RECENT MATHEMATICAL TABLES

Details of recent tables are also to be found in our introductory article of this issue, nos. (33). Dieckvoss and Kox, (34). Comrie; and in N4 Gifford and C. G. S. Tables, N5 C. G. S. and Peters, N6 Peters and Comrie, N7 Smithsonian Tables.

89[D, E].—PROJECT FOR COMPUTATION OF MATHEMATICAL TABLES (A. N. Lowan, technical director). *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments.* Prepared by the Federal Works Agency, Work Projects Administration for the State of New York, conducted under the sponsorship of the National Bureau of Standards. New York, 1940, 1939. xx, 405 p. 20.9×27 cm. Reproduced by a photo offset process. Sold by the U. S. Bureau of Standards, Washington, D. C. $2.00; foreign $5.00.

The main part of this volume (p. 1–400) is occupied with a table of the circular and hyperbolic sines and cosines for the range [0.0000 (0.0001) 1.9999; 9D]. Then follow supplementary tables: II (p. 402–403), circular and hyperbolic sines and cosines for the range [0.0(0.1)10.0; 9D]; and III (p. 404–405) a conversion table for radians and degrees, to 6D.

The expansions of the functions involved terms of the form $U_n(x)=x^n/n!$ The values of this function for the key arguments $x_0=0.01, 0.02, \cdots$ were computed to 15D for $n=0, 1, 2, \cdots p$, where p is such that $U_p(x_0)$ is not greater than one unit in the fifteenth place. The values of $U_n(x)$ for all other arguments were computed with the aid of a recurrence formula, successive applications of which involved self-checking. But the other checks applied included comparison of the values of sin x and cos x at intervals of 0.001 with the values given in Van Orstrand's Table, and in the New York Project's *Tables of Sines and Cosines for Radian Arguments*, computed by an independent method; see RMT **81.** The values of the hyperbolic functions were also checked by combining the values of e^x and e^{-x} in the New York Project's volume *Tables of the Exponential Function e^x*, for $x=$[0.0000(0.0001)1,0000; 18D], and [1.000(0.0001)25000; 15D], and $-$[0.0000(0.0001)2.5000; 18D]. After these tests, the resulting values of the circular and hyperbolic functions were rounded to 9D. The claim that an error in the ninth place does not exceed 0.51 is doubtless well founded.

From the fairly representative bibliography given in RMT **81,** it may be noted that so far as the circular functions are concerned, an appreciable portion of the present table covers new ground. Among other published tables of the hyperbolic sines and cosines, to 5D or more, and for real values of x, are the following (compare N **7,** IV):

THOMAS HOLMES BLAKESLEY (1847–1929), *A Table of Hyperbolic Cosines and Sines . . .* (Published by the Physical Society of London), London, 1890, 6 p. 15×24 cm. Cosh x and sinh x for $x=$[0.01(0.01)4.00; 7D]. Comparison of this table with only the values of the above mentioned Table II showed the following last figure errors in Blakesley: Cosh x—unit errors for $x=1.30$, 1.70, 2.10, 2.60, 2.90, 3.40, 3.60, 3.80, 3.90, and two units for $x=3.10$, and three units for $x=4.00$; sinh x—unit errors for $x=1.50$, 1.60, 1.80, 2.40, 2.60, 2.80, 3.10, 3.20, 3.30, 3.70, and two units for 3.50, 3.60, and 3.90.

JOHANN OTTO WILHELM LIGOWSKI (1821–1893), *Tafeln der Hyperbelfunktionen . . .*, Berlin, 1890, 16.4×24.6 cm. P. 58–61, 67–79, sinh x and cosh x, for $x=$[0.00(0.01)2.00; 6D], [2.00(0.01) 8.00; 5D], with differences.

ANGIOLO FORTI, *Nuove Tavole delle Funzioni Iperboliche . . .*, Rome, 1892. 16.4×23.9 cm. Sinh x and cosh x for $x=$[0.0000(0.0001)2.000(0.0010)2.010; 6D], [2.00(0.01)8.00; 5–7S]. Becker and Van Orstrand note that in these tables there are frequent errors of 1, 2, and 3 units in the last decimal place.

G. F. BECKER and C. E. VAN ORSTRAND, *Smithsonian Mathematical Tables. Hyperbolic Functions,* Washington, Smithsonian Institution, 1909; fifth reprint 1942, p. 88–171. 15×22.9 cm. Sinh x and cosh x for $x=$[0.0000(0.0001)0.1000; 5D], [0.100(0.001)3.000; 5D], [3.00(0.01)6.00; 4D].

K. Hayashi, *Fünfstellige Tafeln der Kreis- und Hyperbelfunktionen . . . mit natürlichen Zahlen als Argument*. Berlin and Leipzig, Gruyter, 1921. 16×23.1 cm. sin x, cos x, sinh x, cosh x, for x=[0(0.0001)0.100(0.001)3.00(0.01)6.3(0.1)10.0; 5D].

Ulfilas Meyer and Adalbert Deckert, *Tafeln der Hyperbelfunktionen. Formeln* Berlin, 1924, p. 6–17. 17×24.3. Sinh x and cosh x for x=[0.000(0.001)3.009; 5D].

J. R. Airey, Br. Ass. Adv. Sci., *Report*, 1926, p. 295–296. 21.5×27.9. Sinh x and cosh x for x=[0.1(0.1)10.0; 15D]; also in Br. Ass. Adv. Sci., *Mathematical Tables*, v. 1, London, B.A.A.S., 1931, Table VI, p. 30. 21.5×27.9 cm.

K. Hayashi, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen . . .*, Berlin, Springer, 1926, p. 13–201. 21×27.3. Sinh x and cosh x for x=[0.00000(0.00001)0:00100; 20D], [0.0010(0001)0.0999; 10D], [0.100(0.001)2.999; 10D], (3.00(0.01) 9.99; 10D], [10.0(0.1)20; 15D], [21(1)39; 15D], [39(1)50; 31–33S], p. 8–201. Also sinh($\pi x/360$), and cosh($\pi x/360$). for x=[0(1)360; 10D], p. 96–166 (alternate pages). Unreliable table. Since

C. E. Van Orstrand gave (Nat. Acad. Sci., Washington, *Memoirs*, v. 14, 1921, p. 40–45) a table for $e^{\pm(n\pi/360)}$ for n=[0(1)360; 23D], values for sinh ($\pi x/360$) and of cosh ($\pi x/360$), considerably more extensive than those of Hayashi, are readily found.

