# Machines for Solving Algebraic Equations

**1. Introduction.** The search for mechanical means of solving algebraic equations has interested mathematicians for well over a century. Two early papers date back to the eighteenth century. Perusing a paper of 1758 by SEGNER,[1] in which the author proposes a universal method of discovering real roots of equations, based on what we should now call drawing the graph of the function $y = \sum a_i x^i$, ROWNING[2] in 1770 considered the possibility of drawing the graph of a polynomial continuously by local motion. Theoretically at least, a number of rulers could be linked together so that the pencil point on the last ruler would trace the required curve. But mechanical limitations of the day caused a reviewer to remark that "as this is a matter of curiosity rather than any use, . . . it is unnecessary to enter any further into it at this time." Theoretical methods developed since that day have depended for their usefulness on the degree of precision in the mechanisms constructed to carry theory into practice, a precision which has greatly increased in modern times.

The early mechanical equation-solvers were restricted to finding the real roots of equations with real coefficients. But certain electrical methods, starting with the one described by LUCAS in 1888, were able to handle complex roots, and even complex coefficients. The modern isograph is an electromechanical device for finding real or complex roots of algebraic equations. In addition to the machines for solving algebraic equations in a single unknown, other similar devices have been invented for the solution of simultaneous linear equations in several unknowns.

Two excellent surveys of earlier mechanisms appeared at the beginning of this century, one by MEHMKE[3] in 1902, revised by D'OCAGNE[3] in 1909, and the other by MORITZ[4] in 1905. A few years later GHERSI,[5] in his book of mathematical curiosities, included an illustrated account of some of the previously discovered hydrostatic and electric solvers of algebraic equations. A comprehensive survey of various dynamical methods of solving algebraic equations was given by RIEBESELL[6] in 1914. The summaries and bibliographies published in these papers have been very helpful in the preparation of this article, and will be made use of below without further acknowledgment.

The diverse methods which have been proposed for solving algebraic equations mechanically, other than the strictly numerical methods based upon the use of calculating machines, fall naturally into about six types, and we shall discuss these in the succeeding paragraphs, as follows: (2) Graphic and visual methods. (3) Kinematic linkages. (4) Dynamic balances. (5) Hydrostatic balances. (6) Electric and electromagnetic methods. (7) Methods of harmonic analysis. Of these the first four are usually restricted to real roots, whereas the last two may be used to find the complex roots of equations. All these types include machines both for algebraic equations in one unknown, and for simultaneous linear equations in several unknowns.

In our description of various devices, it will be less confusing to the reader if in most cases we adopt a standard notation for a polynomial whose zeros are to be found, which may differ in several instances from those used by the authors we quote. Let it be required to determine the roots $z = x + iy$,

($i^2 = -1$), of the algebraic equation

$$(1.1)\qquad f(z) \equiv c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n = 0,$$

where the coefficients $c_m = a_m + ib_m$ may be real or complex. The letter $R$ will denote any convenient upper bound for the absolute values of the roots of $f(z)$. A suitably chosen integral of $f(z)$ will be denoted by $F(z)$, and its zeros by $Z_1, \cdots, Z_{n+1}$. To denote real coefficients we shall write $a_m$ instead of $c_m$. If only real roots are to be found, the variable $z$ will be called $x$. Thus the notation

$$(1.2)\qquad f(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0,$$

will imply the problem of finding real roots of a polynomial equation with real coefficients. In such cases $y$ will often be used to denote $f(x)$.

**2. Graphic and visual methods.** Twenty-five years after Rowning's paper, LAGRANGE[7] described a graphic method of solving algebraic equations. To solve the equation $f(x) = 0$, Lagrange lays off on the $y$ axis ($L_0$) the $n + 1$ directed segments $OB_0 = a_0$, $B_0B_1 = a_1$, $B_1B_2 = a_2, \cdots, B_{n-1}B_n = a_n$. The coordinates of the point $B_m$ are seen to be $(0, b_m)$, if $b_m = a_0 + a_1 + \cdots + a_m$. A horizontal line through $B_n$ intersects the vertical $L_1 (x = 1)$ at the point $C_n(1, b_n)$. The line $B_{n-1}C_n$, with slope $a_n$, meets a suitably selected vertical line $L_x$ in a point $P_{n-1}(x, b_{n-1} + a_n x)$; a horizontal through $P_{n-1}$ meets $L_1$ in $C_{n-1}(1, b_{n-1} + a_n x)$; the line $B_{n-2}C_{n-1}$ with slope $a_{n-1} + a_n x$ meets $L_x$ in $P_{n-2}(x, b_{n-2} + a_{n-1}x + a_n x^2)$. Successive points $P_m$ are constructed in this way on $L_x$ until finally the point $P_0$ is found, whose coordinates are $(x, f(x))$. The locus of $P_0$, for various lines $L_x$, is the graph of the polynomial $y = f(x)$, and the roots are found whenever $P_0$ lies on the $x$-axis.

