5) All people in the exposed compartment E will progress to the asymptomatic compartment A, and they do so at a rate proportional to the size of the compartment E with proportionality constant p_E. Individuals spend on average 3 days in compartment E before progressing to the asymptomatic compartment A, so that $p_E = 1/3$.[1]

(A6) A general principle is that susceptible persons are infected at a rate jointly proportional to the number of susceptibles and the number of people who are infectious. To use that principle here we must take into account that infectious individuals are in both compartment A and compartment M, and we don't want to assume that asymptomatics and people with mild/moderate illness are equally infectious. Thus we would expect that the differential equation for $u'(t)$ would include a "rate out" term that could be written in the form

$$-\frac{\beta_A a(t)}{N} u(t) - \frac{\beta_M m(t)}{N} u(t)$$

where β_A and β_M are positive constants and N is the total (fixed) size of the population.

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[1]See the Appendix at the end of this project for a justification of the assertion that if the average time spent in the E compartment is 3 days, then $p_E = \frac{1}{3}$.

Values we will use for the progression, recovery, and mortality rates are given in the following table.[2] The units on these parameters are $1/\text{day}$.

<table>
<tr><td colspan="11" align="center">Table 1. Progression, recovery and mortality rates.</td></tr>
<tr><td>p_E</td><td>p_A</td><td>p_M</td><td>p_S</td><td>r_A</td><td>r_M</td><td>r_S</td><td>r_C</td><td>m_M</td><td>m_S</td><td>m_C</td></tr>
<tr><td>0.333</td><td>0.333</td><td>0.0833</td><td>0.05</td><td>0.222</td><td>0.0817</td><td>0.175</td><td>0.0607</td><td>0.00167</td><td>0.025</td><td>0.0107</td></tr>
</table>

Here is a sample calculation for the constants p_S, r_S, and m_S, showing how clinical data leads to the values in Table 1. Individuals spend an average of 4 days in compartment S. This means $p_S + r_S + m_S = \frac{1}{4}$. Ten percent of the people with a severe case of COVID-19 will die, and 20% will progress to critical status. The remaining 70% of people in S will move from S to the recovered compartment R. Since $p_S + r_S + m_S = \frac{1}{4}$, we compute p_S, r_S, and m_S individually by multiplying $\frac{1}{4}$ by the fraction of the compartment occupied by individuals who will progress to compartment C, R, and D, respectively:

$$p_S = (0.2)\left(\frac{1}{4}\right) = \frac{1}{20} = 0.05, \quad r_S = (0.7)\left(\frac{1}{4}\right) = \frac{7}{40} = 0.175, \quad m_S = (0.1)\left(\frac{1}{4}\right) = \frac{1}{40} = 0.025.$$

You will be asked below to show how several of the other rate constants are obtained from similar information.

The first set of questions apply to the no-vaccine model:

(a) People in compartment M spend on average 6 days there. Moreover, one percent of people in M die, 50% will progress to state S, and 49% will move from M to the recovered compartment R. From this information, verify the values given above for r_M, m_M, and p_M.

(b) The average time spent in compartment C is 14 days, and 15% of the people in the critical state will die, while the rest will recover. Use this information to verify the values in Table 1 for m_C and r_C.

(c) Next we develop a system of differential equations for $u(t)$, $e(t)$, $a(t)$, $m(t)$, $s(t)$, $c(t)$, $r(t)$, and $d(t)$ corresponding to the compartment diagram above, assumptions (A1)-(A6), and the rate constants in Table 1. We assume a population of total size 100,000, with initial values

$$u(0) = 90,900, \; e(0) = 100, \; r(0) = 9,000, \quad \text{and all other initial values 0.}$$

Temporarily setting aside the equations for $u'(t)$ and $e'(t)$ we begin with

$$a'(t) = -0.555a(t) + 0.333e(t)$$

$$m'(t) = -0.16667m(t) + 0.333a(t)$$

$$s'(t) = -0.25s(t) + 0.0833m(t)$$

$$c'(t) = -0.0714c(t) + 0.05s(t)$$

$$r'(t) = 0.222a(t) + 0.0817m(t) + 0.175s(t) + 0.0607c(t)$$

and

$$d'(t) = 0.00167m(t) + 0.025s(t) + 0.0107c(t).$$

Explain each of the numerical coefficients in these equations by reference to the rates in Table 1 and the compartment diagram. For example, the first term on the right-hand side of the equation for $a'(t)$ is a "rate out" term (hence the negative sign); it has the coefficient $p_A + r_A = 0.555$ in front of $a(t)$.

