Arg.	Unmodified Table		Modified Table	
x	$1/x$	$-\Delta'$	f	$-D$
371	2695423	7246	2695423	7246
...	....	....	....	....
...	....	....	....	....
597	1675047	2801	1675047	2801
598	1672246	2792	1672246	2792
599	1669454	2782	1669454	2782
600	1666672	5537	1666672	2769
602	1661135	5501	1666637	2751
604	1655634	5464	1666562	2732
606	1650170	5428	1666454	2714
608	1644742	5393	1666318	2697
610	1639349	5357	1639349	2679

PAUL HERGET and G. M. CLEMENCE

U. S. Naval Observatory,
Washington, D. C.

[1] Compare W. J. ECKERT, *Punched Card Methods in Scientific Computation*, New York, The Thomas J. Watson Astronomical Computing Bureau, Columbia University, 1940.

RECENT MATHEMATICAL TABLES

138[A].—MATHEMATICAL TABLES PROJECT, New York, A. N. LOWAN, technical director, *Table of Reciprocals of the Integers from* 100,000 *through* 200,009. New York, Columbia University Press, 1943, viii, 201 p. 19.6 × 26.5 cm. Reproduced by a photo offset process. $4.00.

This work is designed to provide a 7-place table of reciprocals between 100,000 and 200,000, which will "expand by tenfold the scope of the existing tables in this interval."

The table is presented in what James Henderson has called the "modern arrangement" where the first five figures of the argument proceed in natural order down the page and the final figures from 0 to 9 are given along the top. Only the significant figures in the reciprocals are recorded. Since the entries are close together the differences vary from 100 at the beginning of the table to 25 at the end. Tables of proportional parts are given at the bottom of each page.

According to the preface we learn that: "Preparation of manuscript tables was begun in December, 1934, by Dr. C. C. Kiess of the National Bureau of Standards. The work was completed under his direction in May, 1939, with the co-operation, at various times, of Messrs. B. F. Scribner, H. R. Mullin, W. G. Esmond, and J. Waldron."

It may be well to add a short account of the development of tables of reciprocals, which have been in the making for more than a century. The first adequate table appears to have been that of PETER BARLOW, published in the first edition of his *Tables* in 1814 [*MTAC*, p. 17], where the reciprocals are given for the first 10,000 integers. A. De Morgan, in the edition of 1840, says: "I cannot ascertain that any tables of square roots, cube roots or reciprocals comparable in extent to those of Mr. Barlow were ever printed before his." In the fourth edition, issued in 1941 by L. J. Comrie, the reciprocals were extended from 10,000 to 12,500. Comrie found "60 errors of a unit in the last decimal, but none greater" in the 1840 edition.

The next significant contribution was R. PICARTE'S, *La division réduite à une Addition*, Paris, 1861, which provided values of the reciprocals from 1000 to 10,000 to 10 significant figures, together with the first nine multiples of them.

W. H. OAKES published his 7-place *Table of the Reciprocals of Numbers from 1 to 100,000*, in London in 1865. The volume under review is a continuation of this work, which was similarly printed in the "modern arrangement," with differences at the side of the page. The more recent table of M. B. COTSWORTH, erroneously entitled "Reciprocals for All Numbers from 1 to 10,000,000," [1] is also a 7-figure table over the same range as that of Oakes.

J. C. HOUZEAU in Acad. r. des Sciences . . . de Belgique, *Bull.*, v. 40, 1875, p. 107, published a 20-place table of reciprocals of the first 100 integers and their first 9 multiples to 12 places. Apparently ignorant of this work, KARL PEARSON, in v. 2 of his *Tables for Statisticians and Biometricians*, London, 1931, p. 257 gave 20-decimal values of the reciprocals to 100. Ten-place reciprocals of the first 1,000 numbers are also given by E. GÉLIN in his *Recueil de tables numériques*, Huy, 1895.

Mention has already been made in this journal (p. 164) of the 17-figure tables for the reciprocals of the first 1,000 integers which are in the possession of the reviewer. He has recently acquired a 9-figure table of the reciprocals of numbers from 1 to 10,000, the work of CHARLES MILLS of Northwestern University.

H. T. D.

[1] A misstatement in the book under review. A more correct title of the work in question is as follows: *Cotsworth's Reciprocals for All Numbers From* 1 *to* 10,000,000 *with Direct Index Reference to Seven Significant Figures, complete for Numbers under* 100,000 *beyond which the extension to* 10,000,000 *is rendered easy by the Additive Differences printed at the foot of each Table*, York, England, M. B. Cotsworth, 1902.—EDITORS.

139[C].—H. S. UHLER, "Natural logarithms of small prime numbers," Nat. Acad. Sci., *Proc.*, v. 29, 1943, p. 319–325. 17.1 × 25.6 cm.

The results here given are supplementary to those in Mr. Uhler's publications already reviewed in *MTAC*, p. 55, 56, 20 (RMT **95**, **96**, **86**). It was thus already noted that J. C. ADAMS (1887) calculated each of the quantities ln2, ln3, ln5, and ln7, to 272D; and that Mr. Uhler extended each of these to 328D and found ln17 to 331D, and ln71 to 213D. In the present paper the following new results are presented: ln11 (329D), ln13 (290D), ln19 (299D), ln23 (295D), ln29 (300D), ln31 (292D), ln37 (291D), ln101 (329D), and ln9901 (303D). Furthermore, each of the following is given to 155D and is supplementary to T. 4 of the monograph on 137-place values of $\ln(1 \pm n \cdot 10^{-p})$, RMT **86**: ln41, ln43, ln59, ln61, ln67, ln73, ln79, ln83, and ln89. Thus for every N, prime and < 100, ln N is given to at least 155D. The descriptions of checks applied tend to inspire confidence in the accuracy of the results. The author wished us to add, however, that in ln43, line 3, 12917, *should read* 12971; a slip due to failure of the printer to make this correction, indicated in proof; there is no error in the reprints.

140[C, L].—WALTER MEISSNER, *Tafel der* ln Γ-*Function mit komplexem Argumentbereich*, Dresden Diss., *Deutsche Mathematik*, v. 4, 1939, p. 537–555. 20.8 × 28.9 cm.

For the 192 complex arguments

$$z = (2m + i\,2n\sqrt{3})/24, \quad \text{and} \quad z = [(2m+1) + i(2n+1)\sqrt{3}]/24,$$

with $m = 6, 7, \cdots, 16, 17$; $n = 0, 1, \cdots, 6, 7$ (triangular net), and for the gamma function

$$\Gamma(z) = e^{u+iv} = 10^{\lambda} \cdot e^{iv},$$

u and v are given to 7D (except for 24 entries to 15D); $\lambda = u \log e$ is given to 7D, and $\phi = v180/\pi$ is expressed in degrees, minutes, and whole seconds. (See F. EMDE, *Z. angew. Math. Mech.*, v. 20, 1940, p. 295). The tables were intended as a supplement to the little tables of $\Gamma(z)$ and $\ln \Gamma(z)$ (p. 111, 109), z real, in the chapter on gamma functions in P. E. BÖHMER, *Differenzengleichungen und bestimmte Integrale*, Leipzig, Köhler, 1939, where the complex case is discussed.

R. C. A.

141[D].—J. C. P. MILLER, *Tables for Converting Rectangular to Polar Coordinates*. London, Scientific Computing Service, 23 Bedford Square, London W. C. 1, 1939, 16 p. 15.3 × 25 cm. Two shillings. Authorized American reprint, Dover Publications, 31 East 27th St., New York 16 [1943]. Seventy-five cents.

The main object of these excellent tables is to facilitate the conversion of rectangular coordinates (x, y) to polar coordinates (r, θ) by means of the relations $r = (x^2 + y^2)^{\frac{1}{2}}$, $\theta = \tan^{-1}(y/x)$.

Denote by l the larger and by s the smaller of $|x|$ and $|y|$, and evaluate $k = s/l$; this is the argument of the tables and evidently lies in the range 0 to 1. Then we have $r = (l^2 + s^2)^{\frac{1}{2}} = l(1 + k^2)^{\frac{1}{2}}$; and $\theta_0 = \tan^{-1} k = \tan^{-1}(s/l)$, $\phi_0 = \cot^{-1} k = \cot^{-1}(s/l) = 90° - \theta°$. Then $\theta = \tan^{-1}(y/x)$ may be determined in terms of θ_0 or ϕ_0 if the result is required in degrees or in terms of θ_0 alone if radians are used, with the help of an "octant" scheme, in which the signs and relative size of x and y constitute the argument.

The small interval of tabulation, namely 0.001, leads to the advantage that the difference between two successive values of any function never exceeds 10; interpolation may therefore be performed mentally. For cases where r is required to three significant figures only, and θ to 0°.1 only, interpolation is unnecessary.

The text discusses "Method with a calculating machine," "Slide rule method" and "Applications of the tables." Under the last heading it is noted that the tables may be used for transformation of harmonic constants a and b, obtained by harmonic analysis, to amplitude c and phase angle ϵ, in accordance with the relation

$$c \sin(nt + \epsilon) = a \sin nt + b \cos nt.$$

The use of the tables for the evaluation of impedance and phase lag or lead from resistance and reactance will immediately suggest itself to electrical engineers. The evaluation of the magnitude and direction of a vector from rectangular components and the conversion of complex numbers from the form $x + iy$ to the form $r = e^{i\theta}$ are further applications.

