Project: Legendre's equation

Prerequisites: Completion of Section 7.3 of *Differential Equations: Techniques, Theory, and Applications*, by MacCluer, Bourdon, and Kriete.

Legendre's equation is

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \tag{1}$$

where λ is a constant. Some of its solutions have widespread applications in the physical sciences – see for example part (n) below. In this project you are invited to find the series solutions for this equation and investigate a famous special case, the **Legendre polynomials**. Since $x_0 = 0$ is an ordinary point for this equation, we can seek power series solutions

$$y(x) = \sum_{k=0}^{\infty} a_k x^k \tag{2}$$

centered at $x = 0$. We use k, rather than n, as the index of summation here to save n for another purpose. Since the only zeros of $1 - x^2$, namely $x = 1$ and $x = -1$, are one unit away from $x_0 = 0$, Theorem 7.3.2 guarantees that these series solutions will have radius of convergence no smaller than 1, and so will converge (at least) on $-1 < x < 1$.

(a) Show that the recursion relation for the series solution in (2) is

$$a_{k+2} = \frac{k(k+1) - \lambda}{(k+2)(k+1)}\, a_k$$

for $k = 0, 1, 2, \ldots$. This tells you how a_0 determines all of the subsequent evenly-indexed coefficients $a_2, a_4, a_6, \ldots$ and how a_1 determines the remaining oddly-indexed coefficients $a_3, a_5, a_7, \ldots$.

(b) Legendre's equation is of most interest when $\lambda \geq 0$. In this case there is a unique nonnegative number b with $\lambda = b(b+1)$, and we'll see that it's convenient to write λ in this form. Show that in terms of b the recursion relation is

$$a_{k+2} = -\frac{(b-k)(b+k+1)}{(k+2)(k+1)} a_k. \tag{3}$$

A fundamental set of solutions. With the recursion relation in hand, let's turn to finding a linearly independent pair of solutions, y_1 and y_2, by requiring that

$$y_1(0) = 1,\ y_1'(0) = 0$$

and

$$y_2(0) = 0,\ y_2'(0) = 1.$$

Since the Wronskian $W(x)$ of y_1 and y_2 will satisfy

$$W(0) = y_1(0)y_2'(0) - y_2(0)y_1'(0) = 1,$$

the linear independence of y_1 and y_2 will be guaranteed. The initial conditions for y_1 say that $a_0 = 1$ and $a_1 = 0$. It then follows from the recursion relation (3) that in the series expansion for y_1, $a_1, a_3, a_5, \ldots$ must all be zero, and the series for y_1 contains only even powers of x. Similarly, the initial conditions for y_2 imply that for the second solution we seek, $a_0 = 0$ and $a_1 = 1$. The recursion relation tells us that in the series expansion for y_2, $a_2, a_4, a_6, \ldots$, are all zero, and the series contains only odd powers of x.

(c) We know the series for y_1 has the form

$$y_1(x) = \sum_{k=0}^{\infty} a_{2k} x^{2k}$$

where $a_0 = 1$. Show that for $k = 1, 2, 3, \ldots$,

$$a_{2k} = (-1)^k \frac{b(b-2)(b-4)\cdots(b-2k+2)(b+1)(b+3)\cdots(b+2k-1)}{(2k)!}. \tag{4}$$

If you are familiar with mathematical induction, the best practice here would be to give an induction proof using the initial value $a_0 = 1$ and the recursion relation (3). Otherwise, it will suffice to calculate a_2, a_4, a_6, and a_8 to demonstrate the pattern.

(d) Now consider the second solution y_2. Here we know that $a_0, a_2, a_4, \ldots$ are all zero and $a_1 = 1$. We can write y_2 as

$$y_2(x) = \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}.$$

Show that

$$a_{2k+1} = (-1)^k \frac{(b-1)(b-3)(b-5)\cdots(b-2k+1)(b+2)(b+4)(b+6)\cdots(b+2k)}{(2k+1)!} \tag{5}$$

for $k = 1, 2, 3, \ldots$.

Polynomial solutions of Legendre's equation. As we have already noted, any power series solution centered at zero to Legendre's equation must converge at least on the interval $-1 < x < 1$. Some solutions may converge for a larger set of x's. One way this can happen is if the power series terminates; that is, if all of its coefficients are zero after a certain point. In this case the solution is just a polynomial, which is defined for all real numbers x.