(d) The differential equation for $u'(t)$ has the form

$$u'(t) = -\frac{\beta_A a(t)}{N} u(t) - \frac{\beta_M m(t)}{N} u(t)$$

where β_A and β_M are positive constants and $N = 100,000$ is the constant fixed size of the population. For our model, the values of β_A and β_M are related to the basic reproduction number R_0 by the formula

(1)

$$R_0 = \frac{\beta_A}{p_A + r_A} + \frac{p_A}{p_A + r_A} \cdot \frac{\beta_M}{p_M + r_M + m_M}.$$

In our analysis we will assume that asymptomatic people and people with mild/moderate illness are equally infectious so that $\beta_A = \beta_M$ (or equivalently $\beta_A/\beta_M = 1$). Here is how we will use Equation (1): If we choose a range of values for R_0, say $R_0 = 1.5$, $R_0 = 1.8$, and $R_0 = 2.2$, we can determine $\beta_A = \beta_M \equiv \beta$ corresponding to our chosen value of R_0. This will give us the differential equation for $u'(t)$. Keep in mind that R_0 is the expected number of additional infections directly caused by one infectious person in an entirely susceptible population. A value of R_0 greater than 1 means the number of cases is expected to increase; we expect that the larger the value of R_0, the more difficult the epidemic will be to control. More mask-wearing, social distancing, and policies that limit riskier activities lead to lower values of R_0. Determine the values of β corresponding to $R_0 = 1.5$, $R_0 = 1.8$, and $R_0 = 2.2$.

(e) Give the differential equation for $e'(t)$ in our no-vaccine model, assuming $R_0 = 1.8$. It should have two pieces: one representing new transmissions coming from U and one representing progressions from E to A. You will need the numerical value of p_E from the above table and the calculated value of β from (d).

(f) You now should have the complete system of equations for $u'(t)$, $e'(t)$, $a'(t)$, $m'(t)$, $s'(t)$, $c'(t)$, $d'(t)$, and $r'(t)$. As a check on your work, what should the sum $u'(t)+e'(t)+a'(t)+m'(t)+s'(t)+c'(t)+d'(t)+r'(t)$ be? Is it?

(g) The *Mathematica* starter "SEIRwithVACCStarter.nb" shows how to solve your no-vaccine system of equations for $0 \leq t \leq 180$ (i.e. over a period of about 6 months) with R_0 value $R_0 = 1.8$, and how to determine total number of deaths, peak hospitalization, peak ICU usage, and total number of infections during this time period. The results are summarized in Table 3 below. By modifying the system of equations for $R_0 = 1.5$ and $R_0 = 2.2$ (only the value of β, which appears in the equations for $u'(t)$ and $e'(t)$ will change), complete the "No Vaccine" column of Table 3. Also graph the solutions for $e(t)$, $a(t)$, $m(t)$, $s(t)$, $c(t)$, and $d(t)$ over this time period. Remark: Since you are not asked for $r(t)$, and $r(t)$ does not appear in any other equation, you may omit the differential equation for $r'(t)$ in your system if you wish.