142[D, E].—MATHEMATICAL TABLES PROJECT, New York, A. N. LOWAN, technical director, *Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments.* New York, Columbia University Press, 1943, xxxviii, 410 p. 19.6 × 26.5 cm. Reproduced by a photo offset process. $5.00.

The main table of this interesting volume is a 400-page table of $\tan x$, $\cot x$, $\tanh x$, $\coth x$, for x = [0.0000(0.0001)2.000] radians. The number of decimal places varies from 5 to 13 for the different functions and the different argument ranges. The greater part of the table gives values to 8 significant figures. Near poles more significant figures are given as noted below. There are auxiliary tables as follows: Table II gives these functions for x = [0.0(0.1)10.0 radians; 10D], conversion tables: degrees, minutes, and seconds to radians and vice versa,[1] integral multiples of $\pi/2$ to 15D. Interpolation coefficients $p(1 - p)/2$ and $p(1 - p^2)/6$ for p = [0.000(0.001)1.000; 6D].

Except near $x = 0$ and $x = \pi/2$, the main table was constructed, one entry at a time, by actual division of the corresponding values of $\sin x$, $\cos x$, $\sinh x$, $\cosh x$, taken from this Project's table of circular and hyperbolic sines and cosines.[2] The values thus obtained were then verified by the reciprocal relations

$$\tan x \cot x = \tanh x \coth x = 1.$$

The resulting table was then differenced, to reveal errors in copying into the manuscript. This meticulous procedure, characteristic of this Project's fine large tables, was made possible by the large man- and machine-power available. Although the computer, to whom a good computing machine is available, can calculate isolated entries in the same way, or better still for the hyperbolic functions, by use of the formulas

$$\tanh x = (e^{2x} - 1)/(e^{2x} + 1), \qquad \coth x = (e^{2x} + 1)/(e^{2x} - 1),$$

where the value of e^{2x} can be taken from this Project's excellent table of the exponential function,[3] nevertheless the present table is more convenient (especially for the circular functions) when some interpolation is required.

The arrangement of the function values is a little confusing to the reader who merely thumbs the pages. The last three digits of each entry are separated from the earlier digits by a space. This makes for a variable number of digits between this space and the decimal, an irritating and perhaps hazardous circumstance. For $x = [0.0000(0.0001)0.1000]$, all four functions are given to 8 decimals. Since $\cot x$ and $\coth x$ are quite large here, these functions are given with great accuracy, and since they are decreasing rapidly, interpolation is difficult. For the sub-range $x = [0.0300(0.0001)0.1000]$ a column of second differences is provided ranging from 74075 to 2000 units of the 8th decimal place. For still smaller values of x direct interpolation is impractical. The rest of the main table gives the four functions to 8 significant figures except for page 316 ($x = [1.5700(0.0001)1.5750]$) where $\cot x$ is given to 13 decimal places. On p. viii of the foreword, by H. T. D., one reads "The figures seen on the printed page may appear dull reading to the uninitiated. . . ." For once, this is not true of p. 316. Here one sees the function $\tan x$ passing through its agonizing crisis at $x = \pi/2$, its onetime companion $\tanh x$ moving slowly ahead with apparent unconcern. Striking as these values of $\tan x$ are, they are of little practical value. To use any one of them, the value of x must be known with extreme certainty. Interpolation is not possible. Perhaps it would have been more practical to tabulate, near $\pi/2$, the function

$$\tan x - (\pi/2 - x)^{-1}$$

which is interpolable. Similarly near $x = 0$ the functions

$$\cot x - 1/x, \qquad \coth x - 1/x,$$

might have been tabulated. The policy of basing the table on 8 significant figures rather than a fixed number of decimals, is presumably to make uniform the relative error, rather than the actual error in the tabular entries. This expedient fails to produce the desired result in the case of $\tanh x$ and $\coth x$. Near the end of the table the $\tanh x$ values are, on the average, nearly ten times as accurate as those of $\coth x$, on the basis either of relative or actual error (since both functions are nearly unity). For the last half of the table, the relative error in $\tanh x$ averages about sixteen percent that of $\coth x$.

The introduction contains (p. xxiii–xxxviii) a bibliography of tables and charts of circular and hyperbolic tangents and cotangents and their logarithms, the inverse hyperbolic functions, the gudermannian and related functions, together with tables of roots of equations involving $\tan x$. No short bibliography could cover the very large number of titles of this description. In the present case only 73 important items are listed. These do not include any tables produced by New York Tables Project. Of these 73 tables only 17 refer to circular or hyperbolic tangents or cotangents given to more than 6 decimal places. Of these 17, only one table is for radian measure. This is the unreliable table of HAYASHI[4] in which $\tan x$ and $\tanh x$ are tabulated up to $x = 50$, the main table being for $x = [0.0010(0.0001)0.100(0.001)3.00(0.01)9.99]$ Thus the table under review fills a real gap in the bibliography of these elementary functions. It will, no doubt, be useful chiefly in problems in which a large number of values of one of the four functions are needed quickly. The fact that linear interpolation gives such good results over most of the table makes it quite useful to the engineer or physicist. It will help to wean away the typical astronomer and civil engineer from his sexagesimal table of log $\tan x$.

D. H. L.

[1] In the reviewer's opinion such conversion tables should be used only for pencil and paper work. With a machine available they are too slow and dangerous to bother with.

[2] *Tables of Circular and Hyperbolic Sines and Cosines*, New York, 1939.

[3] *Tables of the Exponential Function e^x*, New York, 1939.

[4] K. HAYASHI, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen und deren Produkte sowie der Gammafunktion*, Berlin, Springer, 1926.

NOTE.—R. C. A. requests that his statement concerning works of IVES and BENSON in reviews of *MTAC*, 1943, p. 8–10, be considered rather than the one quoted on p. xxx, l. 16–20, from *Scripta Mathematica*, v. 2, 1933, p. 91.

143[D, E, L].—HERBERT BRISTOL DWIGHT (1885–), *Mathematical Tables of Elementary and Some Higher Mathematical Functions including Trigonometric Functions of Decimals of Degrees and Logarithms*, New York and London, McGraw-Hill Book Co., 1941. Third impression (with additions), 1944, viii, 231 + 8 p. 15.2 × 22.8 cm. $2.50. Reproduced by a photo offset process.

A detailed statement of the contents of this volume will best exhibit its value. First differences for interpolation are given in connection with practically all of the tables. After the tables are many references to sources where more extended tables may be consulted. Tables I–V: sine, cosine, tangent, cotangent and their logarithms, and also secant and cosecant for every hundredth of a degree, to 5D, p. 1–101. T. VI–VIII: sine, cosine, for each $0^r.001$, 0 to 2, and tangent, 0 to 1.570, p. 102–113; T. IX–XI: $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $x = [0^r.000(0^r.001)1^r.000$; 4D] and $\tan^{-1} x$ for more extended values, p. 114–121; T. XII: $\ln x$, $x = [1.000(0.001)3.00(0.01)10$; 4D], p. 122–127; T. XIII–XIV, e^x, e^{-x}, $x = [0.000(0.001)5$; 4–6S], p. 128–147; T. XV–XVII: $\sinh x$, $\cosh x$, $\tanh x$, $x = [0.000(0.001)3$; 4–6S], p. 148–165; T. XVIII–XIX: $\sinh^{-1} x$, $\cosh^{-1} x$, $x = [0.000(0.001)2.00(0.01)10$; 4–5S], p. 166–175; T. XX: $\tanh^{-1} x$, $x = [0.000(0.001)1$; 4–5S], p. 176–178; T. XXI: binomial coefficients ${}_1C_r$ to ${}_{25}C_r$, p. 179; T. XXII: $(a^2 + b^2)^{\frac{1}{2}}/a$, $b/a = 0.000(0.001)1$, with a reference to Miller's *Tables* (see RMT **141**), p. 180–181; T. XXIII: factorials $n!$, $n = 1$ to 50, 1–6S, p. 182; T. XXIV: $\log n!$, $n = 1$ to 250, 5–7D, p. 182–183; T. XXV: Gregory-Newton interpolation coefficients, ${}_pC_r$, $r = 2(1)6$, $p = [0.01(0.01)1.00$; 2–7D], p. 184–185; Lagrangean interpolation coefficients for interpolating without columns of differences, p. 186–187; T. XXVI–XXVII: surface zonal harmonics $P_n(x)$, $P_n(\cos \theta)$, $n = 1(1)10$, $x = [0.00(0.01)1$; 1–6D], $\theta = 0°(1°)90°$; p. 188–195; T. XXVIII: first derivatives of the surface zonal harmonics $P_n(\cos \theta)$, $\theta = 0°(1°)90°$ from Farr's article, and therefore very erroneous (see MTE **33**), p. 196–197; T. XXIX: complete elliptic integral of the first kind, p. 199–203; T. XXX: complete elliptic integral of the second kind, p. 204–205; T. XXXI: Bernoulli's numbers (35), p. 206; T. XXXII: Euler's numbers (35), p. 207; T. XXXIII: $\Gamma(n)$, $n = [1.00(0.01)2$; 5D], p. 208–209; T. XXXIV: erf x, $x = [0.000(0.001)1.50(0.01)3$; 5D], p. 212–213; T. XXXV: ber x, bei x, ber′ x, bei′ x, p. 214–217; T. XXXVI: ber x + ibei x, ber′ x + ibei′ x, p. 218–221; T. XXXVII–XXXIX: Riemann zeta function $\zeta(s)$, $s = [-24.0(0-1)24$; 10–11S], also tables of $(s-1)\zeta(s)$ with δ^2, δ^4 and $\zeta'(s)/\zeta(s)$, p. 222–227; T. XXXX: $\log x$, $x = [1.00(0.01)9.99$; 4D], p. 228–229; Index, p. 231. The zeta function tables were furnished by H.T.D.