Fewer construction lines are involved in the graphic method of LILL[8] (1867). If we define the algebraic quantities $y_m$ by the successive relations

$$(2.1)\qquad y_n = 0, \qquad y_{m-1} = -x(a_m - y_m), \qquad m = n, n-1, \cdots, 2, 1,$$

then

$$(2.2)\qquad y_{n-1} = -x a_n, \qquad y_{n-2} = -x(a_{n-1} + x a_n), \cdots, y_0 = a_0 - f(x).$$

The problem is reduced to constructing successively the segments $y_{m-1}$, $m = n, n-1, \cdots, 1$ and finding $x$ by trial so that $y_0 = a_0$. A rectangular framework introduces the coefficients as follows. Starting at $O$, lay off $OA_n = a_n$ as a directed segment along the $x$-axis, lay off $A_nA_{n-1} = a_{n-1}$ as a directed segment parallel to the $y$-axis, lay off $A_{n-1}A_{n-2} = a_{n-2}$ as a directed segment with the positive sense opposite to that of the $x$-axis; and continue at each stage to rotate the positive sense through 90°. Now for any assumed value of $x$ draw a line through $O$ with slope $-x$ intersecting $A_nA_{n-1}$ (extended if necessary) at a point $P_{n-1}$ (this is such that $A_nP_{n-1}$ will equal $y_{n-1}$); draw a perpendicular to this line at $P_{n-1}$ intersecting $A_{n-1}A_{n-2}$ in $P_{n-2}$; draw a perpendicular to $P_{n-1}P_{n-2}$ at $P_{n-2}$, etc., until finally a point $P_0$ is located on $A_1A_0$, such that $P_0A_0 = f(x)$. If $x$ is so chosen that $P_0$ coincides with $A_0$, it will be a root of $f(x) = 0$.

This method of Lill was somewhat modified by CREMONA[9] in 1874, and was reviewed by Moritz.[4]

A graphic method published by CUNYNGHAME[10] in 1886 gives the real roots of an equation of the form $x^n + Ax + B = 0$ to two decimals, if the curve $y = -x^n$ be first drawn on a suitably large scale. The author's notation is cumbersome, but his idea can be expressed simply by the use of coordinates. Let a line of slope $A$ be drawn through $(0, B)$. Then the intersections of the curve $y = -x^n$ with this line $y = Ax + B$ define the real roots of the given equation. In the case of the cubic, the author discusses the computation of the "impossible" (what we should call imaginary) roots. Let $P(x, y)$ be a real intersection of the line $y = Ax + B$ with the cubic $y = -x^3$, and let $m$ be the slope of the line drawn from $P$, tangent to the cubic at another point. Then it is easily shown that the three roots of the equation are $x$ and $-(x/2) \pm \sqrt{m - A}$. Hence if $m < A$, two roots are imaginary, but these are readily computed from the two slopes. The author suggests the use of a special protractor on which slopes can be read directly, and marks the abscissas directly on the curve $C$ instead of along the $x$-axis to simplify reading of the roots.