PART II: Adding Vaccinations to the Model. Now we modify the model from Part I to include vaccinations. In doing this, we will add a parallel chain of compartments to the model, denoted UV, EV, AV, MV, SV, and CV to denote the susceptible vaccinated, exposed vaccinated, asymptomatic vaccinated, mild/moderate vaccinated, severe vaccinated, and critical vaccinated states. The later compartments are

included because we do not assume that our vaccine is perfect, so some vaccinated persons will become infected. The new compartment diagram is as shown below.
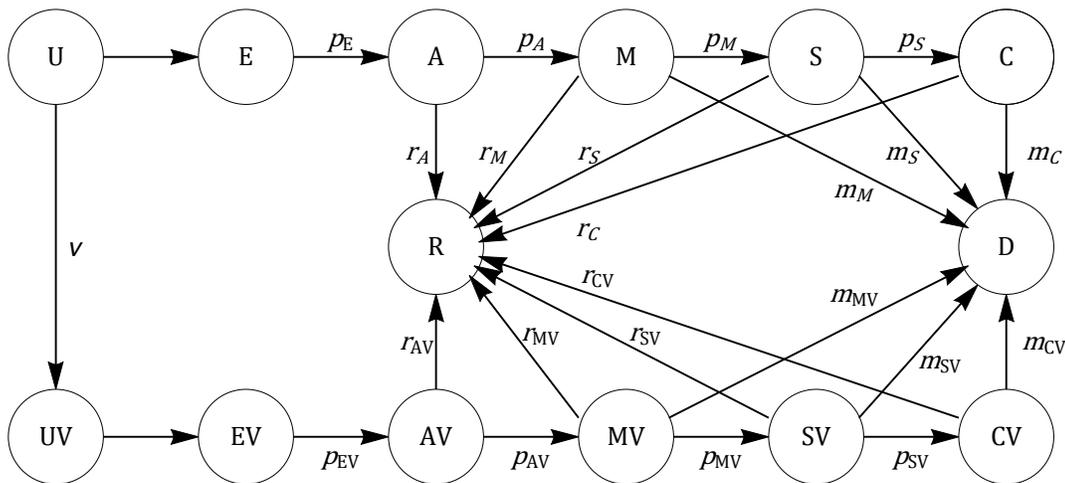
FIGURE 2. Compartment diagram for Vaccine Model.

Three basic assumptions we make about vaccinations:

(A7) Only people in state U are vaccinated; in particular no vaccinations occur in people in state E.

(A8) Vaccination occurs at a rate proportional to the size $u(t)$ of compartment U, with proportionality constant ν.

(A9) There is a time delay between when a vaccine is administered, and when it first takes effect. The amount of this delay will depend in part on whether the vaccine is given in a single-dose, or requires two doses. A person is not considered vaccinated (i.e. moved from compartment U to compartment UV) until this time delay has elapsed.

The number of people in compartments UV, EV, AV, etc. at time t are denoted $uv(t)$, $ev(t)$, $av(t)$, and so on.

(h) For a vaccine with disease-modifying effects, the progression rate p_{EV}, p_{AV}, p_{MV}, p_{SV} and p_{CV} would be decreased from their corresponding "no-vaccine" model progression rates, as given in Table 1. Similarly the mortality rates m_{MV}, m_{SV} and m_{CV} would be decreased from the corresponding mortality rates in the no-vaccine model. The recovery rates r_{AV}, r_{MV}, r_{SV} and r_{CV} would be increased from the corresponding rates in Table 1. Give values for all of these parameters assuming all mortality rates are decreased by 50%, all progression rates are decreased by 50%, and all recovery rates are increased by 100%. Record your answers in the table below. Also give the differential equations for $av'(t)$, $mv'(t)$, $sv'(t)$, $cv'(t)$, and $d'(t)$. To do this, you will need the full compartment diagram above, and the parameter values from Table 2.