Such are the contents of the first edition. In the current edition Preface and Contents are printed from type rather than "offset"; literature notes have been modified on pages 195, 197 and 213; on p. 124 ln 2·269, for ·8193 has been substituted .8194; argument headings on p. 208–209 have been corrected; and 8 pages have been added. On the new p. 195A-B are the first derivatives of surface zonal harmonics $P'_n(x)$, $n = 2(1)8$, $x = [0.00(0.01)1.00$; 2–6S]; the erroneous p. 196–197 persist. Six new pages, 213A-F, are devoted to a table of the normal probability integral.

144[F].—HANSRAJ GUPTA, "On the class-numbers of binary quadratic forms," Tucuman, Argentina, Universidad, *Revista*, s. A, *Matem. y Fisica Teorica*, v. 3, 1942, p. 283–299. 17.8 × 27.1 cm.

An integral binary quadratic form is a homogeneous expression

$$f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 = [a, b, c] \tag{1}$$

with integral coefficients a, b, c. If the greatest common divisor (a, b, c) of a, b, and c is equal to unity, then f is said to be properly primitive. A binary quadratic form $F(y_1, y_2)$ obtained from $f(x_1, x_2)$ by a substitution $x_i = \alpha_{i1}y_1 + \alpha_{i2}y_2$ $(i = 1, 2)$ with integral coefficients and of determinant $\alpha_{11}\alpha_{22} - \alpha_{21}\alpha_{12} = 1$, is said to be properly equivalent to f. Both

f and F have the same discriminant $b^2 - 4ac = -d$. All properly equivalent forms are said to form a class.

The present paper contains a table of the number $h'(d)$ of the classes of properly primitive binary quadratic forms of negative discriminant $-d$, $1 \leqslant d \leqslant 12500$, together with the description of the procedure employed in the computation. It is the most extensive class-number table known to the reviewer. However, it contains no such related information on genera as do, for example, the extensive tables of Gauss.[1] The total number $h(d)$ of classes of discriminant $-d$ is obtained from the well-known formula

$$h(d) = \sum_{k^2|d} h'(d/k^2).$$

There exist closed expressions[2] of theoretical importance for $h'(d)$. None of these, however, are really suitable for computation. The method of construction of the present table is based on the following considerations. As is well known, every properly primitive quadratic form of negative discriminant $-d$ is equivalent to one and only one *reduced* form f for which[3]

$$(2) \qquad -a < b \leqslant a, \qquad 0 < a \leqslant c, \qquad b > 0 \text{ if } a = c, \qquad (a, b, c) = 1.$$

Thus the number $h'(d)$ is the same as the number of properly primitive reduced forms of discriminant $-d$. Should one know the number[4] $D(d, B)$ of reduced forms (2) with $|b| = B$, one may use the relation

$$h'(d) = \sum D(d, B), \qquad 0 \leqslant B \leqslant [d^{\frac{1}{2}}/3]$$

to obtain $h'(d)$.

To compute $D(d, B)$ it suffices to determine the number $N(i, B)$ of decompositions $i = (d + b^2)/4 = a \cdot c$ into two positive factors a and c such that

$$(3) \qquad |b| < a < \sqrt{i}, \qquad (C(a, c), b) = 1,$$

where $C(a, c) = (a, c)$.

For, $D(d, 0) = N(d/4, 0)$. Next let $B \neq 0$.

In case i is not a perfect square, then

$$(4) \qquad D(d, B) = 2N(i, B) + 1, \qquad \text{if } B|i \text{ and } C(B, c) = (B, i/B) = 1,$$

and

$$(5) \qquad D(d, B) = 2N(i, B) \text{ otherwise.}$$

In case i is a perfect square, then formulae (4) and (5) hold if $(B, i) > 1$. If, however, $(B, i) = 1$, the values in (4) and (5) must be increased by $+1$. Thus, in this case:

(6) $D(d, 1) = 2N(i, B) + 2$; $D(d, B) = 2N + 1$ if $B > 1$ and either $(B, i) = 1$ or $B|i$ and $(B, i/B) = 1$; $D(d, B) = 2N$ for other values of B.

These formulae follow at once from (2). The term $+1$ in (4), accounts for the reduced form with $a = B = b$ which is properly primitive only if $C(a, c) = 1$. The unit to be added when i is a perfect square accounts for the form with $a = c$, $b > 0$. This form is properly primitive if $(B, a) = (B, a^2) = (B, i) = 1$. Actual computation may be arranged systematically as follows. First comes the "preliminary table."

i	a, B	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	$C(a, c)$		1	2		4			1	4						2	
224	$N(i, B)$[5]	2	5	1	4	1	3	1	2	0	1	0	1	0	1	0	
	$D(4i-B^2, B)$	2	11	2	8	2	6	2	5	0	2	0	2	0	2	0	
	C		1		3		5				1						15
225	N [5]	2	3	3	2	2	1	1	1	1	0	0	0	0	0	0	0
	D	2	8	7	4	5	2	2	3	3	1	0	1	0	1	1	0

Here, in view of (3), $N(i, B)$ is obtained by counting the number of $C(a, c)$ with $B < a < i^{\frac{1}{2}}$, which are prime to B. The value of D is obtained through the formulae (4)–(6).

Next, the D entries are copied out opposite the proper values of d in another table and are totalled by rows to give $h'(d)$.

Thus, the method of tabulation used by the author is essentially the usual method of construction of the table of reduced forms in which the scheme of computation is so arranged that only the presence of a reduced form $[a, b, c]$ is recorded and not the form itself. One observes with interest that the desirable scheme in this case differs from the excellent arrangement by Wright[6] in tabulating reduced forms themselves. As in Wright the method calls for only a simple succession of elementary operations.

St. Louis University ARNOLD E. ROSS

[1] C. F. GAUSS, Werke, v. 2, p. 450–476. For reviews of this and other similar tables consult D. H. LEHMER, *Guide to Tables in the Theory of Numbers*, Nat. Res. Council, *Bull.* no. 105, 1941.

[2] Compare L. E. DICKSON, *History of the Theory of Numbers*, v. 3, Washington, D. C., 1923 (New York reprint, 1934), chapter VI.

[3] Compare L. E. DICKSON, *Modern Elementary Theory of Numbers*, Chicago, 1939.

[4] The author uses $D((d+B^2)/4, B)$.

[5] The author omits this row and records the values of D directly.

[6] H. N. WRIGHT, "On a tabulation of reduced binary quadratic forms of a negative determinant," California, University, *Publs. in Math.*, v. 1, no. 5, 1914, p. 98–114+2 folding plates.

145[F].—CHAO KO and S. C. WANG, "Table of primitive positive quaternary quadratic forms with determinants $\leqslant 25$," Academia Sinica, *Science Record*, v. 1, nos. 1–2, Aug., 1942, p. 54–58, Chungking, China. [In the 260 pages of this publication 97 pages are devoted to 25 papers announcing results of mathematical research.] 18 × 25.4 cm.

In the study of diophantine equations (1) $f(x) = m$, where $f(x) = \sum_{i,j=1}^{n} a_{ij}x_ix_j$ is a quadratic form in n variables $x_1, \cdots, x_n$ with integral coefficients a_{ij}, it was noticed at an early date that integral solutions of (1) may be obtained from the solutions of a related equation (2) $F(y) = m$, where $F(y)$ is a quadratic form *equivalent* to $f(x)$, i.e., a form obtained from f by a transformation τ: $x_i = \sum_{j=1}^{n} \tau_{ij}y_j$ with integral coefficients τ_{ij} and of determinant $|\tau_{ij}|$ equal to unity. Thus, seeking solutions of (1) and (2) independently would involve needless duplication of effort which would be avoided if one would select for study only one of the unlimited number of equations (1) whose left-hand members f are equivalent. A process of selection of a limited number of representative forms, desirably a single one, for each set (*class*) of equivalent forms is called *reduction*, and the selected representatives are referred to as *reduced* forms. Since the number of classes of quadratic forms of a fixed determinant $|a_{ij}|$ is finite, a limited number of reduced forms suffices to represent all forms of a given determinant.

In the paper under review, the authors have constructed a table of *primitive* (*g c d* of $2a_{ij}$ and a_{ii} is unity) reduced positive quaternary quadratic forms of determinant $D \leqslant 25$, using a method of reduction which gave exactly one representative for each class of equivalent forms. Such a table should be considered an improvement of CHARVE'S[1] original table, since his method of reduction led to more than one reduced form for some classes. The authors were apparently unaware, however, of the existence of a similar table by S. B. TOWNES[2], $D \leqslant 25$. Townes's method of reduction employs ideas of Eisenstein and also leads to a unique representative. Reduced forms in the two tables differ only in the choice of the cross-product coefficients.