Only for certain values of the parameter b will the solution y_1 be a polynomial. From Equation (4) for the coefficient a_{2k} in y_1, we can see that if b is an *even integer*, call it n, then $a_n \neq 0$ but $0 = a_{n+2} = a_{n+4} = \cdots$ and y_1 is a polynomial of (even) degree exactly n,

$$y_1(x) = \sum_{k=0}^{n/2} a_{2k} x^{2k}. \tag{6}$$

Moreover, this is the only way y_1 turns out to be a polynomial of even degree n, and we have from (4)

$$\begin{aligned} a_{2k} &= (-1)^k \frac{n(n-2)(n-4)\cdots(n-2k+2)(n+1)(n+3)(n+5)\cdots(n+2k-1)}{(2k)!} \\ &= (-1)^k \frac{A_{2k}B_{2k}}{(2k)!}, \end{aligned}$$

where

$$A_{2k} = n(n-2)(n-4)\cdots(n-2k+2) \tag{7}$$

and

$$B_{2k} = (n+1)(n+3)(n+5)\cdots(n+2k-1).$$

We would like to somehow express A_{2k} and B_{2k} in "closed form". In the present context, this means expressing them in terms of factorials. First note that if $n - 2k = 0$ (that is, if $k = n/2$, the upper index of summation in (6)), then

$$A_{2k} = A_n = n(n-2)(n-4)\cdots(4)(2).$$

There are $n/2$ factors here, all even, so

$$\begin{aligned} A_n &= 2^{n/2}\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdots(2)(1) \\ &= 2^{n/2}\left(\frac{n}{2}\right)! \end{aligned} \tag{8}$$

(e) Now suppose that $k < n/2$ so that $n - 2k$ is not zero. Show that

$$A_{2k} = \frac{2^k\left(\frac{n}{2}\right)!}{\left(\frac{n}{2}-k\right)!}.$$

Notice that when $k = n/2$, this agrees with Equation (8), since $0!$ is defined to be 1. Suggestion: Multiply and divide the expression (7) for A_{2k} by

$$(n-2k)(n-2k-2)(n-2k-4)...(4)(2).$$

(f) Show that

$$B_{2k} = \frac{(\frac{n}{2})!(n+2k)!}{2^k n!(\frac{n}{2}+k)!},$$

and conclude that

$$\begin{aligned} a_{2k} &= (-1)^k \frac{1}{(2k)!} A_{2k} B_{2k} \\ &= \frac{\left((\frac{n}{2})!\right)^2}{n!} \frac{(-1)^k (n+2k)!}{(2k)!\left(\frac{n}{2}-k\right)!\left(\frac{n}{2}+k\right)!}. \end{aligned} \tag{9}$$

(g) It is customary (and more convenient) to write the sum (6) for $y_1(x)$ "backwards", starting with the leading term $a_n x^n$ and ending with the constant term a_0. You can do this by defining a new index of summation j via the equation $2k = n - 2j$. With this convention, Equation (6) becomes

$$y_1(x) = \sum_{j=0}^{n/2} a_{n-2j} x^{n-2j}.$$

Insert this change of index into the formula (9) for a_{2k} to show that

$$a_{2k} = a_{n-2j} = \frac{(-1)^{n/2}\left((\frac{n}{2})!\right)^2}{n!} \frac{(-1)^j (2n-2j)!}{j!(n-j)!(n-2j)!}$$

so that

$$y_1(x) = \frac{(-1)^{n/2}\left((\frac{n}{2})!\right)^2}{n!} \sum_{j=0}^{n/2} (-1)^j \frac{(2n-2j)!}{j!(n-j)!(n-2j)!} x^{n-2j}. \tag{10}$$

To summarize so far: If n is an even integer, y_1 as given by (10) is a polynomial solution having degree n of Legendre's equation

$$(x^2-1)y'' - 2xy' + n(n+1)y = 0$$

satisfying $y_1(0) = 1$ and $y_1'(0) = 0$.