J. R. Airey, Br. Ass. Adv. Sci., *Report*, 1928, p. 308–316. 21.5×27.9. Sinh πx and cosh πx, for x=[0.00(0.01)4.00; 15D]; also in Br. Ass. Adv. Sci., *Mathematical Tables*, v. 1, London, B.A.A.S., 1931, Table V, p. 28–29. These tables "are required in the computation of the elliptic theta functions with imaginary argument, and of the gamma function with complex argument."

K. Hayashi, *Fünfstellige Funktionentafeln Kreis-, zyklometrische, Exponential-, Hyperbel-, . . . Funktionen . . .*, Berlin, Springer, 1930, p. 3–41, 60–64. 16.5×24.7 cm. sinh x and cosh x for x=[0.00(0.01)10.00; 5D]. Also sinh πx and cosh πx for x=[0.00(0.01)0.1(0.1)10.0; 5D], 7/6, 13/6, 19/6, 5/4, · · · , 7/2, · · · , 23/6 [21 miscellaneous values]. Unreliable table.

L. J. Comrie, table of sinh πx and cosh πx for x=[0.0000(0.0001)0.0100; 15D], Br. Ass. Adv. Sci., *Mathematical Tables*, v. 1, London, B.A.A.S., 1931, Table IV, p. 24–25; intended for use as an auxiliary table to Table V of Airey (1928).

F. E. Fowle, ed., *Smithsonian Physical Tables*, eighth rev. ed., first reprint, (*Smithsonian Misc. Coll.*, v. 88) Washington, Smithsonian Institution, 1934, p. 41–47. 14.9×22.8 cm. Sinh x and cosh x for x=[0.00(0.01)3.0; 5D], [3.0(0.1)5.0; 4D].

K. Hayashi, *Tafeln für die Differenzenrechnung sowie für die Hyperbel-, Besselschen, elliptischen und anderen Funktionen*, Berlin, Springer, 1933, p. 38–47. 21.2×27.7 cm. Sinh πx and cosh πx for x=[0.01(0.01)0.99; 6D], [1.00(0.01)10.00; 5–8S].

Samata Sakamoto, *Tables of Gudermannian Angles and Hyperbolic Functions*, Tokyo, 1934. 12.6×18.7 cm. Table III, p. 112–137 gives sinh x and cosh x for x=[0.000(0.005)0.100(0.010) 0.20(0.01)3.00(0.05)4.0(0.1)10.0; 5D]. Table IV, p. 138–199, has sinhπx, coshπx for x=[0.00(0.01)0.10(0.10)5.00(0.50)9.00, 10.00; 10D to 2.4, then 13S to 20S]. A comparison of this latter table for the range 0 to 4 in Airey's table of 1928 showed the following errors made by Sakomoto: for x=0.02 (cosh πx), and 0.40 (sinh πx) unit errors in the last place; x=0.80 (cosh πx) *for* 6.21314 32607 *read* 6.21314 32657; x=3.30 (sinh πx and cosh πx) *for* 13900.543 · · · *read* 15900.543 · · · .

J. R. Airey, assisted by L. J. Comrie, "The circular and hyperbolic functions, argument $x/\sqrt{2}$." *Phil. Mag.*, s. 7, v. 20, 1935, p. 721–726 and 726–731. sin ($x/\sqrt{2}$), cos ($x/\sqrt{2}$) and sinh ($x/\sqrt{2}$), cosh ($x/\sqrt{2}$), each for x=[0.0(0.1)20.0; 12D]. The calculations of these functions for x=2 to 20 were based on their values when x=1, which are given to 20D.

Further, C. A. Bretschneider, gave sinh 1 and cosh 1 each to 105D, *Archiv d. Math. U. Physik*, v. 3, 1843, p. 28–29.

We note also two other tables which may be regarded as sort of supplementary to the present list, as well as to the bibliography in RMT **81**. The first table is in

Vladimir Vassal, *Nouvelles Tables donnant avec Cinq Décimales les Logarithmes Vulgaires et Naturels des Nombres . . . et des Fonctions Circulaires et Hyperboliques pour tous les Degrés du*

Quart de Cercle de Minute en Minute, Paris, 1872, p. 67–111. For every sexagesimal minute there is a column giving the corresponding number of radians x to 5D, and another column for the values of u, to 5D, such that $\sinh u = \tan x$, $\cosh u = \sec x$, $\operatorname{sech} u = \cos x$, $\tanh u = \sin x$, etc. The corresponding values of the circular and hyperbolic functions to 5D, may then be read off. There are similar tables, for every sexagesimal, and centesimal, minute, in

L. POTIN, *Formules et Tables Numériques relatives aux Fonctions Circulaires, Hyperboliques, Elliptiques*, Paris, 1925, p. 450–494, and 496–595.

Hence, not only in the Circular but also in the Hyperbolic Sines and Cosines of the volume under review there are important additions to previous ranges of values. The U. S. Bureau of Standards has performed very notable service, not only in making this, and a dozen other admirable volumes emanating from the New York group, available to scientists, but also at nominal charges. R. C. A.

90[D].—PROJECT FOR COMPUTATION OF MATHEMATICAL TABLES (A. N. Lowan, technical director), *Table of arctan x.* Prepared by the Federal Works Agency, Work Projects Administration for the State of New York, conducted under the sponsorship of the National Bureau of Standards. New York, 1942, xxv, 169 p. 20.9×27.1 cm. Reproduced by a photo offset process. Sold by the National Bureau of Standards, Washington, D. C. \$2.00; foreign \$2.50.

This volume gives values to 12 decimals of the definite integral

$$\arctan x = \int_0^x \frac{du}{1+u^2}$$

with second differences for the following ranges of x: 0(0.001)7, p. 2–71; 7(0.01)50, p. 72–114; 50(0.1)300, p. 115–139; 300(1)2000, p. 140–156; 2000(10)10000, p. 157–164.

As with all major tables published by this Project, this table is more extensive and accurate than any previously published table of its sort. The foreword to this volume, by W. G. Bickley, (at whose suggestion the table was produced) gives many instances of the utility of this table. The intimate connection of the function with the natural logarithm is stressed. For instance, armed with tables of these two functions, the computer may evaluate the integral of any rational function. Among the many applications of arctan x might have been mentioned the gudermannian

$$gdx = \arctan(\sinh x).$$

As pointed out in the foreword, this table is not intended to be used for ordinary trigonometry.