Extending the graphic idea from the trinomial to numerical equations of four or five terms, MEHMKE[11] describes an apparatus which was displayed in a mathematical exhibit in Munich in 1893. The theory is based on the fact that if four curved scales in space are cut by a plane, then the four readings will satisfy a functional relation. In Mehmke's model, three of the scales are uniform vertical scales $A$, $B$, $C$, parallel to each other but not coplanar. A line is determined by a string fastened between a point marked $u$ on the $A$ scale and a point marked $w$ on the $C$ scale. A plane is then determined by viewing this line through a sliding eyepiece set at $v$ on the $B$ scale. For an equation of four terms, a non-uniform curved scale, suitably graduated, is viewed through the eyepiece and seen to cut this plane in one or more points which define the required roots. For an equation of five terms a one-parameter family of such curves is used. The curves are constructed as follows. Let the given equation be

$$f(x) = x^m + ux^n + vx^p + wx^q = f. \tag{2.3}$$

On a particular curve corresponding to the parameter $f$, the point marked $x$ is such that its projection on a horizontal plane cutting the $A$, $B$, $C$ scales in a triangle $ABC$ would be the centroid of masses $x^n$, $x^p$, and $x^q$ placed at $A$, $B$, $C$, respectively; and its vertical projection $s$ on one of the three parallel scales $A$ or $B$ or $C$ is made to be $s = (f - x^m)/(x^n + x^p + x^q)$. Hence the intersection of this curve with the visual plane locates the root of the equation

$$s = \frac{ux^n + vx^p + wx^q}{x^n + x^p + x^q} = \frac{f - x^m}{x^n + x^p + x^q}. \tag{2.4}$$

**3. Kinematic linkages.** The mathematical theory of kinematic linkages, discussed long before by Rowning, awaited the day of precision machinery before it could be considered practical. In the meantime various theoretical devices were discussed in mathematical papers. The equiangular linkage described by KEMPE[12] in 1873 is a device for obtaining real roots of equations with real coefficients. It is constructed of $n + 1$ links $L_0, L_1, \cdots, L_n$, of lengths $l_0, l_1, \cdots, l_n$, joined consecutively at points $P_0, P_1, \cdots, P_{n-1}$, so that the first link $OP_0$ lies along the positive $x$-axis with its left end $O$ at the origin, and so that each of the consecutive links is constrained to make the same exterior angle $\phi$ with its predecessor. The horizontal projection of the directed link $L_m(=\overrightarrow{P_{m-1}P_m})$ is then equal to $l_m \cos m\phi$, and hence the horizontal projection of the directed line $\overrightarrow{OP_n}$ joining the two end-points of the linkage is

$$l_0 + l_1 \cos \phi + l_2 \cos 2\phi + \cdots + l_n \cos n\phi. \tag{3.1}$$

To find the roots of an algebraic equation $f(x) = 0$ it is necessary first to rewrite the equation in the form (3.1) and then to find those angles $\phi_i$ for which the free end $P_n$ of the linkage lies on the $y$-axis. If $R$ is any convenient upper bound of the roots of $f(x)$, we may set $x = R \cos \phi$, and then calculate the Fourier coefficients of the trigonometric polynomial $f(R \cos \phi)$. The lengths $l_m$ can be taken as any convenient multiples of these coefficients, which are linear combinations of the given coefficients $a_m$ of $f(x)$.

Although Kempe did not attempt a description of a mechanical method for producing his equiangular linkage, such an attempt was made later by BLAKESLEY. In 1907 he published a paper entitled "Logarithmic lazytongs and lattice-works,"[13] and then in 1912 he used the lazytongs idea to devise a mechanical construction for Kempe's equiangular linkage.[14] The construction which he gave was erroneous, however, and did not really produce an appropriate linkage except in special cases. The following is Blakesley's construction. On opposite sides of each of the segments $P_{m-1}P_m$ of Kempe's open polygon $OP_0P_1 \cdots P_n$, two isosceles triangles $P_{m-1}A_mP_m$ and $P_{m-1}B_mP_m$ are drawn forming a kite-shaped quadrilateral $P_{m-1}A_mP_mB_m$. The points $A_0$ and $B_0$ are arbitrary points on the perpendicular bisector of $OP_0$, but the other vertices $A_i$ and $B_i$ are chosen successively as the points where $A_{i-1}P_{i-1}$ and $B_{i-1}P_{i-1}$, produced, meet the perpendicular bisector of $P_{i-1}P_i$. Then $2n - 2$ rods,—one through $A_{i-1}P_{i-1}A_i$ and one through $B_{i-1}P_{i-1}B_i$, $(i = 1, 2, \cdots, n - 1)$,—are hinged to each other in pairs at $P_{i-1}$ and are further hinged at their common extremities $A_i$ and $B_i$. Two rods connect $A_0$ and $B_0$ to $O$, and two rods connect $A_n$ and $B_n$ to $P_n$, but the vertices of Kempe's polygon are *not* connected rigidly to each other by rods $P_{i-1}P_i$. The joints $P_i$ in this configuration can be shown to form an equiangular linkage as the lattice is articulated. Blakesley's error arose in asserting that the ratios of the segments $OP_0$, $P_0P_1$, $P_1P_2$, $\cdots$, etc., remain constant. If this were true we should indeed have a machine for applying Kempe's theory. But in fact, only the ratios of alternate segments remain constant in the general case, although all ratios happen to do so in such special cases as the logarithmic lazytongs in which the successive lengths form a geometric progression. Blakesley's device fails as an exact solver of algebraic equations.