There are a number of errata in the article under review; on p. 56f, $D = 20$, *for* (1,1,2,20), *read* (1,1,2,10); $D = 21$, *for* $(1\ 2\ 3\ 3\ 0_4\ {-1}\ 0)$, *read* $(1\ 2\ 3\ 4\ 0_4\ {-1}\ 0)$; and $D = 24$, *for* $(2\ 2\ 8\ 3\ {-1}\ 0_3\ {-1}\ 0)$, *read* $(2\ 2\ 3\ 3\ {-1}\ 0_3\ {-1}\ 0)$. On p. 58, the fourth determinant is 9 and not 8. In addition to these errata, noted by the authors in an errata list at the end of the issue of the *Record*, we note that the entries $(2\ 2\ 3\ 4\ {-1_2}\ 0_3\ {-1})$ for $D = 16$ and

$(2\ 2\ 3\ 3\ 0\ -1_2\ 0\ -1_2)$ and $(2\ 2\ 3\ 5\ -1_2\ 0_3\ -1)$ for $D = 24$ are incorrect, being actually of determinants 28, 11 and 36 respectively. A representative for one of the twenty-four classes of determinant 24 is missing, and the entry $(2\ 3\ 3\ 3\ -1\ 0_2\ -1\ 0_2)$ of $D = 25$, should read $(2\ 2\ 3\ 3\ -1\ 0_2\ -1\ 0_2)$.

To restore completeness of the table one may use the following reduced forms from Townes in place of the entries referred to above: the form $(2\ 2\ 3\ 3\ 1_2\ 0\ 1_3)$ for $D = 16$, and the forms $(2\ 2\ 3\ 4\ 1_2\ 0\ 1_3)$, $(2\ 3\ 3\ 3\ -1_2\ 1\ 0\ 1_2)$, and $(1\ 2\ 4\ 4\ 0_5\ 2)$ for $D = 24$.

The count of the number of classes yields 14 for $D = 16$ and 24 for $D = 24$. We have, in the above, used the well-known notation $(a\ b\ c\ d\ r\ s\ t\ u\ v\ w) = ax_1^2 + bx_2^2 + cx_3^2 + dx_4^2 + 2rx_2x_3 + 2sx_1x_3 + 2tx_1x_2 + 2ux_1x_4 + 2vx_2x_4 + 2wx_3x_4$. A run of k zeros is indicated by 0_k for brevity. One may note that the entry $(1\ 1\ 4\ 4\ 0_5\ 2)$ for $D = 16$ in Townes's table should read $(1\ 1\ 4\ 5\ 0_6\ 2)$.

St. Louis University ARNOLD E. ROSS

[1] L. CHARVE, "Table des formes quadratiques quaternaires positives réduites dont le déterminant est égal ou inférieur à 20," Acad. d. Sci., Paris, *Comptes Rendus*, v. 96, 1883, p. 773–775.

[2] S. B. TOWNES, "Table of reduced positive quaternary quadratic forms," *Annals Math.*, s. 2, v. 41, 1940, p. 57–58.

EDITORIAL NOTE: Mr. Ross was formerly called ARNOLD CHAIMOVICH, as we learn from his mimeographed English translation from the Russian of P.S. NAZIMOV's Prize Essay (Moscow, 1884): *Applications of the Theory of Elliptic Functions to the Theory of Numbers*, Evanston, Ill., 1928.

146[F].—D. H. LEHMER, "Ramanujan's function $\tau(n)$," *Duke Math. J.*, v. 10, 1943, p. 483–492.

The numerical function $\tau(n)$ is the coefficient of x^n in the power series expansion of $x[(1-x)(1-x^2)(1-x^3)\cdots]^{24}$ This function has long been of interest in the theory of numbers, particularly in the arithmetical applications of the elliptic modular functions. If m, n are coprime,

$$\tau(mn) = \tau(m)\tau(n); \qquad (1)$$

if p is prime

$$\tau(p^{\alpha+1}) = \tau(p)\tau(p^\alpha) - p^{11}\tau(p^{\alpha-1}), \quad \alpha \geq 1. \qquad (2)$$

Hence $\tau(n)$ is calculable from the values of $\tau(p^\alpha)$, which in turn are calculable for $\alpha = 2, 3, \cdots$ when $\tau(p)$ is known. An explicit formula for $\tau(p)$ is lacking. Ramanujan conjectured (from scanty numerical evidence, namely, the 10 cases determined by $p < 30$) that $|\tau(p)| < 2p^{11/2}$, so that $p^{-11/2}\tau(p) = 2\cos\theta_p$, θ_p real. This unproved conjecture, whose truth was regarded by Ramanujan as "highly probable," is called by Hardy the "Ramanujan hypothesis." The author's purpose is best described in his own words: "In seeking to disprove the Ramanujan hypothesis I have examined all primes $p < 300$, that is, the first 46 primes, as well as $p = 571$, and I find that in all these cases the hypothesis is true. There is only one 'near miss' which occurs at $p = 103$."

The table gives the value of $\tau(n)$ for $n \leq 300$. For composite n the values were computed by (1), (2). For n prime a recurrence formula in which the arguments of $\tau(n)$ decrease by successive pentagonal numbers is used. The formula is obtained by an application of Euler's power series expansion of $\Pi(1-x^n)$, $n = 1, 2, \cdots$ to the logarithmic derivative of the generating identity for $\tau(n)$, and is

$$(n-1)\tau(n) = \Sigma[n - 1 - \tfrac{1}{2}25(3m^2+m)]\,\tau\,[n - \tfrac{1}{2}(3m^2+m)],$$

the summation referring to $1 \leq |m| < a_n$, $6a_n = 1 + (1+24n)^{\frac{1}{2}}$. This formula has practical computational advantages over Ramanujan's recurrence formula involving triangular numbers, derived similarly from Jacobi's expansion of $\Pi(1-x^n)^3$.

The calculations for $\tau(n)$, n prime, were checked by Ramanujan's congruence

$$\tau(n) \equiv \sigma_{11}(n) \bmod 691,$$

where $\sigma_{11}(n)$ is the sum of the 11th powers of the divisors of n. The table as a whole was checked by comparison at frequent intervals of the values modulo 691 of $\Sigma\tau(s)$, $\Sigma\sigma_{11}(s)$, $s \leq x$. Further congruences, to be discussed in another paper, also were used as checks. In addition to the table and the method of computing and checking it, the paper discusses problems connected with the orders of $\tau(n)$, $\Sigma\tau(s)$, neither of which is known.

E. T. BELL

California Institute of Technology

147[I].—ARTHUR SHEPARD LITTLE (1872–), *A Table of Interpolation Multipliers, for obtaining through the means of calculating machines, Intermediate Rates Bond Values at yield intervals of one ten-thousandth percent.* Boston, Financial Publishing Co., [1927], 29 p. 17.5 × 21.6 cm. $4.00.

The main table of this work gives the four coefficients A, B, C, D, in the following special case of Lagrange's interpolation formula:

$$f(a + 2hx) = A(x)f(a) + B(x)f(a + h) - C(x)f(a + 2h) + D(x)f(a + 3h)$$

where

$$\begin{aligned} A(x) &= -(2x - 1)(x - 1)(2x - 3)/3, \\ B(x) &= 2x(x - 1)(2x - 3), \\ C(x) &= x(2x - 1)(2x - 3), \\ D(x) &= 2x(2x - 1)(x - 1)/3, \end{aligned}$$

for x = [0.000(0.001)0.500; 9D]. The reason for the peculiar "$2x$" in the above is that the table is intended for use with bond tables having an argument interval of 0.05 percent. There is also on p. 29 a small table giving the three coefficients in the corresponding three-term formula:

$$f(a + 2hx) = A(x)f(a) + B(x)f(a + h) - C(x)f(a + 2h)$$

where

$$\begin{aligned} A(x) &= (2x - 1)(x - 1), \\ B(x) &= -4x(x - 1), \\ C(x) &= -x(2x - 1), \end{aligned}$$

for x = [0.00(0.01)0.50; 4D]. The author's 8 pages of introduction are interesting (not to say amusing) but inadequate. None of the above formulas appears in the book and not even one small example is worked out to show the business man how to use these tables. Instead there are repeated warnings against failing to notice that formulas (which are not given) contain minus signs so that the crank of the calculating machine must be turned in the negative direction at the proper time. To make this point very clear the coefficients $C(x)$ in the two tables are printed in red and the column headings bear the injunction "Turn Off," while the other columns are headed "Turn On."

The author does not indicate how he discovered these interpolation formulas. There is no reference to Lagrange or to any other source. The author believes that his method is "absolutely original and is now being made public for the first time."

The Lagrange type of interpolation formula is, I believe, destined to become more and more used by computers to whom a good calculating machine is available. The more traditional formulas employing differences are products of an age of pencil and paper calculation. Many tables are printed without differences. When the successive differences are available or worked out, the various terms of the interpolation formula vary so in size that they are a constant source of irritation if not of actual error. With the Lagrange type, the separate terms are of the same order of magnitude and are automatically accumulated in the product register. A small table of Lagrangean interpolation coefficients, such as the one under review,[1] should be in the library of every computer.