The introduction contains the usual formulas for arctan x including the well known infinite series of Gregory and Euler, and a discussion of the problem of interpolation. Besides the usual Everett's formula, using second differences, the following formula is available for this function:

$$\arctan x = \arctan x_0 + \theta - \theta^3/3 + \cdots$$

where $x = x_0 + h$, x_0 being a tabulated argument, and $\theta = h/(1+xx_0)$. Inverse interpolation, by two methods, is not difficult, so that the table can be used to find tan y for y in radians.

At the end of the volume are auxiliary tables of $p(1-p)$ and $p(1-p^2)/6$ for use in interpolation together with two tables for converting degrees, minutes, and seconds to radians and vice versa.

A peculiar feature of the main table is the fact that the last 50 percent of it, in spite of the coarseness of the argument x near the end, is devoted to less than four percent of the range of the function arctan x. An alternative arrangement in which the range of x is 0(.0001)1 would have been sufficient in view of the relation

$$\arctan x = \frac{\pi}{2} - \arctan(1/x).$$

According to the introduction, the application of this formula for $x > 1$ "would generally be quite laborious, as it would involve finding the reciprocal of x, and then interpolating for that argument." The reviewer believes that with the argument in interval as fine as .001, the consequent saving in the labor of interpolation would more than offset the trouble of finding $1/x$. The present arrangement requires a second difference of as much as 649519 units in the twelfth decimal place near $x=1/\sqrt{3}$, so that linear interpolation is correct to only 6 decimal places. The alternative arrangement would have given 8 decimal places and, of course, simpler second difference interpolation, not to mention 100 instead of 164 pages. Problems for which the present table would seem to have been arranged, namely those involving the arctangent of large integers are not mentioned in the introduction or foreword nor are they known to the reviewer.

There is given (p. xxiii–xxv) a bibliography of 15 tables (mostly small) of arctan x and related functions together with a few errata.

There is every reason to believe that this excellent table, produced by subtabulation and checked to sixth differences, is as free from errors as are the other dozen major tables of this Project.

D. H. L.

91[K].—(i) WILLIAM FLEETWOOD SHEPPARD (1863–1936), *The Probability Integral* . . . Completed and edited by the Br. Ass. Adv. Sci., Committee for the Calculation of Mathematical Tables. Cambridge Univ. Press, publ. for the B.A.A.S., 1939. xi, 34 p. 21×28 cm. Portrait frontispiece of Sheppard. 8/6.

(ii) PROJECT FOR THE COMPUTATION OF MATHEMATICAL TABLES (A. N. Lowan, technical director), *Tables of Probability Functions.* Prepared by the Federal Works Agency, Work Projects Administration for the State of New York, conducted under the sponsorship of the National Bureau of Standards, New York. v. 1, 1941, xxviii, 302 p.; v. 2, 1942, xxi, 344 p. 20.9×27 cm. Reproduced by a photo offset process. Sold by the U. S. Bureau of Standards, Washington, D. C. \$2.00 + \$2.00; foreign \$5.00.

These works which are intimately related to one another, provide new and authoritative tables of what have been called the probability or error functions, that is to say the function Ae^{-kx^2}, where k may be either 1 or $\frac{1}{2}$, and its integral. The multiplier is usually chosen so that the integral over the infinite range is unity.

Although these functions have been the subject of interest for more than a century, there is as yet no standard notation for either of them. Perhaps one reason for this difficulty is found in the fact that while the statistician, a rather recent addition to the scientific fraternity, is interested in the function $\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$ and its integral, the physicist finds more use for the function e^{-x^2} and its integral. Hence it would seem that each person who has encountered the functions anew, has given new symbols for each of their several forms. This same lack of uniformity is encountered in the volumes under review.

Sheppard in his work adopts the following notations, which have become fairly common in English works because of their use in *Biometrika*:

$$z_x = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2},\ \tfrac{1}{2}(1-\alpha_x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{1}{2}t^2}dt, \text{ that is to say,}$$

$$\alpha_x = \frac{1}{\sqrt{2\pi}}\int_{-x}^{x} e^{-\frac{1}{2}t^2}dt = \frac{2}{\sqrt{2\pi}}\int_0^x e^{-\frac{1}{2}t^2}dt, \qquad F(x) = \tfrac{1}{2}(1-\alpha_x)/z_x,$$

$$L(x) = -\log_e\left\{\tfrac{1}{2}(1-\alpha_x)\right\}, \qquad l(x) = \log_{10}\left\{\tfrac{1}{2}(1-\alpha_x)\right\}.$$

Before describing the contents of Sheppard's tables it will be necessary to explain the interpolation scheme which he employs. In most tables a difference method is used based upon the Gregory-Newton formula, or one of its variants, that is to say, the formula:

$$f(x+\theta h) = f(x) + \theta\Delta f(x) + \frac{\theta(\theta-1)}{2!}\Delta^2 f(x) + \cdots.$$

where h is the interval of the argument. One of the great advantages of this formula, or its variants such as the common Everett's formula which employs only central differences, is found in the fact that the value of h does not appear explicitly in it. Hence a computer can provide an interpolation scheme merely by giving a set of differences. The disadvantage is found in the binomial-coefficient multipliers of the successive differences, which must either be computed or read from tables.

Sheppard for his interpolation scheme decided to make use of the Taylor's expansion of a function, that is to say,

$$(x + \theta h) = f(x) + \theta f_1(x) + \theta^2 f_2(x) + \cdots + \theta^n f_n(x) + \cdots$$

where we employ the abbreviation $f_n(x) = \dfrac{h_n}{n!} f^{(n)}(x)$.

There is obviously some advantage gained in the ease of interpolation by replacing the binomial coefficients by powers, but it is also clear that the labor of computing the original table is very greatly increased since h appears explicitly in the multipliers and derivatives must be computed instead of differences. But when, as in the present tables, as many as 16 differences would be required were a full interpolation computed, it is readily seen that there is great advantage gained in the use of these so-called *reduced derivatives*.

Table I in Sheppard's work gives the values of $F(x)$ over the range $x=[0.00(0.01)10.00;\ 12D]$ together with the reduced derivatives $h^n F^{(n)}/n!$, n ranging from 5 in the early part of the table to 3 in the latter part.

Table II provides the values of $F(x)$ for $x=[0.0(0.1)10.0;\ 24D]$ together with the functions $h^n F^{(n)}/n!$, where n ranges from 16 in the early part of the table to 13 in the latter part.

Table III gives eleven values of $L(x)$ for $x=[0(1)10;\ 24D]$.

Table IV, gives the values of $L(x)$ over the range $x=[0.0(0.1)10.0;\ 16D]$, with corresponding values of $h^n L^{(n)}/n!$, n ranging from 10 to 8.