It may be of interest nevertheless to note Blakesley's alternate method of transforming the given equation $f(x) = 0$ into a Fourier series. This he does by setting $x = \tan \phi$, and computing the Fourier series for the trigonometric polynomial $(\cos^n \phi) \cdot f(\tan \phi)$. Separate linkages represent the cosine and sine terms of the series, and their extremities are fastened at right angles to each other to produce a single kinematic linkage.

An entirely different mechanism for solving equations of the $n^{\text{th}}$ degree, based essentially on the graphic method similar to that of Lagrange (1795), was published in the same year (1912) by MUIRHEAD.[15] First, by a transformation of the polynomial which replaces a negative root by a positive root, or a root $<1$ by its reciprocal, the problem is reduced to finding a root $>1$. Two parallel shafts $s$ and $s'$ are separated by the variable distance $x-1$. In the same plane, on a line $PP'$ perpendicular to $s$ and $s'$ (which cuts $s$ in $P$ and $s'$ in $P'$, and which we shall call the axis), two centers $O$ and $O'$ are fixed so that $OP = P'O' = 1$, $OP' = PO' = x$. To find a root $>1$ of the equation (1.2), mark a directed distance $PP_n = a_n$ on the shaft $s$ and fix it mechanically, let a rod $OP_n$ intersect $s'$ in $P_n'$, lay off and fix mechanically the directed distance $P_n'P'_{n-1} = a_{n-1}$ on the shaft $s'$, let a rod $O'P'_{n-1}$ meet $s$ in $P_{n-1}$, and lay off $P_{n-1}P_{n-2} = a_{n-2}$ on $s$, etc. To avoid mechanical interference, the successive rods can be placed in parallel planes instead of in the same plane. The machine is articulated by pulling the shafts $s$ and $s'$ apart. Whenever the point $P_0$ or $P_0'$ lies on the axis, the corresponding root $x$ can be read off.

In the same paper Muirhead described a similar mechanism for solving a set of simultaneous linear equations. Each of the variables $x$, $y$, $z$, etc., requires a pair of shafts ($X$, $X'$, and $Y$, $Y'$, and $Z$, $Z'$ etc.), separated by a unit distance, whose distances from a fixed reference shaft are constrained to be $x$ and $x+1$, $y$ and $y+1$, $z$ and $z+1$, etc. Each equation ($a_ix+b_iy+c_iz+d_i=0$, for example) requires a set of rods of which consecutive pairs are hinged on the $X$, $Y$, $Z$ shafts, and are forcibly separated by directed distances $a_i$, $b_i$, $c_i$ along the $X$, $Y$, $Z$ shafts. The first rod is fixed at $O$ on the reference shaft, and the last is separated by the distance $d_i$ from $O$ on the reference shaft. Ways are devised to avoid mechanical interference.

Two years earlier (in 1910) NÄBAUER[16] described a rather different kinematic machine for solving simultaneous linear equations. The fundamental unit in his machine consists of a horizontal rod bearing two equal gears of radius $r$ in vertical planes, one of which rests and rolls on a horizontal turntable, and can be slid back and forth on its axis so as to rest at a required distance $ar$ from the center of the turntable. As the turntable turns through an angle $x$, the two equal gears turn through an angle $ax$. To avoid slipping, nine gear circles, for $a = 1, 2, 3, \cdots, 9$, are provided on the turntable. A separate turntable mechanism is provided for each of the variables $x$, $y$, $z$, and the contributions $ax$, $by$, $cz$ are added by differential gears to produce a given sum $l$. Three such sums $a_ix + b_iy + c_iz = l_i$ can be formed with three sets of turntables, mounted so that each of the variables $x$, $y$, $z$ has a vertical axis for itself. If the sums $l_1$, $l_2$, $l_3$ are prescribed, then the variables $x$, $y$, $z$ are determined. Since friction alone is insufficient to produce the desired constraint, gears are used. Since the gears restrict the coefficients $a_i$, $b_i$, $c_i$ to one-digit integers, a method of successive approximation is outlined which overcomes this difficulty. To a person familiar with early models of the differential analyzer, this machine demonstrates some of the same ideas and the same difficulties.