D. H. L.

[1] Perhaps one of the best elementary introductions to the subject may be found in K. PEARSON'S *Tracts for Computers*, nos. II–III, Cambridge, University Press, 1920, with illustration examples and short tables of coefficients in the 4 to 11-point "Lagrangean

formulae." LAGRANGE gave the formula now associated with his name in Paris, École Polytechnique, *J.*, v. 2, 1795, p. 274–278. But this formula was discovered twice earlier, namely: (1) by EDWARD WARING, R. So. London, *Trans.*, v. 69, 1779, p. 59–67, and also in generalized forms; (2) by L. EULER, *Opuscula Analytica*, v. 1, 1783, p. 184, setting forth the general Lagrangean formula for equal intervals.

Reference may be given also to T. L. KELLEY, *The Kelley Statistical Tables*, New York, Macmillan, 1938. There are cubic (4-point, [0.000(0.001)1; 10D]), quintic (6-point, [0.00(0.01)1; 10D]), and septic (8-point, [0.0(0.1)9; 11D]) interpolation tables. See RMT **130**. And finally we may note MATHEMATICAL TABLES PROJECT, *Table of Lagrangean Coefficients*, Ordnance Department, 1941, 40 p. 22×26.5 cm. This edition of a 5-point table [0.000(0.001)2; 7D] was not available to the public until prepared for distribution by the Marchant Co., Oakland, California, in 1942. This is now published on p. 383–403 of MATHEMATICAL TABLES PROJECT, *Table of Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments*, New York, Columbia University Press, 1943; see RMT **151**. The PROJECT has in the press an elaborate volume, entitled *Tables of Lagrangian Interpolation Coefficients* (xxxii, 390 p.).

EDITORIAL NOTE.—We include here a review of this table published before 1933, for two reasons, because only the slightest attention has been paid to it in any mathematical periodical, and because there is little doubt that the author, working most of his life in a clerical capacity for the First National Bank and one of its affiliates, in St. Louis, was an independent discoverer of the utility of these so-called Lagrange interpolation coefficients. He published at least three other volumes (1915–26), listed in the Library of Congress Catalogue, and a good many articles in *Journal of Accountancy*.

148[I].—A. J. THOMPSON, *Tables of the Coefficients of Everett's Central-Difference Interpolation Formula. Tracts for Computers*, No. V. Edited by E. S. Pearson and published by the Department of Statistics, University College, London, Cambridge University Press, second ed., 1943. viii, 32 p. 16 × 23.3 cm. 5 shillings. See also MTE **41**.

The first edition of Mr. Thompson's tables was published in 1921 and was quickly sold out. The present edition differs considerably from the first (xvi, 21 p.). The size of the introduction is cut by half and its contents (somewhat paradoxically) are increased. Also there are five tables of coefficients in the new edition, the first three columns of Table I covering the whole of the single table in the original edition. The reputation of the distinguished author guarantees the accuracy of the figures published. It may be expected, therefore, that the tract will be in great demand.

A considerable portion of the original Introduction was devoted to numerical examples illustrating in varying degrees the same type of application of Everett's interpolation formula and coefficients. In the second edition the number of numerical examples is reduced to just a single one. In addition to this, the author gives to the reader the benefit of his wide experience in dealing with various difficult situations. A straight application of Everett's formula is possible when the table, in which it is desired to interpolate, contains the central differences. This, of course, is far from being always the case. A convenient method is indicated by which explicit use of the central differences can be avoided.

Other material in the new Introduction deals with the construction of mathematical tables. The first step suggested is to compute directly the values of the function to be tabulated for a basic framework of widely spaced values of the argument. These computations should yield some three or four places of decimals in excess of the accuracy of the proposed table. Once the framework is completed, the use of Tables II, III, IV and V of the new edition of Thompson's tract will permit a relatively easy process for filling of the gaps by reducing the original increment of the argument in the ratio of one to ten or more. The coefficients of Everett's formula are used to compute both the values of the function and the corresponding central differences.

Table I gives the values of the first four coefficients for values of the argument $0 \leqslant \theta \leqslant 1$ proceeding by increments of .001. It also gives the central differences of the coefficients which could be used for interpolation among the values published. Table II contains coefficients of the same orders as Table I but proceeding at intervals of θ equal to .01. Table III gives the values of six coefficients, ϵ_2 to ϵ_{12}, with complete differences, but proceeds at increments of θ equal to .1. Table IV has the same order of coefficients and the same increment of θ as Table III but the range of θ is from -5 to $+6$. The primary purpose of this table is to facilitate interpolation in other tables near the ends of the range of their arguments, where the central differences of higher orders cannot be made available without previously extending the range of the original table. In Table V the coefficients extend to ϵ_{16} and proceed at increments of θ equal to .2.

Those who have frequent occasions to interpolate in tables in which linear or quadratic interpolation is inadequate, and have had to do so without the benefit of Mr. Thompson's tract, may think that its title is not satisfactory and that it should be changed to "Interpolation Made a Pleasure," or to "A Hard Life Made Easy."

J. Neyman

University of California, Berkeley

149[I, J].—H. E. Salzer, "Coefficients for numerical differentiation with central differences," *J. Math. Phys.*, M.I.T., v. 22, 1943, p. 115–135. 17.5 × 25.5 cm.

This paper provides the coefficients for the expansion of the nth derivative of $\phi(x)$, when $\phi(x)$ is expanded in terms of central differences, that is to say, for the coefficients of $\delta^{2\nu-1}$ and $\delta^{2\nu}$ in the expansions:

$$\omega^{2n-1}\phi^{(2n-1)}(x) = \sum_{\nu=n}^{\nu=k} A_{2\nu-1}^{2n-1}\delta^{2\nu-1} + R_1,$$

$$\omega^{2n}\phi^{(2n)}(x) = \sum_{\nu=n}^{\nu=k} A_{2\nu}^{2n}\delta^{2\nu} + R_2,$$

where R_1 and R_2 are remainders, whose explicit forms are found in standard treatises on finite differences. For the mean differences of odd order, we replace the customary notation, $\mu\delta^{2\nu-1}$ or $\square\delta^{2\nu-1}$, by the simpler notation $\delta^{2\nu-1}$. The original expansion of $\phi(x)$ in terms of central differences is usually referred to as the Newton-Stirling series.

The author defines the coefficients in terms of $B_m^{(n)}(x)$, the Bernoulli polynomial of mth order and nth degree, as follows:

$$A_{2\nu-1}^{2n-1} = \frac{1}{(2\nu-2n)!}B_{2\nu-2n}^{(2\nu)}(\nu), \quad \text{and} \quad A_{2\nu}^{2n} = \frac{n}{\nu(2\nu-2n)!}B_{2\nu-2n}^{(2\nu)}(\nu).$$

It is unfortunate that he did not also state them in terms of the familiar differential coefficients of zero, namely,

$$A_{2\nu-1}^{2n-1} = D^{2n-1}0^{[2\nu]-1}/(2\nu-1)!, \quad \text{and} \quad A_{2\nu}^{2n} = D^{2n}0^{[2\nu]}/(2\nu)!.$$

Coefficients of even order can be immediately written down from those of odd order by the obvious relationship

$$A_{2\nu}^{2n} = (n/\nu)\,A_{2\nu-1}^{2n-1}.$$

The author used the following new identity in a partial check of his computations:

$$A_{2\nu-1}^{2n-1} = \frac{1}{2\nu-1}\left[(2n-1)A_{2\nu-2}^{2n-2} - (\nu-1)^2\frac{1}{2n}A_{2\nu-2}^{2n}\right].$$

In order to achieve his purpose in making "this table the 'ultimate' in coefficients for numerical differentiation," the author computes the coefficients as far as the 52nd derivative. For the first 30 derivatives the author gives the exact values, in common fractions, up to the difference of 30th order, and also for some coefficients of differences beyond the 30th. "For all derivatives beyond the 30th, exact values are given for coefficients of differences going as far as some difference between the 41st and 52nd. Elsewhere, that is, for most of the coefficients of the 31st to 42nd differences, 18 significant figures are given, with accuracy to within 0.6 unit in the last significant figure."

To put it otherwise, the derivative formulas can now be written to at least 18 significant figures for the first 32 derivatives up to differences of 42nd order, the 33rd, 34th, 35th, 36th, 37th, 38th, 39th, 40th, and 41st derivatives up to orders 43, 44, 45, 46, 47, 48, 49, 50, and 51 respectively, and the derivatives of orders from 42 to 52 to differences of 52nd order.

The coefficients were checked to about 10 significant figures by means of the recurrence formula given above.

Although it is difficult to see where these tables would be used in the calculation of derivatives of such high order, since differences beyond eight or ten are almost never encountered, nevertheless the coefficients themselves are related to some interesting functions and could quite conceivably be of importance in problems other than those for which the table has been computed. We are greatly indebted to the author for providing us with these new constants. H. T. D.

150[L].—CARL AUGUST HEUMAN (1870–), "Tables of complete elliptic integrals," *J. Math. Physics*, M.I.T., v. 20, 1941, p. 127–206. 17.5 × 25.4 cm.

The first two tables in this paper give values to 6D of the complete elliptic integrals of the first and second kinds, $F_0(\alpha)$ and $E_0(\alpha)$, for $\alpha = 0°.0(0°.1)90°$, with first differences; except that the differences are omitted for $\alpha > 65°$ in the case of $F_0(\alpha)$. The functions $F_0(\alpha)$ and $E_0(\alpha)$ are $2/\pi$ times the Legendre integrals F and E. Linear interpolation is sufficiently accurate except in the range $\alpha > 65°$ for $F_0(\alpha)$. To fill this gap there is given a table of an auxiliary function in which linear interpolation can be used:

$$G_0(\alpha) = F_0(\alpha) + (2/\pi) \ln (90° - \alpha°)$$

for $\alpha = 65°.0(0°.1)90°$, with first differences.