Table V provides values of $l(x)$, for $x=[0.0(0.1)10.0;\ 12D]$ together with the corresponding values of $h^n l^{(n)}/n!$ with n ranging from 7 to 6.

Table VI gives $l(x)$ for $x=[0.00(0.01)10.00;\ 8D]$, together with the central differences δ^2.

The following quotation from the "Introduction" indicates the method of computation and the accuracy of the tables.

"Table II was evidently constructed from Laplace's continued fraction and the derivatives calculated from its convergents. Table I was obtained by subtabulating Table II to interval (0.01). Sheppard used the fact that any tabular entry is the sum of the next tabular entry and its reduced derivatives, all taken positively; while the reduced derivatives of any entry are simple linear functions, with known coefficients, of these quantities, already calculated in Table II at interval (0.1). He never completed his subtabulation. This has been done on the Association's National machine by Mr. F. H. Cleaver using a method devised by Mr. D. H. Sadler. Mr. W. L. Stevens gave great help in supervising the calculations. All the tables have now been checked, Table III by recalculation, Tables II, IV and V by summing the function and its reduced derivatives for each value of the function; the latter process does not of course ensure the accuracy of the last figure. It is an example of the remarkable accuracy of Sheppard's work that not a single error was discovered in any of the entries in Table II."

The name of Sheppard is familiar to every student of statistics. He was an authority on methods of graduating data and most textbooks give an account of what is called the method of *Sheppard's corrections* for the adjustment of the values of moments computed from discrete data.

In the American volumes the following notations were adopted:

$$H'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}, \quad H(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt; \quad Q(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \quad P(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-\frac{1}{2}t^2} dt.$$

The first volume is devoted to the evaluation of $H'(x)$ and $H(x)$, the second to $Q(x)$ and $P(x)$. Both volumes contain introductory explanations about the method of computation of the functions and the use of the tables.

Table I, v. 1, gives the values of $H'(x)$ and $H(x)$ for $x=[0.000(0.0001)1.0000(0.001)5.600; 15D]$. Since $H'(x)$ at $x=5.600$ is 27×10^{-15} and $H(x)$ is $1-2\times10^{-15}$, one page of values is given at the end of the table showing the change in the argument for each unit change in the fifteenth place of these tabulated values. Neither argument reaches 6.000.

Table II, v. 1, gives the values of $H'(x)$ and $1-H(x)$ for $x=[4.00(0.01)10.00; 8S]$. It is interesting to note that $H'(10)=4.1976562\times10^{-44}$ and $1-H(10)=2.0884876\times10^{-45}$.

The last page of v. 1 gives the constants π, $1/\pi$, $\sqrt{\pi}$, $1/\sqrt{\pi}$, $\sqrt{2\pi}$, $1/\sqrt{2\pi}$, $2/\sqrt{\pi}$, $\log \pi$, $\log\sqrt{2\pi}$, e, $\log e$, and $\log 2$, to 16D.

Table I, v. 2, gives the values of $Q(x)$ and $P(x)$ for $x=[0.0000(0.0001)1.0000(0.001)7.800; 15D]$. Since $Q(x)$ at $x=7.800$ is 25×10^{-15} and $1-P(x)$ is 6×10^{-15}, one page of values is given showing the argument which corresponds to one unit change in the fifteenth place in these tabulated values. $Q(x)$ is zero to 15D for $x=8.285$ and $P(x)$ is unity to 15D for $x=8.112$.

Table II, v. 2, provides values of $Q(x)$ and $1-P(x)$ for $x=[6.00(0.01)10.00; 7S]$. One may observe that the first significant figure for both $Q(x)$ and $P(x)$, when $x=10.00$, is in the 23rd place. These values are far beyond conceivable use in statistics since the realistic range for data seldom exceeds three standard deviations, and these tables extend the range to 10 standard deviations. However, one can never tell to what other uses such fundamental values may be adapted.

The method of computing the tables in both volumes was essentially the same and made use of the well known properties of the Hermite polynomials. As with the probability functions themselves, notations differ for the Hermite polynomials. It will be observed that the nth derivative of e^{-kx^2} is the product of e^{-kx^2} by a polynomial of nth degree. These polynomial multipliers are called Hermite polynomials after the French mathematician, Charles Hermite (1822–1901) who first studied them. Two forms are to be observed, those corresponding to $k=\frac{1}{2}$ and those corresponding to $k=1$. The writer prefers the notation:

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-\frac{1}{2}x}, \qquad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx} e^{-x^2}.$$

One may observe that the two forms are connected by the relationship

$$h_n(x) = 2^{-\frac{1}{2}n} H_n\ (x/\sqrt{2}).$$

It may also be proved that they satisfy the following recurrence relationships:

$$h_{n+2}(x) - xh_{n+1}\ (x) + (n+1)h_n(x) = 0, \qquad H_{n+2}\ (x) - 2xH_{n+1}(x) + 2(n+1)H_n\ (x) = 0.$$

Returning to the computational problem, let us first write Taylor's series in the form

$$f(x \pm ph) = f(x) \pm \frac{ph}{1!} f'\ (x) + \frac{(ph)^2}{2!} f''(x) + \cdots,$$

where h is the tabular interval. Then if p is set equal to 1 and $f(x)=H(x)$, we obtain the expansion

$$H\ (x \pm h) = h(x) \pm hH'(x) + \frac{h^2}{2!} H''\ (x) + \cdots \pm \frac{h^n}{n!} H^{(n)}(x) \pm \cdots.$$

To begin with $H'(x)$ was computed for the 60 "key arguments" $x=[0.0(0.1)6.0; 25D]$. By means of the properties of the Hermite polynomials, derivatives of higher order were next computed for the same arguments so that 25-decimal accuracy would be obtained in $10^{-k}H^{(k)}(x)/k!$

Beginning then with the expansion for $H(x\pm h)$ and noting that $H(0)=0$, it was easy to evaluate $H(0.1)$. From this new value $H(0.2)$ and $H(0.0)$ were computed, and then in succession $H(0.3)$ and $H(0.1)$, $H(0.4)$ and $H(0.3)$, etc. The second computation of each previously computed value was used as a check on the computations of the derivatives employed. This process was continued until $H(6.0)$ was obtained, which was then checked with its direct evaluation from the asymptotic expansion of $H(x)$. With these key values of $H(x)$ and $H'(x)$ subtabulation was then employed to complete the table, Taylor's series being used in the computation.

Because of effective tests applied to check the entries, including differencing and double proofreading, the authors believe that "this table is entirely free from error."