Table 2. Progression, recovery and mortality rates.										
p_{EV}	p_{AV}	p_{MV}	p_{SV}	r_{AV}	r_{MV}	r_{SV}	r_{CV}	m_{MV}	m_{SV}	m_{CV}

(i) To determine the new equation for $u'(t)$ when vaccinations are incorporated into the model, first observe that infections in state U can occur by contact between susceptible unvaccinated people and people in states A, M, AV, and MV (as before we assume that people with severe or critical illness are hospitalized and do not contribute to new infections). As in part I we assume that people in states A and M are equally infectious, as are people in states AV and MV. The differential equation for $u'(t)$ and $uv'(t)$ have the form

$$u'(t) \quad = \quad - \text{ rate of new transmissions to compartment } E - \text{ rate of new vaccinations}$$
$$= \quad -\frac{\beta a(t)}{N}u(t) - \frac{\beta m(t)}{N}u(t) - \frac{\beta^* av(t)}{N}u(t) - \frac{\beta^* mv(t)}{N}u(t) - \text{ rate of new vaccinations}$$

and

$$uv'(t) \quad = \quad - \text{ rate of new transmissions to EV} + \text{ rate of new vaccinations}$$
$$= \quad -\sigma\left(\frac{\beta a(t)}{N}uv(t) - \frac{\beta m(t)}{N}uv(t) - \frac{\beta^* av(t)}{N}uv(t) - \frac{\beta^* mv(t)}{N}uv(t)\right) + \text{ rate new vaccinations}$$

As in Part I, we will assume that the constant size of our population is $N = 100,000$. The constants β and β^* reflect both the infectiousness of the disease (for people in states A and M, and AV and MV, respectively) and susceptibility to infection of persons in states U and UV. For a vaccine with disease-modifying effects that decrease infectiousness among vaccinated people we have $\beta^* < \beta$. When the vaccine is also preventative, vaccinated persons are less susceptible, and we reflect this in our equation for $uv'(t)$ by means of a constant $\sigma < 1$. We will use $\sigma = \frac{1}{2}$ (i.e. 50% less susceptibility). Record the "rate of new transmissions to E" and the "rate of new transmissions to EV" using the value of β corresponding to $R_0 = 1.8$, $\beta^* = \frac{1}{2}\beta$, $\sigma = \frac{1}{2}$, and $N = 100,000$.

(j) To account for the time lag between when the vaccine is administered and when it takes effect the equations we will use for $u'(t)$ and $uv'(t)$ will be *delay differential equations* with a time lag of either 14 days (appropriate for a vaccine administered as a single shot; Johnson and Johnson's candidate vaccine now in Phase 3 Trials may be of this type) or 30 days (the Pfizer vaccine requires 2 doses, 21 days apart, with effectiveness being reached about 7 days after the second dose). The delay differential equation for $u'(t)$ looks like

$$u'(t) = -\frac{\beta a(t)}{N}u(t) - \frac{\beta m(t)}{N}u(t) - \frac{\beta^* av(t)}{N}u(t) - \frac{\beta^* mv(t)}{N}u(t) - \nu u(t - \lambda),$$

where the time lag λ will be either 14 or 30 and the constant ν is the pace, or rate, at which vaccination is done. Similarly the delay differential equation for $uv'(t)$ has the form

$$uv'(t) = -\sigma\left(\frac{\beta a(t)}{N}uv(t) - \frac{\beta m(t)}{N}uv(t) - \frac{\beta^* av(t)}{N}uv(t) - \frac{\beta^* mv(t)}{N}uv(t)\right) + \nu u(t - \lambda).$$

In general the theory of delay differential equations is rather more complicated that that of ordinary differential equations. Luckily, numerical solutions of systems containing delay differential equations with a constant delay can still be done using *Mathematica*, and that is what we will do. The main new feature we will see is that instead of just providing initial conditions for our system, we will need

initial history functions—notice that the delay equation for $u'(t)$ requires that we know $u(t - \lambda)$ and that when $0 \le t \le \lambda$, $t - \lambda$ is negative. Record the delay differential equations for $u'(t)$ and $uv'(t)$ assuming that $\beta = 0.33$ (corresponding in part (d) to $R_0 = 1.8$), and that the vaccine reduces both infectiousness and susceptibility by 50%. All constants in your equations except for ν and λ should be specified. Also give the differential equations for $e'(t)$ and $ev'(t)$, using the progression rates p_E and p_{EV} from Tables 1 and 2, respectively.