Two decades later a large machine for the mechanical solution of simultaneous equations was built at the Massachusetts Institute of Technology. An interesting description of the machine by WILBUR[17] is accompanied by plates showing the machine and some of the details of its operation. Ten plates $P_j$, one for each of ten homogeneous variables, are mounted to rotate about parallel horizontal axes fastened in a rigid frame. Each plate contains ten slots perpendicular to its axis, nine of which are provided with midrunners on which a positive or negative coefficient $a_{ij}$ may be set approximately by hand, and then adjusted with micrometer screws to .0005 in. or 4 significant figures. The front slot on each plate is used for an accurate reading of sin $\theta_j$, the sine of the angle of rotation of the plate from the horizontal. The nine slots in a given plate are each associated with one of nine given linear equations—which can be considered as homogeneous in ten unknowns to be read as sin $\theta_j$, or as non-homogeneous in nine ratios of these unknowns. The ten midrunners corresponding to a single equation lie in a vertical plane and are connected by a 60 ft. flexible steel tape. Fastened at a point $A$, the tape rides on 10 ball-bearing top runners, from each of which it loops down to a pulley on a midrunner. Then it passes down through a clamp $C$ to the bottom, where it passes under a set of 10 bottom runners, from each of which it loops up to a second pulley on the midrunner, and is finally made fast at its end $B$. When the clamp $C$ is released, all the plates are free to rotate arbitrarily, since any decrease in length of an upper loop is compensated by an increase in the corresponding lower loop, and vice versa. The total lengthening of $CB$, measured along the tape, is

$$2\sum a_{ij}\sin\theta_j + 2D_i \sin\theta_0 \tag{3.2}$$

if the "coefficients" $a_{ij}$ are set as displacements on nine of the plates $P_j$, and the "constants" $D_i$ are set as displacements on the tenth plate $P_0$. To solve the equations

$$\sum a_{ij}x_j + D_i = 0 \tag{3.3}$$

the clamp $C$ is made fast and one of the plates is rotated with an oscillating motion which constrains the others to move with it. Then the unknowns $x_j = (\sin\theta_j)/\sin\theta_0$ may be read from the machine.

Having found $x_i$ (usually to within 1% of the largest unknown, but this depends upon the stability of the system), the left-hand sides of (3.3) are computed accurately to as many figures as desired, with a computing machine. If these values $d_i$ do not approximate 0 sufficiently, they may be reset on the "constant" plate $P_0$ as new constants in a set of equations with the same coefficients $a_{ij}$, of which the solutions are the errors of the first approximations. By repeating this procedure, as high a degree of accuracy may be obtained as is desired.

**4. Dynamic balances.** A paper by BÉRARD[18] in 1810 was the first to suggest using an ordinary balance for finding real roots of algebraic equations. His idea was improved by LALANNE[19] in 1840. The principle of the Lalanne balançe was to have weights, proportional to the coefficients $|a_m|$, exert forces placed at directed distances from a fixed reference line of $+x^m$ if $a_m$ is positive, or of $-x^m$ if $a_m$ is negative, so as to produce moments $a_m x^m$ whose sum should be in equilibrium. For convenience in staying within a finite portion of the plane, a weight $10|a_2|$ at distance $\pm x^2/10$ can replace a weight $|a_2|$ at distance $\pm x^2$, etc. Balanced on a horizontal knife-edge is a horizontal rectangle in which guide curves, representing $\pm x^m$ divided by a suitable power of 10, are marked either by rigid curved wires or by slots in a sheet of metal. From the guide curves hang appropriate weights representing the coefficients. The constant term $c_0$ is represented by a weight $|c_0|$ hung on one of two fixed hooks on opposite sides of the knife-edge. By means of a horizontal indicator slide, perpendicular to the knife-edge, all the hooks are constrained to hang at the same distance $x$ from an initial reference line. As the slide is moved from left to right, the real roots are read off at positions of equilibrium.