It seems unfortunate that neither of the major tables contain differences, particularly since fourth differences are negligible and δ^2 would have been sufficient. However, when derivatives of the second and higher orders are neglected, the following easily applied formulas are available for interpolation:

$$H(x_0 + ph) = H(x_0) + phH'(x_0), \qquad H'(x_0 + ph) = H'(x_0)[1 - 2x_0ph].$$

Six-place accuracy or better is obtained by these formulas. Nevertheless, the publication of differences would have added greatly to the value of the tables and the omission is very much to be regretted in a work of such importance.

H. T. D.

92[K, L].—A. N. LOWAN, N. DAVIDS, A. LEVINSON, "Table of the zeros of the Legendre Polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula," Amer. Math. So., *Bull.*, v. 48, 1942, p. 739–743. 15.4×24.2 cm. The tables were computed by the Project for Computation of Mathematical Tables, conducted by the W.P.A., New York.

Legendre's polynomial of the nth order, or zonal surface harmonic of the first kind, may be defined by

$$z = P_n(x) = \frac{1}{2^n n!} \frac{d^n(x^2 - 1)^n}{dx^n},$$

which is a particular solution of Legendre's equation

$$(1 - x^2) \frac{d^2z}{dx^2} - 2x \frac{dz}{dx} + n(n + 1)z = 0.$$

Zonal harmonics P_n were first introduced in 1784 by Adrian Marie Legendre (1752–1833), in a paper published in *Mémoires des Savants Étrangers*, v. 10, 1785, and applied to the determination of the attractions of solids of revolution. $P_n(x) = 0$ has n distinct roots between -1 and $+1$, arranged symmetrically about $x = 0$. While for small values of n the expression for P_n is comparatively simple, when n is as large as 16 we get

$$P_n(x) = (1/2^{15})(300\ 540\ 195x^{16} - 1\ 163\ 381\ 400x^{14} + 1\ 825\ 305\ 300x^{12} - 1\ 487\ 285\ 800x^{10} + 669\ 278\ 610x^8 - 162\ 954\ 792x^6 + 19\ 399\ 380x^4 - 875\ 160x^2 + 6435).$$

Finding the zeros of even just this P_{16}, to 15D, is no small piece of computation. The Project calculated such zeros by successive approximations, combining synthetic division with the Newton-Raphson method. They were checked by using the relations between roots and coefficients.

Gauss's method of mechanical quadrature was set forth in his paper published in *Commentationes Societatis Regiae Scientiarum Gottingensis Recentiores*, v. 3, 1816; also, with certain misprints not in the original article, in *Werke*, v. 3, 1866, p. 165–196. If $x_1, x_2, \cdots, x_n$ are the zero values of $P_n(x)$, Gauss stated, in effect, the theorem

$$\int_{-1}^{1} f(x)dx = a_1f(x_1) + a_2f(x_2) + \cdots + a_nf(x_n)$$

where $a_i = \int_{-1}^{+1} \frac{P_n(x)dx}{P_n'(x_i)(x - x_i)}$, and $f(x)$ is a polynomial of degree not greater than $2n-1$ (C. G. J. Jacobi, 1826). If $f(x)$ is not a polynomial, but is with its derivatives continuous within the range, we have an approximation to the quadrature which becomes closer as n increases.

The table before us contains the non-negative zeros and corresponding a's, $n = 2, \cdots, 16$, to 15D. Let us see to what extent the results here are new. In that fine work of B. P. Moors, *Valeur Approximative d'une Intégralé Définie*, Paris, 1905, viii, 195 p. +11 folding plates, the non-negative zeros and a's are given for $n = 2(1)10$; 16D. These values are to one more decimal place than the one under review; otherwise the tables are in agreement, except that for $n = 9$ the fifteenth digits of a_0 and a_1 differ by 3–4 units each. Moors is here in error. B. deF. Bayly in

Biometrika, v. 30, 1938, p. 193, gave the zeros and a's for $P_{12}(x)$ to 13D. With his correct results we found two errors in the zeros of the table under review, as well as last figure errors of Bayly in six of the a's; Mr. Lowan pointed out another error in $P_{11}(x)$. See MTE **6**. Thus there were no dependable results beyond the range of Moors, and Bayly, when the new table added the range $n=11$, 13(1)16; 15D.

Gauss's results for the interval (0, 1) are reproduced in somewhat modified form for the interval $(-1, +1)$, $n=[2(1)7$; 16D], in Heine, *Kugelfunktionen*, v. 2, Berlin, 1881, p. 15–16 (the last three figures here in each of the zeros for $n=4$ are erroneous); these were copied from Heine, with the errors, in E. W. Hobson, *Spherical and Ellipsoidal Harmonics*, Cambridge, Univ. Press, 1931, p. 80–81. In H. J. Tallquist, *Grunderna af Teorin för Sferiska Funktioner, jamte Användningar inom Fysiken*, Helsingfors, 1905, p. 400 are given the zeros for $n=2(1)8$; 7D (in $n=8$ *for* 0.9602898, *read* 0.9602899). For the interval $-1/2$ to $+1/2$ E. J. Nyström calculated the zeros and a's $n=2(1)10$; 7D, *Acta Math.*, v. 54, 1930, p. 191. Nyström notes (p. 190) an error in Gauss, for $n=2$, where in place of $a''=1,887\cdots$, should be $a''=0,887\cdots$.

For a discussion and comparison of different methods of mechanical quadrature see especially the work of Moors, with an excellent historical survey, referred to above; but *Tract for Computers*, no. X, contains a very brief treatment by J. O. Irwin, *On Quadrature and Cubature or On Methods of Determining Approximately Single and Double Integrals*, Cambridge, 1923.

R. C. A.

I have already had occasion to use this table to good effect. Readers who are unacquainted with the literature of the subject and who may wish to use the table will, however, be confused by the fact that the description of the table does not agree in its notation with that of the table itself, and may be misled by the fact that the formula given for the remainder is seriously incorrect. This note is an attempt to make this valuable table just a little more useful.