(k) The *Mathematica* starter "SEIRwithVACCStarter.nb" shows how to solve numerically the system of equations for $u(t)$, $e(t)$, $a(t)$, $m(t)$, $s(t)$, $c(t)$, $uv(t)$, $ev(t)$, $av(t)$, $mv(t)$, $sv(t)$, $cv(t)$, and $d(t)$ with progression, recovery, and mortality rates as given above (in Tables 1 and 2), $\beta^* = \frac{1}{2}\beta$, $\sigma = 0.5$, $\beta = 0.33$ (corresponding to $R_0 = 1.8$) and with the following pairs for the time lag λ and and the vaccination rate ν:

$$\lambda = 14, \ \nu = 0.005; \quad \lambda = 30, \ \nu = 0.005.$$

As before, we use $N = 100,000$, $u(0) = 90,900$, $e(0) = 100$, $r(0) = 9,000$, and all other variables at $t = 0$ are 0. Key conclusions are shown in Table 3 below. By modifying these sample calculations in the *Mathematica* starter, solve the delay differential equations system when the β value is changed to correspond to the reproduction numbers $R_0 = 1.5$ and $R_0 = 2.2$, for both of the given pairs of λ and ν. Summarize the resulting conclusions by completing Table 3. Also include the graphs of $d(t)$, $s(t)$, $c(t)$, $sv(t)$, and $cv(t)$ for $0 < t < 180$.

Table 3: Comparing results at different R_0 values, $0 \le t \le 180$.			
	No Vaccine	14 day delay, $\nu = .005$	30 day delay, $\nu = .005$
$R_0 = 1.5$			
Total Deaths			
Peak hospitalization			
Peak ICU usage			
Total number of infections			
$R_0 = 1.8$			
Total Deaths	2632	866	1152
Peak hospitalization	1661	458	688
Peak ICU usage	655	181	271
Total number of infections	59162	23668	30291
$R_0 = 2.2$			
Total Deaths			
Peak hospitalization			
Peak ICU usage			
Total number of infections			

Also do one more calculation for the following scenario: Progression, recovery, and mortality rates as given in Tables 1 and 2, $\beta^* = \frac{1}{2}\beta$, $\sigma = 0.5$, with the values of β corresponding to $R_0 = 1.8$, time lag $\lambda = 14$ and the vaccination rate $\nu = 0.01$, so that the only change from the "14 day delay" calculations in Table 3 is a doubling of the rate of vaccination. Summarize the key metrics in the table below:

Table 4: A faster pace of vaccination, $0 \le t \le 180$.	
	14 day delay, $\nu = 0.01$
$R_0 = 1.8$	
Total Deaths	
Peak hospitalization	
Total number of infections	

(l) Models such as this one can help inform policy decisions. At least to start out, vaccine demand will greatly outpace supply. Based on the information you obtained in Table 3, would you advocate for a policy that allocates greater vaccine supply to areas where COVID-19 is spreading more severely? To answer this, you might compare the percent decrease in deaths/hospitalizations from the no-vaccine to the vaccine models for each R_0 value. Use the 14 day delay and $\nu = 0.005$ from Table 3.

(m) Compare the total deaths, peak hospitalizations, and total infections for two different vaccine models, using the β value corresponding to $R_0 = 1.8$:

$$\text{Option A: 14 day delay}, 50\% \text{ efficacy}, \ \nu = 0.01$$

and

$$\text{Option B: 30 day delay}, 90\% \text{ efficacy}, \ \nu = 0.005.$$

Notice that for the first case, you have already done the calculations in (k)—see your answers in Table 4. For Option B, interpret 90% efficacy as meaning that progression and mortality rates are decreased by 90% from the values given in Table 1, $\beta^* = 0.1\beta$, and $\sigma = 0.1$. Further assume that recovery rates are increased by a factor of 3 from those given in Table 1. You may find it helpful to summarize these new rates in Table 5 below. Using these values, write the new system of delay differential equations corresponding to Option B, and follow the outline in the *Mathematica* starter to fill out Table 6.