An extensive table of complete elliptic integrals of the third kind $\Lambda_0(\alpha, \beta)$ is also given, p. 160–197, where

$$\Lambda_0(\alpha, \beta) = F_0(\alpha)E(\alpha', \beta) - \{F_0(\alpha) - E_0(\alpha)\}F(\alpha', \beta).$$

Here $\alpha' = \pi/2 - \alpha$ and $F(\alpha, \beta)$ and $E(\alpha, \beta)$ are the incomplete integrals of the first and second kinds. Tabular values are for $\alpha = 0°(1°)90°$, $\beta = 0°(1°)90°$, and for $\alpha = 0°.0(0°.1)5°.9$, $\beta = 80°(1°)89°$, all to 6D.

Heuman obtained his values of $F_0(\alpha)$ and $E_0(\alpha)$ directly from values of log F and log E in Table I of Legendre's *Exercices de calcul intégral sur divers ordres de transcendantes et sur les quadratures*, Paris, 1816, Tome 3, and *Traité des fonctions elliptiques*, Paris, 1826, Tome 2, with the aid of Vega's ten-place *Thesaurus Logarithmorum*. The values of $\Lambda_0(\alpha, \beta)$ were computed from Legendre's Table I, to 12D–14D, for each tenth of a degree, and Table IX, the latter being a table of the incomplete integrals $F(\alpha, \beta)$ and $E(\alpha, \beta)$ to 9D–10D, for each degree. A list of 42 errors in Legendre's tables is given, most of them being in the incomplete integrals.

An appendix to the paper contains applications to the motion of a spherical (or more general) pendulum, and the gyroscopic pendulum. In the former case a chart is given for the apsidal angle as function of the two independent variables z_1 and z_2, which are the greatest and least heights attained by the bob during its motion. The gyroscopic pendulum is considered in the case where the spindle starts at an inclination θ_0 with zero initial velocity. Two charts are given, for the apsidal and zonal angles respectively, as functions of the two independent variables θ_0 and μ where μ is the coefficient of stability.

P. W. KETCHUM

151[L].—MATHEMATICAL TABLES PROJECT, New York, A. N. LOWAN, technical director, *Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments*, New York, Columbia University Press, 1943, xliv, 403 p. 19.6 × 26.5 cm. Reproduced by a photo offset process. \$5.00.

The main part of this useful volume consists of a table of real and imaginary parts of $J_0(z)$ and $J_1(z)$ as functions of $z = \rho e^{i\vartheta}$, for $\rho = [0.00(0.01)10; 10D]$, $\vartheta = 0°(5°)90°$. The last twenty pages contain a table of the five-point Lagrangean Interpolation Coefficients for $\rho = [0.000(0.001)1; 10D]$, twice previously published (*MTAC*, p. 94). There is also a short foreword by H. BATEMAN, and a long Introduction by Mr. LOWAN, followed by a Bibliography (65 items) of Tables of Bessel functions and of applications of Bessel functions of a complex argument of the form $ri^{\frac{1}{2}}$. The four pages of "Contour Lines" for $J_0(z) = J_0(\rho e^{i\phi}) = U(\rho, \phi) + iV(\rho, \phi), J_0(z) = J_0(\rho e^{i\phi}) = Re^{i\theta}, J_1(z) = J_1(\rho e^{i\phi}) = u(\rho, \phi) + iv(\rho, \phi)$, $J_1(z) = J_1(\rho e^{i\phi}) = Se^{i\sigma}$, are of considerable interest. The publication of the Bessel function tabulation makes possible the numerical solution of a large number of problems, which have hitherto only been discussed symbolically. Apart from tables with pure imaginary arguments, and for arguments in the form $ri^{\frac{1}{2}}$, DINNIK'S short tables[1] of $J_0(z)$ and $J_1(z)$ are the only ones previously published.

Bessel functions are ubiquitous in mathematical physics, from acoustics to stochastics. When the wave equation in more than one dimension is separated in rectangular coordinates, the separated equations have an irregular singular point at infinity, and the solutions are exponential or trigonometric functions. If the coordinate system involves a point center or a single axis of revolution, such as for spherical or circular cylindrical coordinates for instance, the radial equation has a regular singular point at the origin and an irregular singular point at infinity. The general solution of this type equation is the confluent hypergeometric function, but in the great majority of cases of practical interest, the special case of the Bessel equation is the result. There are other more complicated coordinate systems in which the wave equation separates, but the resulting ordinary differential equations have a more complicated system of singular points, and the solutions (Mathieu functions, Lamé functions and the like) are not well enough tabulated to make them readily usable in specific problems.

The peculiar position of the Bessel function is again evident when one studies the applications of the Laplace transform, whether in wave or diffusion problems or in probability theory. In many coordinate systems the characteristic integral expression

$$\int e^{iz\cos\phi + in\phi} d\phi$$

occurs. This can be shown to be proportional to one or another of the solutions of the Bessel equation of order n, depending on the choice of the contour for the integral.

Perhaps part of the reason for the preeminence of the Bessel function in mathematical physics is that it is neither too complex nor too simple. A solution of the wave equation expressed in terms of trigonometric functions would not be adequate to use in studying the behavior of waves from point or line sources, which are best expressed in terms of Bessel functions. On the other hand, although a solution of the oblate spheroidal waves from a circular disk would be even more valuable than a solution for a point source, the oblate spheroidal functions needed for such a solution are so much more difficult to compute than are the Bessel functions, that no adequate tables are yet available from which to obtain complete numerical results. Bessel function solutions are complex enough to exhibit certain general wave properties, and yet simple enough to tempt the computer.

As the Bibliography well suggests, tables of Bessel functions for real and for pure imaginary arguments are now fairly numerous. A great number of problems can be solved numerically by their use, and the tables under review do not represent an advance in this direction. There are a number of important problems, however, whose solution requires values of the Bessel functions for complex values of the argument; and in this field the present *Table* represents a distinct advance.

Uses for the functions, for the argument $\rho\sqrt{i}$, are fairly well known (the ber, bei and related functions), and a number of tables have been published of these functions.[2] Problems relating to the behavior of electromagnetic waves in cylindrical conductors (skin-effect) require the use of these functions, as also do calculations of the fluctuation of temperature in a cylinder whose surface temperature is a sinusoidal function of time (furnace with thermostatic control). The present tables are more complete than any of the earlier ones of ber x, bei x (the B.A.A.S. Tables, for instance) for this special case of $\vartheta = 45°$.

For other values of ϑ, however, the present tables are unique, and their publication makes possible calculations of considerable immediate interest. The behavior of microwaves in cylinders of dielectric material in which there is energy dissipation is one application; and the scattering of electromagnetic waves from such a cylinder is another. The attenuation of acoustic waves down hollow cylinders[3] can now be calculated, and computing the transmission of sound in a shallow sea with absorbing bottom is made considerably easier. In fact, wherever cylindrical waves cause energy dissipation, either in the medium or at the boundary, Bessel functions of complex argument are encountered, and the present tables are of use. The companion volume, giving values of N_0 and N_1 for complex values of the argument, will also be most useful when it is published.

The reviewer has not had the leisure to test the accuracy of the present Tables. However, the accuracy of the previous publications of the Mathematical Tables Project is well known.

PHILIP M. MORSE

Massachusetts Institute of Technology

[1] A. N. DINNIK, All Russian Central Committee, [Communications], 1922, p. 121–126; reprinted in K. HAYASHI, *Fünfstellige Funktionentafeln . . .* , Berlin, Springer, 1930, p. 105–109.

[2] A. G. WEBSTER, B.A.A.S., *Report*, 1912, p. 56–68; ber x and bei x and their derivatives, $x = [0.0(0.1)10.0; 9D]$, with 7 differences. For corrections by H. G. SAVIDGE see B.A.A.S., *Report*, 1916, p. 122; there are also corrections by H. B. DWIGHT. SAVIDGE gave tables of ker, kei and their first derivatives and other tables in B.A.A.S., *Reports*, 1915, p. 36–38, and 1916, p. 108–121. An abridgement of some of the tables in these Reports is given by H. B. DWIGHT in (a) *Tables of Integrals and Other Mathematical Data*, New York, Macmillan, 1934; see RMT **154**; and (b) *Mathematical Tables*, New York, McGraw-Hill, 1941; see RMT **143**. The latter has 5-place tables of functions and derivatives for $\vartheta = 45°, 135°$.

[3] P. M. MORSE, "The transmission of sound inside pipes," Acoustical So. Am., *J.*, v. 11, 1939, p. 205.

152[L].—E. O. POWELL, "An integral related to the radiation integrals," *Phil. Mag.*, s. 7, v. 34, Sept. 1943, p. 600–607. 17 × 25.3 cm.