The subscript i used above runs, in the table, from 1 to n when n is even, and from 0 to $n-1$ when n is odd. Since the roots x_i are symmetrically situated with respect to the origin, the table gives only the non-negative roots

$$x_1, x_2, x_3, \cdots, x_{n/2} \quad \text{if } n \text{ is even,}$$
$$x_0, x_1, x_2, \cdots, x_{(n-1)/2} \quad \text{if } n \text{ is odd,}$$

arranged in order of increasing magnitude. Opposite each such x is found the corresponding a. The other x's and a's are given by

$$x_{n-i} = -x_i, \qquad a_{n-i} = a_i, \qquad i = 1, 2, \cdots, n-1,\text{—if } n \text{ is odd;}$$
$$x_{n-i+1} = -x_i, \qquad a_{n-i+1} = a_i, \qquad i = 1\ \ 2, 3, \cdots, n,\text{—if } n \text{ is even.}$$

With these notations Gauss' formula becomes

$$\int_p^q f(x)dx = \tfrac{1}{2}(q-p)\sum_{i=\epsilon}^{n+\epsilon-1} a_i F(x_i) + R_n(f)$$

where $\epsilon=0$ or 1 according as n is odd or even and where

$$F(u) = f\left(\frac{p+q}{2} + \frac{q-p}{2}\right)u.$$

As for the remainder term $R_n(f)$, the authors give

$$R_n(f) = \frac{f^{(2n)}(\xi)}{(2n+1)!}(q-p) \qquad (p \leqq \xi \leqq q) \tag{1}$$

whereas the correct formula, due to Markov and Mansion,[1] is

$$R_n(f) = \frac{(n!)^4 f^{(2n)}(\xi)}{[(2n)!]^2(2n+1)!}(q-p)^{2n+1}. \tag{2}$$

It is clear that when the range of integration is large (1) will be much too small. Since

$$\frac{2(n!)^4 4^{2n}}{\pi(2n)!(2n+1)!} = \frac{\int_0^{\pi/2} \sin^{2n+1}\theta d\theta}{\int_0^{\pi/2} \sin^{2n}\theta d\theta} < 1$$

a somewhat simpler form of (2) may be given as follows

$$(3) \qquad |R_n(f)| < \frac{2\pi|f^{(2n)}(\xi)|}{(2n)!}\left(\frac{q-p}{4}\right)^{2n+1} \sim \frac{\sqrt{8\pi n}}{e}\left\{\frac{(q-p)e}{8n}\right\}^{2n+1}.$$

Since we are concerned, after all, with very moderate values of n, it is perhaps more to the point to write (2) in the form

$$R_n(f) = \frac{(q-p)^{2n+1}f^{(2n)}(\xi)}{(2n)!U_n}$$

and to tabulate[2] the integers U_n as follows

n	U_n	n	U_n	n	U_n
2	180	7	176679360	12	182811491808400
3	2800	8	2815827300	13	2920656969720000
4	44100	9	44914183600	14	46670906271240000
5	698544	10	716830370256	15	745904795339462400
6	11099088	11	11445589052352	16	11922821963004219300

A reference should have been made to an important note by Uspensky[3] on the subject of $R_n(f)$. He shows that $R_n(f)$ possesses an asymptotic development in every way comparable with the celebrated Euler-Maclaurin formula which latter may be thought of as giving the asymptotic development of the remainder of the trapezoidal rule.

[1] A. A. Markov, *O Nekotorykh Prilozheniiakh Algebraicheskikh Nepreryvnykh Drobeĭ.* [On some Applications of Algebraic Continued Fractions], Doctoral diss., St. Petersburg, 1884, p. 68; A. A. Markov, "Sur la méthode de Gauss pour le calcul approche des intégrales," *Math. Annalen*, v. 25, 1885, p. 429; and P. Mansion, "Détermination du reste dans la formule de quadrature de Gauss," Acad. Royale d. Sci. d. Lettres. et d. Beaux-Arts de Belgique, *Bulletins*, s. 3, v. 11, 1886, p. 303. Also in A. A. Markov, *Differenzenrechnung*, Leipzig, 1896, p. 68; Gauss's numerical results are given on p. 70.

[2] These values up to U_7 were given by Gauss, *Werke*, v. 3, p. 193–195.

[3] "On an expansion of the remainder in the Gaussian quadrature formula," Amer. Math. So. *Bull.*, v. 40, 1934, p. 871–876.

D. H. L.

93[I].—(i) A. N. Lowan, H. E. Salzer, A. Hillman, "A table of coefficients for numerical differentiation," Amer. Math. So., *Bull.*, v. 48, Dec., 1942, p. 920–924.

(ii) W. G. Bickley, J. C. P. Miller, "Numerical differentiation near the limits of a difference table," *Phil. Mag.*, s. 7, v. 33, Jan., 1942, p. 1–14 +4 folding plates.

Numerical differentiation presents two problems depending on whether (a) the given values of the function are known to a high degree of precision, as would be the case, for example, if one wishes to find the second derivative of the gamma function from a six place table of that function, or (b) the values are determined by experiment and are subject to considerable uncertainty.

Well known solutions of problem (a) which date back to Newton depend on interpolation to the given values by means of a polynomial of arbitrary degree n. The final result is an expression for $f^{(m)}(x)$ as a linear combination of successive differences of the function values:

$$\omega^m f^{(m)}(x) = \sum_{q=m}^{n} A_{m,q}\Delta^q f,$$

where ω is the tabular interval, $\Delta^q f$ is the q-th difference found from tabular values at the equally spaced points $x_1^{(q)}, x_2^{(q)}, \cdots, x_{q+1}^{(q)}$, and the coefficients $A_{m,q}$ depend only on the position of x relative to $x_1^{(q)}, x_2^{(q)}, \cdots, x_{q+1}^{(q)}$. Various particular methods are obtained by varying this relative position. In case these points are symmetrical about x, one gets a formula in terms of "central differences." Central difference formulas are usually preferred whenever they can be used, but they are inapplicable to points at or near the ends of the table. In such a case one ordinarily uses the formula in terms of "forward differences," where $x_1^{(q)} = x$. Paper (i) is concerned with the calculation of the coefficients $A_{m,q}$ for this case, and gives a table of these coefficients for the first 20 derivatives up to those of the 20th difference; the coefficients of the first 12 of these had been already given in one of the tables of paper (ii).

Paper (ii) raises an objection to the use of advancing differences when x is not at the end of the table but is too near for central differences, on the grounds that forward differences do not use all the available tabular values nearest x. To meet this objection the authors have developed new general formulas in terms of "mixed differences," in which x can have any tabular position from $x_1^{(q)}$ to $x_{q+1}^{(q)}$. These formulas include central and backward and forward differences as special cases. In applications one uses central differences for derivatives of orders up to the first for which $x_1^{(q)}$ (or $x_{q+1}^{(q)}$) reaches the limit of the table; for higher derivatives the mixed differences for which $x_1^{(q)}$ (or $x_{q+1}^{(q)}$) is at the end of the table. Analogous formulas are also obtained for x midway between two tabular points; these enable one to subdivide the given tabular interval. The authors have computed the coefficients $A_{m,q}$ of these mixed difference formulas for differences up to those of the 12th order for the first 4 derivatives when x itself is a tabular point; also for the first three derivatives when x is midway between two tabular points. In all these cases the coefficients are rational numbers and their values are given exactly. The tables displaying these coefficients have an unusually convenient arrangement. The central differences from a horizontal line along the middle of the page with the mixed differences arranged on either side to form a triangular array. In a particular problem the values to be taken from the table will occur first along this horizontal line and then along the diagonal. Arrows are placed in the table to help guide the eye along the proper diagonal. Illustrative examples in the use of the tables are also included.