Table 5. Progression, recovery, mortality rates; high efficacy vaccine.										
p_{EV}	p_{AV}	p_{MV}	p_{SV}	r_{AV}	r_{MV}	r_{SV}	r_{CV}	m_{MV}	m_{SV}	m_{CV}

Table 6: Option B; 30 day delay, $\nu = 0.005$, $0 \leq t \leq 180$.	
	30 day delay, $\nu = 0.005$
$R_0 = 1.8$	
Total Deaths	
Peak hospitalization	
Total number of infections	

Suppose you had a vaccine as in Option B, originally intended to be given as a two-dose shot. Ongoing vaccine shortages are occurring, and if just a single dose of the vaccine is given it behaves as in Option A. From a public health standpoint, would you advocate for using the vaccine this way?

To put this question in perspective, we note that on December 31, 2020, an article in The Washington Post discussing the approval in Britain of the AstraZeneca vaccine (a 2 dose vaccine) states that "[Britain Health Minister] Hancock said that in the interest of injecting as many people as possible as quickly as possible, it would be sufficient to give a first dose of the AstraZeneca vaccine and allow more than the usual 21 days between shots. ...immunity comes from around two weeks after the first dose, and then the second dose should be taken up to 12 weeks later ..." On the same day Dr. Anthony Fauci in the United States indicated that a similar plan regarding the Pfizer vaccine was under "intense discussion", while independently Pfizer released a statement that their vaccine had not been evaluated on a dosing schedule different from the "2 doses, separated by 21 days" plan.

Appendix. Here we provide justification of the following assertion: *In a compartment model where members of a compartment exit that compartment at a rate proportional to the population of the compartment with proportionality constant k, then the average time spent in the compartment is $1/k$.*
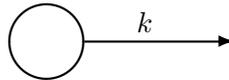
We will assume that our time unit is days. The differential equation $y'(t) = -ky$ has solution $y = y(0)e^{-kt}$. The fraction of the original population that leaves the compartment on the first day is

$$\frac{y(0) - y(0)e^{-k}}{y(0)} = 1 - e^{-k}.$$

The fraction of the original population that leaves the compartment on the second day is

$$\frac{y(0)e^{-k} - y(0)e^{-2k}}{y(0)} = e^{-k} - e^{-2k}.$$

Continuing, we see that the fraction of the original population that leaves the compartment on the third day is $e^{-2k} - e^{-3k}$, and in general the fraction that leaves on the jth day is $e^{-(j-1)k} - e^{-jk}$. This leads to a

calculation of the approximate average time spent in the compartment as

$$1(1 - e^{-k}) + 2(e^{-k} - e^{-2k}) + 3(e^{-2k} - e^{-3k}) + \cdots = 1 + e^{-k} + e^{-2k} + e^{-3k} + \cdots = (1 - e^{-k})^{-1},$$

where the last equality follows from using the geometric series formula to compute the infinite sum $1 + e^{-k} + e^{-2k} + e^{-3k} + \cdots$. Using the Taylor series for e^x we have

$$e^{-k} = 1 - k + \frac{k^2}{2!} - \frac{k^3}{3!} + \frac{k^4}{4!} - \cdots$$

and thus

$$(1 - e^{-k})^{-1} \approx \frac{1}{k}$$

when k is small. Now imagine doing the same kind of calculation with a shorter time period of length $T < 1$. The expected time spent in the compartment is

$$T(1 - e^{-kT}) + 2T(e^{-kT} - e^{-2kT}) + 3T(e^{-2kT} - e^{-3kT}) + \cdots = T(1 + e^{-kT} + e^{-2kT} + e^{-3kT} + \cdots) = \frac{T}{1 - e^{-kT}}.$$

Taking the limit as $T \to 0$ using L'Hopital's Rule gives

$$\lim_{T \to 0} \frac{T}{1 - e^{-kT}} = \lim_{T \to 0} \frac{1}{ke^{-kT}} = \frac{1}{k}$$

for the time spent in the compartment.