COLLIGNON,[20] in 1873, extended Lalanne's balance in a theoretical way to a complex variable, hanging weights at points representing the complex numbers 1, $z$, $z^2$, etc.; but he was not successful in devising a practical machine to move the hooks continuously in the prescribed manner. In 1881 EXNER[21] devised a turning balance in which the weights are placed on spiral curves.

Systems of levers were hinged together in a machine designed by BOYS[22] in 1886. In this machine a set of $n + 1$ horizontal axes are placed at levels 0, 1, 2, $\cdots$ $n$, above a fixed pivot. The even-numbered axes are equally spaced in a fixed vertical plane $s$ and the odd-numbered ones are equally spaced at intermediate levels in a parallel vertical plane $s'$ at distance $x + 1$ from $s$. Intersecting the $m^{\text{th}}$ axis at right angles there is a horizontal lever with weight pans at distances marked $+1$ and $-1$, of which the one whose sign is that of $a_m$ is to carry a weight $|a_m|$ so as to produce a moment $a_m$. Positive moments are clockwise on the left-hand even axes and counterclockwise on the right-hand odd axes. The $m-1^{\text{th}}$ lever is supported by a sliding joint fastened to the positive pan on the $m^{\text{th}}$ lever which is at distance $x$ from the $m-1^{\text{th}}$ axis. Thus the weight $|a_n|$ on the $n^{\text{th}}$ lever produces a moment $a_n x$ on the $n - 1^{\text{th}}$ axis, to which is added the moment $a_{n-1}$ from the scale pan. A total moment of $(a_{n-1} + a_n x)x$ is communicated to the $n - 2^{\text{th}}$ lever, to which is added the moment $a_{n-2}$ of the weight $|a_{n-2}|$, etc. Finally, on the bottom $0^{\text{th}}$ lever the total moment is $f(x)$. The two vertical planes $s$ and $s'$ are to be moved apart until equilibrium is reached. Real positive roots can thus be read off within limits determined by the size of the machine.

Ten years later (1896) GRANT[23] described a similar machine, with which roots from 1 to $\infty$ could be read off on a reciprocal scale, on which the number $x$ is marked at a distance $1/x$ from the mark $\infty$. Instead of alternating the $n + 1$ horizontal axes in two banks, as Boys did, Grant makes them all lie in an inclined plane, displacing them horizontally by worm gears from an initial vertical reference plane through the 0-axis in such a way that the $m$-axis is displaced $m$ times as far from reference plane as the 1-axis is displaced. Scale pans are again hung at distances $\pm 1$ from the axes, but the supports come at distances $1/x$ from the axes. This machine, like that of Boys, is used only to find the real roots, which are read at equilibrium positions. It is interesting, in view of later inventions, to note Grant's statement that "an imaginary root is as impossible mechanically as it is arithmetically."

Although a paper published by SKUTSCH[24] in 1902 refers primarily to hydrostatic balances, and will be discussed later in that connection, it should be mentioned here for its description of a kinematic circular balance. Like spokes of a wheel, a set of $n + 1$ rods of length $r$ emanate from a center $O$ to points $A_0, A_1 \cdots A_n$; the whole figure being in a vertical plane. At each endpoint $A_m$ is a geared pulley wheel (also in a vertical plane) around which is hung a chain suspending a weight pan with a weight adjusted to produce a moment $a_m$ on the $m^{\text{th}}$ gear. A rod $C_mD_m$ of length $2l$ is fastened rigidly to this $m^{\text{th}}$ gear at its midpoint $A_m$. Connecting adjacent points $C_m$ and $D_{m+1}$ is a link $C_mD_{m+1}$, making an angle $\gamma$ with $C_mD_m$ and an angle $\delta$ with $D_{m+1}C_{m+1}$. The ratio $x = \sin \gamma / \sin \delta$ is adjusted to be the same for each of the $n$ links. Then a balance of moments is obtained when $x$ is a root of $f(x) = 0$.