In problem (b), the methods given in these papers are unsuitable, in general. In fact no method based on the theory of interpolation, in which an approximating curve is passed through the given values, is suitable; for, to force the approximating curve to pass exactly through the given values is not desirable and will usually introduce wide oscillations in the derivatives. Some other method of approximation having the effect of smoothing out of the given data must be used in this case.

Among references to topics in paper (i) are the following: L. M. Milne-Thomson, *The Calculus of Finite Differences*, London, Macmillan, 1933, Chap. 7, p. 157–159; H. T. Davis, *Table of the Higher Mathematical Functions*, Bloomington, Ind., v. 1, 1933, p. 73–77; E. T. Whittaker and G. Robinson, *The Calculus of Observations, A Treatise on Numerical Mathematics*, 3d ed. London, Blackie, 1940, p. 62–65. The references in paper (ii) are to K. N. Bradfield and R. V. Southwell, "Relaxation methods applied to engineering problems. I—the deflexion of beams under transverse loading," R. So. London, *Proc.*, v. 161A, 1937, p. 155–181; L. J. Comrie, *Interpolation and Allied Tables*, London, H. M. Stationery Office, 1936. (Reprinted from the *Nautical Almanac for 1937*.), D. C. Fraser, "On the graphic delineation of interpolation formulae," Inst. Actuaries, *Jn.*, v. 43, 1909, p. 235–241; J. F. Steffensen, *Interpolation*, Baltimore, Williams & Wilkins, 1927.

P. W. KETCHUM

94[A, D, E].—H. S. UHLER, "A new table of reciprocals of factorials and some derived numbers," Connecticut Acad. Arts and Sci., *Trans.*, v. 32, 1937, p. 381–434. 16.2×24.4 cm. Compare RMT 86.

The main table may be regarded as the superposition of two tables one of which is limited (for $n > 14$) to 475D, so that it terminates with $1/214!$, and the other is defined by 70S, and includes all values of $1/n!$ from $n=1$ to $n=369$. The upper limit was arbitrarily chosen so that the table would be adequate for the evaluation of e^{100} to about 100S. D. H. Lehmer gave the value of e to 707D (*Amer. Jn. Math.*, v. 48, 1926, p. 139–143), in order to match the 707-place value of π found by Shanks before 1874. H. S. Uhler reprints Lehmer's value and shows that it is in agreement with his own to 478D.

Other results given are e^{-1} (to 477D), e^2 (to 257D), e^4 (to 256D), e^6 (255D), e^8 (255D), e^{10} (253D), e^{-10} (258D), e^{100} (117D), e^{-100} (40S), sin 1 (477D), cos 1 (477D), sin 10 (212D), cos 10 (212D), cos 20 (212D), sin 100 (72D), cos 100 (72D), sin 200 (72D), cos 200 (72D).

The elaborate tests applied for checking the accuracy of the calculations are fully described, and tend to inspire unlimited confidence in the results.

The work of H. S. Uhler has checked the accuracy of the following earlier substantial results:

C. A. Bretschneider, *Archiv d. Math. u. Phys.*, v. 3, 1843, p. 27–34, e, e^{-1}, sin 1, cos 1, sinh 1, cosh 1, each to 105D.

J. W. L. Glaisher, Cambridge Phil. So., *Trans.*, v. 13, 1883, p. 247. $1/n!$, $n=13(1)50$, to 28D. For $n=20$, 27, 41 and 50 "Glaisher's numbers end with 9, 7, 5, and 6 instead of 8.436, 6.974, 4.449, and 5.468 respectively."

C. E. Van Orstrand, Nat. Acad. Sci., *Memoirs*, v. 14, no. 5, 1921. $1/n!$, $n=1(1)74$, to 108D, p. 12–13; e, e^2, e^4, e^6, e^8, e^{10}, each to 42D, p. 16–17; e^{-10} to 52D, p. 27; e^{-100} to 19D, p. 28; sin 10, cos 10, cos 20, cos 100, sin 100 each to 23D, p. 47–48.

J. T. Peters and Johann Stein (1871–), *Anhang mathematischer Tafeln* in *Zehnstellige Logarithmentafel*, v. 1, Berlin, 1922. e^{-1} to 72D, p. 12; e^2, e^4, e^6, e^8, and e^{10} each to 32S, p. 12; sin 1, cos 1, each to 52D, p. 60.

R. C. A.

95[A, C].—H. S. Uhler, "Log π and other basic constants," Nat. Acad. Sci., *Proc.*, v. 24, 1938, p. 23–30. 17.5×25.8 cm.

The Table here includes the following results all most carefully checked: (1/2) log 2π and log π, each to 214D; ln π to 213D; log 2, log 3, log 5, log 7, log 17, each to 230D; ln 17 to 224D; ln 71 to 213D; log 71 to 110D; ln 113 to 213D; log 113 to 110D; π^{-1} to 253D; π^2 to 261D. Errors in Parkhurst's tables are noted. The value of π^{-1} was calculated by J. W. Wrench, Jr.; the most extensive earlier correct value was by G. Paucker, to 137D, *Archiv Math. Phys.*, v. 1, 1841, p. 1. π^2 was given correctly to 217D by S. Z. Serebrennikov, Akademiîa Nauk, *Classe physico-mathématique, Mémoires*, s. 8, v. 19, 1906, p. 4.