The following function is tabulated

$$Rl(x) = \int_1^x \ln y dy/(y-1) = \sum_1^\infty (-1)^{n+1}(x-1)^n n^{-2} \qquad (0 \leqslant x \leqslant 2),$$

$$= \ln x \ln(1-x) - \pi^2/6 + \sum_1^\infty x^n n^{-2} \qquad (0 \leqslant x \leqslant 1);$$

$Rl(x) + Rl(1/x) = 1/2(\log x)^2$. From these relations the values of $Rl(x)$ were computed for $x = [0.00(0.01)2.00(0.02)6.00; 7D]$, with second central differences, negative throughout. The author writes as follows: "It is hoped that the tabular values will be found correct to six decimal places; the seventh should not usually be in error by more than one unit." The "differences are sufficient to give seven-figure accuracy when x is greater than 0.2, six-figure between 0.1 and 0.2; below this limit they increase very rapidly."

The integral $\int_1^x \ln y dy/(y-1)$ was recently encountered in a physical problem.[1] This integral is equal to $Ri_{-1}[\ln(1/x)] - \pi^2/6$, where $Ri_n(x) = \int_x^\infty dy/y^n(e^y-1)$, an integral tabulated by AIREY for 6 values of n (see *MTAC*, p. 140, no. **46**). DEBYE gave[2] a short table of $(1/x)\int_0^x y dy/(e^y-1) = \pi^2/(6x) - (1/x)Ri_{-1}(x)$.

[1] S. R. FINN and E. O. POWELL, "The chemical and physical investigation of germicidal aerosols. II: The aerosol centrifuge," *J. Hygiene*, v. 42, 1942, p. 364.

[2] P. DEBYE, "Interferenz von Röntgenstrahlen und Wärmebewegung," *Annalen d. Phys.*, s. 4, v. 43, 1914, p. 85–86.

153[L].—F. VANDREY, "Tafel der acht ersten Kugelfunktionen zweiter Art," *Z. angew. Math. Mech.*, v. 20, 1940, p. 277–279. 20.8 × 29.5 cm.

These tables emanated from the aerodynamic research laboratory at Göttingen. They present the values of the Legendre spherical functions of the second kind $Q_n(x)$, from $Q_0(x)$ to $Q_7(x)$, for $x = [0.00(0.01)1.00;\ 5D]$. By virtue of the relation $Q_n(-x) = (-1)^{n+1}Q_n(x)$ the table covers also the range $-1 \leqq x \leqq 0$.

In calculating the table the values of the function

$$Q_0(x) = \tanh^{-1} x = \tfrac{1}{2} \ln [(1 + x)/(1 - x)]$$

were taken from the 9-place table of K. HAYASHI, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen*, Berlin, Springer, 1926, p. 9–87. $Q_1(x)$ to $Q_4(x)$ were calculated from the following equations: $Q_1(x) = xQ_0(x) - 1$, $Q_2(x) = P_2(x)Q_0(x) - (3/2)x$, $Q_3(x) = P_3(x)Q_0(x) - (5/2)x^2 + \tfrac{2}{3}$, $Q_4(x) = P_4(x)Q_0(x) - (35/8)x^3 + (55/24)x$. Q_5 to Q_7 were then determined by means of the recurrence formula,

$$nQ_n(x) + (n - 1)Q_{n-2}(x) - (2n - 1)xQ_{n-1}(x) = 0.$$

In the tables a dot placed after the upper part of the fifth figure indicates that this figure has been increased. Linear interpolation to 3 places is possible in $Q_0(x)$ to $Q_6(x)$ from 0 to about 0.86, and in $Q_7(x)$ from 0 to 0.69.

154[L, M].—H. B. DWIGHT, *Tables of Integrals and other Mathematical Data*, New York, Macmillan, 1934, x, 222 p. 14 × 21.5 cm. $1.75. See also RMT **30**.

The first important American table of integrals was the one by the brilliant Harvard physicist, B. O. PEIRCE (1854–1914) which started in a small way as a 32-page pamphlet (1889), and was bound in with the second edition of W. E. BYERLY'S *Elements of the Integral Calculus*, Boston, 1889. After tremendous labor this, *A Short Table of Integrals*, was expanded to a book of 144 pages (Boston, 1910). The third edition (156 p.), revised by W. F. OSGOOD, appeared in 1929. After 479 indefinite integrals, there are 44 miscellaneous definite integrals, followed by more than 60 formulae under the heading Elliptic Integrals. Then follow (p. 73–115, nos. 570–938) various auxiliary formulae in trigonometric, hyperbolic, elliptic, and Bessel functions; series; derivatives; Green's theorem and allied formulae; table of mathematical constants; general formulae of integration; note on interpolation. The final pages (116–154) contain various tables including those of the probability integral, of elliptic integrals, of hyperbolic functions, and of $\Gamma(n)$.

And now we are considering a volume by a professor of electrical engineering at Massachusetts Institute of Technology. In a general way the volume of Dwight is not greatly dissimilar to that of Peirce but it is considerably more elaborate. Dwight has much more detailed numbering of formulae[1] (with many cross references), and careful statements of ranges for which they are valid. On p. 219–220 are listed 35 volumes of tables, treatises, etc., to which references are often given as authorities for statements made, or as sources where more elaborate results appear. Peirce's tables in this list are naturally referred to several times. Tables 1050, of ber, bei, ber′, bei′, ker, kei, ker′, kei′, are taken from those of Webster and Savidge in B.A.A.S. *Reports*, 1912, p. 56–68; 1915, p. 36–38; 1916, p. 122. Dwight has corrected the value for bei 8.9 given by Webster as $-8.002\cdots$ instead of $-28.002\cdots$. Dwight made this correction first in Amer. Inst. Electr. Engs., *Jn.* v. 42, 1923, p. 830. But he failed to correct Webster's value for bei′ 3.7 0.134 686 760 which should have been 0.131 486 760 (Dwight, A.I.E.E., *Trans.*, v. 58, 1939, p. 787). On pages 215–217 of Table 1050 are original tables of $\mathrm{ber}_n\, x$, $\mathrm{bei}_n\, x$, $\mathrm{ber}'_n\, x$, $\mathrm{bei}'_n\, x$, $\mathrm{ker}_n\, x$, $\mathrm{kei}_n\, x$, $\mathrm{ker}'_n\, x$, $\mathrm{kei}'_n\, x$, for $n = 1(1)5$; $\mathrm{ber}_n\, x + i\, \mathrm{bei}_n\, x = J_n(xi^{\frac{3}{2}}) = i^n I_n(xi^{\frac{1}{2}})$; $\mathrm{ker}_n\, x + i\, \mathrm{kei}_n\, x = i^{-n}K_n(xi^{\frac{1}{2}})$. The first four of these tables, 3D–6D, were first published in A.I.E.E., *Trans.*, v. 42, 1923, p. 858, and reprinted in *Trans.*, v. 48, 1929, p. 814–815; of the second four, mostly 6D–8D, the greater part of the values were calculated by W. H. HASTINGS as thesis work at M.I.T. in 1927.

The section on Bessel Functions, nos. 800–845, is a very useful collection of recurrence formulae, identities (functions and integrals), with real and complex arguments. The notation of Gray and Mathews and Macrobert's *A Treatise on Bessel Functions*, London, 1922 (1931), is here employed.

The section of formulae on Elliptic Functions and Integrals occupies nos. 750–789.2 and in nos. 1040–1041 there are two brief tables of K and E. In K, 87°.6, *for* 4.562, *read* 4.561.

The series and formulae nos. 1–50 include some dealing with Bernoulli and Euler numbers.

This volume, by the author of several mathematical tables, is an excellent and exceedingly useful one, compiled with great care. The "Contents" and "Index" are wholly adequate. Some other errata are noted in MTE **32**.

R. C. A.

[1] We have already drawn attention (*MTAC*, p. 66) to a valuable volume, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, 1922 (corrected reprint 1939), 314 p. The Mathematical Formulae (p. 1–219) were prepared by the physicist EDWIN P. ADAMS. He writes, "In order to keep the volume within reasonable bounds, no tables of indefinite and definite integrals have been included. For a brief collection, that of the late Professor B. O. Peirce can hardly be improved upon; and the elaborate collection of Bierens de Haan show how inadequate any brief tables of definite integrals would be. A short list of useful tables of this kind, as well as of other volumes, having an object similar to this one, is appended." Nevertheless, there is much in common with the volumes under review. Dwight doubtless received more than one suggestion from this volume which has a goodly number of literature references. In the useful chapter on infinite series, p. 109–144, there is still one error, as Mr. W. D. Lambert, of the Coast and Geodetic Survey, has recently pointed out; on p. 122, under 6.42, no. 4, the third and fifth terms of the right-hand member should each be preceded by the sign —. Legendre and Bessel functions are considered at some length on p. 191–219.

155[L, M].—A. H. HEATLEY, "A short table of the Toronto function," R. So. Canada, *Trans.*, v. 37, sect. III, 1943, p. 13–29. 16.2 × 25 cm.

By "Toronto function" the author designates the three-parameter function $T(m, n, r)$ defined in terms of the four-parameter function

$$T(m, n, p, a) = \int_0^\infty t^{m-n}e^{-p^2t^2}I_n(2at)dt$$

by means of the relation

$$T(m, n, r) = 2r^{n-m+1}e^{-r^2}p^{m-n+1}T(m, n, p, a),$$

where $r = a/p$. Many relations between the Toronto function and other functions such as the confluent hypergeometric function $M(\alpha, \gamma, x)$, Bessel functions of fractional orders, the error function, etc., are summarized in Table 1-A. The author gives also (without derivation) a number of recurrence formulae between three T's whose parameters m and n differ by integers, as well as a number of formulae for special values of these parameters. He also gives the differential equation satisfied by T when considered as a function of r and an asymptotic expansion based on the confluent hypergeometric function.