Still another dynamic root-finder was described by PEDDIE[25] in 1912. A set of $n + 1$ pulleys, numbered from 0 to $n$, are so arranged in a vertical plane that the $n^{\text{th}}$ pulley is fixed, the $n - 1^{\text{th}}$ is supported at its center by a string passing over the $n^{\text{th}}$ pulley, and each successive pulley is supported by a string passing over the previous one, until the string over the last pulley (numbered 0) carries a weight to pull the strings all taut. The other ends of the $n + 1$ strings are unwound from drums and are all kept parallel at an inclination of $\theta$ with the horizontal, where $x = 1 + \sin \theta$ is the root to be found. If a length of string $k$ is unwound from the pulley numbered $m$, then it can be shown that the pulley numbered $m - 1$ will unwind a length of string $kx$, and that this will finally cause a length $kx^m$ to unwind from the bottom pulley (numbered 0). Hence if the machine is set by unwinding lengths of string equal to the coefficients $a_n, a_{n-1}, \cdots, a_0$ from the drum ends of the respective strings, the total displacement of the weight will be $f(x)$. At angles $\theta$ for which this displacement is 0, the root is read as $x = 1 + \sin \theta$. The method is obviously limited to finding real roots between 0 and 2, but can be used for finding the real roots of any polynomial if suitable preliminary transformations are first applied to the polynomial. A modification of this principle uses springs instead of gravity to pull the pulleys into position, and the pulleys are all placed on a movable arm which can turn from $\theta = 0°$ to $\theta = 90°$.

For further details on various dynamic balances, the reader is referred to Riebesell's paper.[6]

**5. Hydrostatic balances.** Closely related to the dynamic balances just discussed are the hydrostatic balances in which upward forces can be introduced which are proportional to the volumes displaced by appropriately chosen solids. A description by Ghersi of some of these methods has already been referred to above.[5]

The method proposed by DEMANET[26] in 1898 is adapted to the solution of the cubic equations $x^3 + x = c > 0$ and $x^3 - x = c > 0$. A cylindrical vessel and a conical vessel, both open at the top, are connected underneath by a pipe so that water in the two vessels will reach the same level. If the tangent of the semi-vertical angle of the cone is $\sqrt{3/\pi}$, then the volume of liquid in the cone will be $x^3$ when the surface is $x$ units above the vertex. If the cylinder has unit cross-section, it will then contain $x$ cubic units of water. Hence if $c$ cubic units of water be added above the level of the vertex of the cone, the depth $x$ will be the required root of $x^3 + x = c$. For the equation $x^3 - x = c$, a cylinder is used to displace a volume $x$ of liquid within the cone. The cubic $y^3 \pm py = q$ can be solved by first writing $y = x\sqrt{p}$.

MESLIN'S[27] contribution in 1900 was a horizontal dynamic balance. Solids whose volumes, cut off at a level $x$, are proportional to $x, x^2, x^3, \cdots$, are hung on a lever at directed distances $a_1, a_2, a_3$ from the fulcrum, and the constant $a_0$ is represented by a weight $|a_0|$ in one of two pans fastened on opposite ends of the lever. Before adding the weight $|a_0|$, the weights of the various solids have to be balanced by a suitable weight in a scale pan. Then the solids are suspended in two interconnected vessels, the weight $|a_0|$ is added to the proper scale pan, and water is made to flow from a third vessel until the balance is again restored to equilibrium. The depth $x$ to which the solids are submerged gives the root. In his paper Meslin discusses in some detail the exact dimensions of the different solids to be used in his balance.

Two hydraulic methods were proposed by EMCH[28] in 1901 for extracting the $n^{\text{th}}$ root of any number. The first method, which was extended in a second paper to the solution of an arbitrary equation of the $n^{\text{th}}$ degree, was essentially the same as that of Meslin, and involved the depth of immersion of certain solids in water. The other, less accurate, depended upon the time required to empty a suitable vessel (whose interior is a surface of revolution), through a small orifice of area $a$ in the bottom.