James Stirling and Abraham De Moivre by different methods arrived at the following remarkable and very useful approximation[1] when x is large: $x! \approx (2\pi)^{\frac{1}{2}}(x)^{\frac{1}{2}}(e)^{-x}x^x$. De Moivre gave, in effect, the expansion

$$\ln x! = (1/2)\ln 2\pi + (x+1/2)\ln x - x + \frac{B_1}{1\cdot 2}\cdot\frac{1}{x} - \frac{B_2}{3\cdot 4}\cdot\frac{1}{x^3} + \frac{B_3}{5\cdot 6}\cdot\frac{1}{x^5} - \cdots + R$$

where B_1, B_2, B_3, $\cdots$ denote the Bernoulli numbers. Log $x!$ for $x=1(1)3000$, to 33D may be read off from F. J. Duarte, *Nouvelles Tables de log x!*, Geneva and Paris, 1927; logarithms to 61D for all numbers to 100 and of primes from 100 to 1100 may be found in Sharp's table (1717), and logarithms to the base e and to 48D, in Wolfram's table (1778); the Bernoulli numbers B_i up to $i=110$ are known; also log π to 61D (Sharp, 1717); and ln π to 48D in J. T. Peters and J. Stein, *Anhang mathematischer Tafeln* in *Zehnstellige Logarithmentafel*, v. 1, Berlin, 1922, p. 1. Hence computations in this connection to more than about 60D, for $x>1100$, call for further basic values. For this purpose H. S. Uhler has now provided (1/2) log 2π, log π, ln π, and log 2, while J. C. Adams gave ln 2, all to over 200D (1878 and 1887).

The formula underlying the calculation of ln π was based on an approximation for π due to Ramanujan (*Quart. Jl. Math.*, v. 45, 1914, p. 366; and *Collected Papers of Srinivasa Ramanujan*, Cambridge, 1927, p. 35), which, with a correction factor f making the formula exact, is as follows: $\pi = (355/113)(1-.0003/3533)f$, a rough value for $1-f$ being 347×10^{-18}. Hence the actual computation of ln π reduced to that of ln of 71,113, $1-.0003/3533$ and f. It was shown that the computation of ln 71 and ln 113 could be made to depend wholly on ln 2, ln 3, ln 5, ln 7, already calculated, and on certain rapidly converging series.

R. C. A.

[1] J. Stirling, *Methodus Differentialis*, London, 1730, p. 137; second ed., 1764, p. 137; English edition by F. Holliday, 1749, p. 121. A. de Moivre, *Approximatio ad Summam Terminorum Binomii* $(a+b)^n$ *in Seriem expansi*, London, 1733; rev. transl. in A. de Moivre, *Doctrine of Chances*, London, second ed., 1738, p. 235–242; third ed., 1756, p. 243–250; for a facsimile of the 1733 publication see R. C. Archibald, "A rare pamphlet of Moivre and some of his discoveries," *Isis*, v. 8, 1926, p. 677–683. See also C. Tweedie, *James Stirling* . . . , Oxford, 1922, p. 119, 203–205.

96[A, C, D, E].—H. S. Uhler, "Recalculation of the modulus and of the logarithms of 2, 3, 5, 7 and 17," Nat. Acad. Sci., *Proc.*, v. 26, 1940, p. 205–212. 17.5×25.8 cm.

In the calculation of the table in RMT **95** the series

$$\ln \frac{p}{q} = 2\left\{\frac{p-q}{p+q} + \frac{1}{3}\left(\frac{p-q}{p+q}\right)^3 + \frac{1}{5}\left(\frac{p-q}{p+q}\right)^5 + \cdots\right\} \equiv 2S[(p-q)/(p+q)],$$

with $p-q=1$, played an important role. With $p=5041=71^2$, $5040=2^4\cdot 3^2\cdot 5\cdot 7$

$$\ln 71 = 2\ln 2 + \ln 3 + (\ln 5 + \ln 7)/2 \quad S(1/10081).$$

Similarly for $p=226$, ln 113 involves $S(1/451)$. Thus in the present paper, we have $S(1/5)$, $S(1/239)$, $S(1/2449)$, $S(1/4999)$, and $S(1/8749)$, in connection with ln 2, ln 3, ln 5, ln 7 and ln 17. J. C. P. Adams calculated the first four of these to 262D (1878 and 1887); see MTE **8**. These are here extended, with certainty on the author's part, to 328D. The values are also given of the following: arctan (1/451) to 215D; arctan (1/577) to 335D; arctan (1/2449), arctan (1/4999), and arctan (1/8749) each to 330D; and arctan (1/10081) to 216D.

Adams found M correct to 271D (1887). From his own ln 2 and ln 5 Uhler determined M, correct to 328D.

Five other values found in RMT **94** are here extended, viz: e^{10} to 289D; e^{-10} to 293D; and sin 10, cos 10, cos 20, each to 284D. These latter ranges are also supplementary to results in RMT **81**.

R. C. A.

97[A, K].—H. S. Uhler, "The coefficients of Stirling's series for log $\Gamma(x)$," Nat. Acad. Sci., *Proc.*, v. 28, 1942, p. 59–62. 17.5×25.8 cm.

When n is a positive integer, the asymptotic series of RMT **95** becomes

$$\ln\Gamma(x) = (1/2)\ln 2\pi + (x+1/2)\ln x - x + \sum_{m=1}^{\infty}(c_m/x^{2m-1}) + R,$$

where $c_m \equiv (-1)^{m-1}B_m/[(2m-1)(2m)]$. The table of the paper contains the first 71 values of c_m, many of which have recurring periods within the range of the table; c_{15} is given to 103S. Values of 100! to 158S, and of ln (100!) to 156S, are also given.

MATHEMATICAL TABLES—ERRATA

In this issue we have referred to Errata in RMT **89** (Blakesley, Forti, Hayashi, Sakamoto), RMT **92** (Lowan *et al.*, Moors, Bayly, Gauss, Heine, Hobson, Tallquist), RMT **94** (Glaisher), RMT **95** (Parkhurst, Serebrennikov), UMT **2** (Airey), N **4** (Gifford, C. G. Survey), N **5** (C. G. Survey), N **6** (Gifford), and in the first article of this issue (Callet, Brandicourt and Roussilhe, Jordan, Service Géog. 1914).

5. U. S. Coast and Geodetic Survey, Special Publication, no. 231, *Natural Sines and Cosines to Eight Decimal Places*, 1942; see RMT **77.**

End-figures are missing cos 1°44′41″ and 42″, namely: 0 and 5 respectively.

L. J. C.

Sin 36° *for* 0.587 78255, *read* 0.587 78525.

F. W. Hoffman, 689 East Ave., Pawtucket, R. I.

6. A. N. Lowan, N. Davids, A. Levenson, "Table of the zeros of the Legendre polynomials," 1942; see RMT **92.**

$n=11$, *for* $x_2=0.519096129110681$ *read* $x_2=0.519096129206812$
$n=12$, *for* $x_1=0.125333408511469$ *read* $x_1=0.125233408511469$
$n=12$, *for* $x_2=0.367831498918180$ *read* $x_2=0.367831498998180$

A. N. Lowan, and R. C. A.