(h) However, it turns out that the most commonly used solution is not y_1, but a constant multiple of it, namely cy_1, where the constant c is chosen so that the coefficient b_n of x^n in $cy_1(x)$ is

$$b_n = \frac{(2n)!}{2^n (n!)^2}.$$

Show that the value of c that accomplishes this is

$$c = (-1)^{n/2} \frac{n!}{2^n \left((\frac{n}{2})!\right)^2}$$

and that with this choice of c you arrive at the n^{th} **Legendre polynomial** of even degree n, denoted by $P_n(x)$, defined to be $cy_1(x)$, and given by the formula

$$P_n(x) = cy_1(x) = \frac{1}{2^n} \sum_{j=0}^{n/2} (-1)^j \frac{(2n-2j)!}{j!(n-j)!(n-2j)!} x^{n-2j}.$$

The reason for preferring this particular choice of c (or equivalently, the choice of leading coefficient b_n mentioned above) is that with it you have the very nice fact that $P_n(1) = 1$. (You are asked to verify this in part (m) below.) As a side benefit, the formula for $P_n(x)$ looks simpler than the formula (10) for $y_1(x)$.

(i) What about our second linearly independent solution

$$y_2(x) = \sum_{k=0}^{\infty} a_{2k+1} x^{2k+1}?$$

It's clear from the expression (5) for a_{2k+1} that y_2 will be a polynomial exactly when $b = n$, some *odd integer*. In this case, we see that for $k = 1, 2, 3, ...,$

$$a_{2k+1} = (-1)^k \frac{(n-1)(n-3)(n-5)\cdots(n-2k+1)(n+2)(n+4)(n+6)\cdots(n+2k)}{(2k+1)!}.$$

Thus $a_n \neq 0$, but $0 = a_{n+2} = a_{n+4} = ...$, and y_2 is a polynomial of (odd) degree exactly n,

(11) $$y_2(x) = \sum_{k=0}^{(n-1)/2} a_{2k+1} x^{2k+1}.$$

Here we know $a_1 = 1$, and for $k = 1, 2, 3, ...$ we can abbreviate a_{2k+1} as

$$a_{2k+1} = (-1)^k \frac{C_{2k+1} D_{2k+1}}{(2k+1)!},$$

where

$$C_{2k+1} = (n-1)(n-3)(n-5)\cdots(n-2k+1)$$

and

$$D_{2k+1} = (n+2)(n+4)(n+6)\cdots(n+2k).$$

Your tasks are to express D_{2k+1} and C_{2k+1} in closed form, and to conclude that

$$a_{2k+1} = \frac{\left(\left(\frac{n-1}{2}\right)!\right)^2}{n!} \frac{(-1)^k (n+2k)!}{(2k+1)!\left(\frac{n-1}{2}-k\right)!\left(\frac{n-1}{2}+k\right)!}.$$

(Suggestion: Find a closed form for $(n+1)(n+3)(n+5)...(n+2k-1)$. Then multiply and divide D_{2k+1} by this expression and $n!$ to find a closed form for D_{2k+1}. For both D_{2k+1} and C_{2k+1}, keep in mind that $n-1$ is even.)

(j) As with y_1, the polynomial we really want is not $y_2(x)$, but $cy_2(x)$, where c is again chosen to make the coefficient of x^n in $cy_2(x)$ equal to

$$\frac{(2n)!}{2^n (n!)^2}.$$

Some caution is required: This c is different from the c that worked for $y_1(x)$. In fact, let's call it $\tilde{c}$ to emphasize this. Remarkably, you should reach almost the same result, namely the n^{th} Legendre polynomial P_n, now of *odd* degree n, given by

$$P_n(x) = \tilde{c} y_2(x) = \frac{1}{2^n} \sum_{j=0}^{(n-1)/2} (-1)^j \frac{(2n-2j)!}{j!(n-j)!(n-2j)!} x^{n-2j}.$$

To arrive at this formula for $P_n(x)$, you will need to

(i) Write the sum (11) "backwards" (as you did in the case of even n by introducing a new index of summation j, but this time defined by $2k+1 = n-2j$).

(ii) Calculate $\tilde{c}$.