The short table included with the article gives 5-place values of $T(m, n, r)$ for the following values of the parameters:

$m = -\frac{1}{2}$; $n = -2(0.5)2$; $r = 0(0.2)2$; 5, 6, 10, 25, 50,
$m = 0$; $n = -2(0.5)2$; $r = 0(0.2)4$; 5, 6, 10, 25, 50,
$m = \frac{1}{2}$; $n = -2(0.5)2$; $r = 0(0.2)2$; 4, 5, 6, 10, 25, 50,
$m = 1$; $n = -2(0.5)2$; $r = 0(0.1)2(0.2)3$.

The computations of $T(m, n, r)$ were carried out on the basis of the relations in Table 1-A for sufficiently small values of r and on the basis of the asymptotic expansion for sufficiently large r (mostly for $r > 4$).

The computed values were checked with the aid of the recurrence formulae and occasional differencing. In order to compute $T(m, n, r)$ for small values of r, the author found it necessary to prepare two small tables of the function $M(\alpha, \gamma, x)$ and $e^{-x}M(\alpha, \gamma, x)$. The first lists $M(\frac{1}{4}, 3, x)$, $M(\frac{1}{4}, 2, x)$, $M(\frac{1}{4}, 1, x)$, $M(\frac{5}{4}, 2, x)$ and $M(\frac{9}{4}, 3, x)$ for 11 values of x ranging from 0 to 4 at irregular intervals. The second lists $e^{-x}M(\frac{3}{4}, 3, x)$, $e^{-x}M(\frac{3}{4}, 2, x)$, $e^{-x}M(\frac{3}{4}, 1, x)$, $e^{-x}M(\frac{7}{4}, 2, x)$ and $e^{-x}M(\frac{11}{4}, 3, x)$ for x ranging between 0 and 4 at the same intervals. There is a gap in the values of $T(-\frac{1}{2}, n, r)$ between $r = 2$ and $r = 5$ and a gap in the values of $T(\frac{1}{2}, n, r)$ between $r = 2$ and $r = 4$.

The reviewer wonders whether it would not have been more expedient to base the computation of $T(m, n, r)$ for small values of r directly on the differential equation rather than on its expression in terms of the confluent hypergeometric functions whose series expansion is so slow that the author had to take 19 terms of the series.

Conceivably the method developed by C. LANCZOS ("Trigonometric interpolation of empirical and analytical functions," *J. Math. Physics*, M.I.T., v. 17, 1938, p. 123–199) might have yielded expressions valid in a range of r including the gaps above noted.

ARNOLD N. LOWAN

156[L, S].—L. D. ROSENBERG, "Metod rascheta zvukovykh poleĭ, obrazovannykh raspredelennymi sistemami izluchateleĭ, rabotai͡ushchimi v zakrytykh pomeshchenii͡akh" [Method of calculating sound fields generated by distributed systems of radiators operating in closed places] *Zhurnal Tekhnicheskoĭ Fiziki*, v. 12, 1942, p. 247–248. 16.6 × 25.9 cm.

These pages at the end of the article contain a table of $T(u, \alpha) = \int_{\alpha}^{\frac{1}{2}\pi+\alpha} e^{-u\sin x}dx$, for α = [0.0(0.1)0.7, 0.785; 5D], u = 0.0(0.1)1.0, 1.5, 2.0, 3.0, 3.5, 4, 5, 6, 8, 10, 15, 20, 30, 40, 50, 70, 100.

157[M].—A. N. LOWAN and J. LADERMAN, "Table of Fourier coefficients," *J. Math. Phys.*, M.I.T., v. 22, 1943, pp. 136–147. 17.5 × 25.5 cm.

In this paper we find the tabulated values of the two functions

$$S(k, n) = \int_0^1 x^k \sin n\pi x \, dx, \qquad \text{and} \qquad C(k, n) = \int_0^1 x^k \cos n\pi x \, dx,$$

over the range $n = 1(1)100$, and $k = 0(1)10$. The tables are given to 10 decimal places with a plus sign indicating that the eleventh decimal place is 5 or larger. This is a rather interesting innovation, since the usual rule is to provide in the last place a figure which is not in error by more than .5 units, the sign of the error being either plus or minus. In this connection it might be worth while calling attention to the device employed by L. M. MILNE-THOMSON and L. J. COMRIE in their *Standard Four-Figure Mathematical Tables*, London, 1931, who say with respect to their tables that "a further indication of the figures omitted is given by placing a 'high' dot after the last figure if these omitted figures, in units of the last figure retained, lie between .1666... (i.e. $\frac{1}{6}$) and .5000..., and a 'low' dot if they lie between .5000... and .8333... (i.e. $\frac{5}{6}$)."

The two functions tabulated satisfy the recurrence formulas:

$$S(k, n) = S(1, n) - \frac{k(k-1)}{n^2\pi^2} S(k-2, n), \qquad C(k, n) = \tfrac{1}{2}k\, C(2, n) - \frac{k(k-1)}{n^2\pi^2} C(k-2, n);$$

and the "cross relations":

$$S(k, n) = S(1, n) - \frac{k}{n\pi} C(k-1, n), \qquad C(k, n) = -\frac{k}{n\pi} S(k-1, n).$$

As to the accuracy of the table we may quote the authors, who say: "The computations were performed originally to twenty decimal places with such extreme care, that the 'cross relation' checks on the twenty-place values did not reveal a single error. The present table is an abridged version of the original twenty-place manuscript."

The table is designed to facilitate the computation of Fourier coefficients for functions which may be represented by polynomials of degrees not exceeding 10.

Since the table under review is unique in the field of harmonic analysis, which has been developed vigorously during the past forty years by astronomers, physicists, meteorologists, and economists, it may be of interest to indicate its relationship to other tables.

In harmonic analysis the interest has focused upon the two sums,

$$A_n = \frac{2}{N}\sum_{t=1}^{N} f_t \cos\frac{2n\pi t}{N}, \quad \text{and} \quad B_n = \frac{2}{N}\sum_{t=1}^{N} f_t \sin\frac{2n\pi t}{N};$$

or the corresponding integrals,

$$A_n' = \frac{2}{a}\int_0^a f(t)\cos\frac{2n\pi t}{a}\,dt, \quad \text{and} \quad B_n' = \frac{2}{a}\int_0^a f(t)\sin\frac{2n\pi t}{a}\,dt,$$

which are usually to be evaluated by finite integration.

Hence most tables designed for use in harmonic analysis provide values of the functions

$$M\cos\frac{2n\pi}{p} \quad \text{and} \quad M\sin\frac{2n\pi}{p}.$$

The following is a bibliography of such tables:

H. H. Turner, *Tables for facilitating the use of harmonic analysis.* London, Milford, 1913, 46 p.

L. Zipperer, *Tafeln zur harmonischen Analyse periodischer Kurven,* Berlin, Springer, 1922, 12 p.

L. W. Pollak, *Rechentafeln zur harmonischen Analyse,* Leipzig, Barth, 1926, 23 + 17 + 240 p.

L. W. Pollak, *Handweiser zur harmonischen Analyse,* Prague, Bursík & Kohout, 1928, 72 + 98 p.

P. Terebesi, *Rechenschablonen für harmonische Analyse und Synthese.* Berlin, Springer, 1930, 13 + 11 p. + 12 tables.

A. Hussmann, *Rechnerische Verfahren zur harmonischen Analyse und Synthese, mit Schablonen für eine Rechnung mit 12, 24, 36, oder 72 Ordinaten.* Berlin, Springer, 1938. 28 p. + 12 folding tables.

K. Stumpff, *Tafeln und Aufgaben zur harmonischen Analyse und Periodogramrechnung.* Berlin, Springer, 1939, vii + 174 p.

Since the demands of practical statistics do not usually require many decimal places, the above tables are computed mainly to three or four significant figures. Thus the last table, which is typical, contains three-place values of the two functions for $M = 1$, $p = 3(2)39$, $n = 0(1)\frac{1}{2}p$; $p = 2(4)40$, $n = 0(1)\frac{1}{4}p$; $p = 4(4)40$, $n = 0(1)\frac{1}{4}p$; $M = 0(1)1000$, $n = 1$, $p = 8$, 12, 16, and 24; $M = 2(2)100$, $p = 21(1)27$, 29, 30(1)35, 37(1)39, $n = 0(1)\frac{1}{4}p$; $M = 1(1)100$, $p = 20(4)40$, $n = 0(1)\frac{1}{4}p$.

Since occasionally greater accuracy is required than is provided by these tables the reviewer has prepared in manuscript form an 8-place table of the two functions for $M = 1$, $p = 5(1)75$, $n = 0(1)p$.

It will be clear that the table which is the subject of this review will be useful in harmonic analysis in those instances when the statistical function $f(t)$ can be approximated by a polynomial of degree not exceeding ten. The values of A' and B' are then immediately approximated by appropriate sums of the the tabulated values of $C(k, n)$ and $S(k, n)$, when we use the transformation $t = ax$. The table was computed during 1940–41 in connection with work of the Mathematical Tables Project.

H. T. D.