Even and odd together. The only difference between the formulas for P_n with n even, and P_n with n odd, is the upper limit of summation. We can combine these formulas into one by introducing $[n/2]$, the greatest integer not exceeding $n/2$. When n is even, $[n/2] = n/2$, and when n is odd, $[n/2] = (n-1)/2$. This allows us to write

(12) $$P_n(x) = \frac{1}{2^n} \sum_{j=0}^{[n/2]} (-1)^j \frac{(2n-2j)!}{j!(n-j)!(n-2j)!} x^{n-2j}$$

for any nonnegative integer $n = 0, 1, 2, \cdots$.

(k) Calculate the polynomial $P_n(x)$ explicitly for $n = 0, 1, 2, 3, 4, 5$. Then use a CAS to plot the graph of each of these functions for $-1 \leq x \leq 1$. (See the beginning comment in part (ii) of (n) below.)

(l) Having gotten this far, it's easy to derive another useful expression for $P_n(x)$, called **Rodrigues' formula**, which states that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

On the right-hand side you can use the Binomial Theorem to write

$$(x^2-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{2n-2k} \tag{13}$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(i) Show that the n^{th} derivative of x^{2n-2k} is equal to zero if $[n/2] < k \le n$, but is equal to

$$\frac{(2n-2k)!}{(n-2k)!}x^{n-2k}$$

if $0 \le k \le [n/2]$.

(ii) Use (13) and your result from (i) to show that

$$\frac{d^n}{dx^n}(x^2-1)^n = \sum_{k=0}^{[n/2]} (-1)^k \frac{n!(2n-2k)!}{k!(n-k)!(n-2k)!}x^{n-2k}.$$

(iii) Compare your result in (ii) with Equation (12) to finish the proof of Rodrigues' formula.

(m) Suppose that f and g are functions that can be differentiated any number of times. The product rule from calculus says that

$$(fg)' = fg' + f'g.$$

Differentiating again, using the product rule where necessary on the right, we find

$$(fg)'' = fg'' + 2f'g' + f''g.$$

Similarly, a third differentiation gives

$$(fg)''' = fg''' + 3f'g'' + 3f''g' + f'''g.$$

One can continue in this way (mathematical induction works well for this) to show that for any positive integer n,

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)} \tag{14}$$

where $f^{(k)}$ denotes the k^{th} derivative of f and $\binom{n}{k}$ is the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

(i) Since $x^2 - 1 = (x-1)(x+1)$, Rodrigues' formula states that

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}[(x-1)^n(x+1)^n].$$

Use this and (14) to show that

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^{n-k}(x+1)^k.$$

(ii) Use your result in (i) to show that $P_n(1) = 1$ and $P_n(-1) = (-1)^n$.

(n) Legendre's equation and polynomials arise in a variety of applications; for instance, when solving the three-dimensional wave equation and Laplace's equations in spherical coordinates. Steady-state-temperature functions satisfy Laplace's equation. It can be shown that a function modeling the temperature in degrees Celsius of a solid ball of radius 10 cm whose upper hemispherical surface is held at 100 degrees Celsius and whose lower hemispherical surface is held at 0 degrees Celsius is given, in spherical coordinates, by

$$T(\rho,\phi) = 50 + 25\sum_{k=0}^{\infty} \frac{(-1)^k (2k)!(4k+3)}{2^{2k}(k+1)(k!)^2}\left(\frac{\rho}{10}\right)^{2k+1} P_{2k+1}(\cos\phi).$$

(i) According to this model, what is the temperature of the ball at any point in the "equatorial plane" $\phi = \pi/2$.

(ii) Your computer-algebra system should have Legendre polynomials "built-in". E.g., in *Mathematica*

$$\texttt{LegendreP}[\texttt{n},\texttt{x}] \ = P_n(x).$$

Let

$$T_n(\rho,\phi) = 50 + 25\sum_{k=0}^{n} \frac{(-1)^k (2k)!(4k+3)}{2^{2k}(k+1)(k!)^2}\left(\frac{\rho}{10}\right)^{2k+1} P_{2k+1}(\cos\phi).$$

To two decimal places, compute approximate temperatures $T_{10}(5,\pi/4)$, $T_{10}(10,\pi/4)$, $T_{10}(5,3\pi/4)$, $T_{10}(10,3\pi/4)$, $T_{100}(5,\pi/4)$, $T_{100}(10,\pi/4)$, $T_{100}(5,3\pi/4)$, and $T_{100}(10,3\pi/4)$.