The radius $r$ at height $x$ is obtained as follows:

$$t = \sqrt[n]{x}, \qquad dt = \pi r^2 dx / a\sqrt{2gx} = (\sqrt[n]{x}/nx)dx, \tag{5.1}$$

or

$$r^2 = (a\sqrt{2g}/n\pi)\sqrt[n]{x}/\sqrt{x}. \tag{5.2}$$

A new approach to the same problem was made in 1902 by SKUTSCH[24] who considered not only horizontal positions of equilibrium on a hydrostatic balance, but also positions of equilibrium at an angle $\sin^{-1}\eta$ with the horizontal. After defining a balance as "a kinematic chain set up for the application of arbitrary forces, and used for finding relationships between the magnitude and position of these forces in equilibrium," Skutsch describes balances of MASSAU[29] (1878), Grant, and Meslin. He then uses a construction, similar to that of Meslin, but instead of the single depth variable $x$, he uses two variables $\xi$, $\eta$, and solves the equation $\sum a_\nu(\xi + a_\nu \eta)^\nu = 0$. By means of a float, the inclination $\sin^{-1}\eta$ is related to the depth by the relation $\eta = \xi/c$. This makes it possible to get the roots as functions of $\eta$ and avoids the difficulties created by negative coefficients.

A hydrostatic machine for solving a system of simultaneous linear equations was developed by FUCHS in three papers.[30,31,32] The last and longest of these contains the important elements in the description of his machine. A system of stationary containers, partly filled with water, are arranged in rows and columns. Each row corresponds to an equation $a_i x + b_i y + \cdots = l_i$. Each column corresponds to a variable $x$, $y$, etc. The containers in any column are interconnected so that the water will rise in all of them an amount equal to the corresponding unknown: $x$, $y$, etc. In each row, hollow floating cylinders with cross-sections $a_i$, $b_i$, are fastened to a beam which is loaded with a weight proportional to $l_i$. The cylinders are thus depressed into their containers so that the total displaced water, nearly $a_i x + b_i y + \cdots$, is balanced by the load $l_i$. Provision is made for negative coefficients by letting water flow into the corresponding float and weight it down, instead of being displaced by the float.

Other variations of the same idea provide for inner and outer water levels for each term, one to be filled for positive and the other for negative terms. Changes in level indicate the values of the variables. Another variation interchanges the role of rows and columns so that a loading $l_i$ of water is run into the inner cylinders of one column and the values of the unknowns are read off as the weights thus applied to the bars in the several rows. Still another machine uses a system of containers in which water communicates by rows, and oil, floating on the water, communicates in columns.

More recently (1940) these hydraulic principles were applied by SCHUMANN[33] in designing a mechanism for calculating regression equations and multiple correlation coefficients, which is also adaptable to the solution of simultaneous linear equations. He estimates that a skilled computer working with slide rule and adding machine requires $5m^2 + m^3/4$ minutes to solve a problem in $m$ variables, whereas his machine will solve the problem in $6m$ minutes. To each independent variable corresponds a horizontal beam $B_j$ (somewhat like the plates in Wilbur's machine[17]) from which the given values $y_{ij}$ of the unknowns $y_j$ protrude as directed spikes in a horizontal plane. On the end of each spike is a pulley $P_{ij}$. A rotation of $\beta_j$ in the beam for a variable $y_j$ lifts the $i^{\text{th}}$ pulley a distance $y_{ij} \sin \beta_j$ and releases twice that length from a cord supporting the pulley. Each equation is represented by a cord. The $i^{\text{th}}$ cord, with one end displaced upward by an amount $x_i$, then comes down to engage the pulley on the spike $y_{i2}$, then up over another pulley and back down for the next variable $y_{i3}$, etc., and finally ends in a heavy float $F_i$ suspended in mercury. If we set $k_j = 2 \sin \beta_j$, then the float $F_i$ will be lifted an amount $x_i - \sum y_{ij} k_j$ when the beams $B_j$ rotate through $\beta_j$ and the hooks for the dependent variables are raised by $x_i$. The tension in each cord is proportional to that displacement. If there are more equations than independent variables, the coefficients $k_j$ which the machine indicates are shown to be the regression coefficients. By articulating one beam at a time with the others clamped, a set of rotations $\gamma_j$ can be found from which the multiple correlation coefficients can be computed with relative ease.

**6. Electric and electromagnetic devices for finding real and complex roots of algebraic equations.** A fascinating series of papers published in 1888 by LUCAS gives the reader an opportunity to watch an important new scientific idea being born. A first paper[34] presents a generalization of Rolle's theorem. In this paper a set of unit masses, placed at the zeros $Z_1, Z_2, \cdots, Z_p$ of a polynomial $F(z)$ in the complex plane, are assumed to repel a unit mass $N$ at $z$ with forces inversely proportional to the distances $|Z_j - z|$ from $N$. The components $P$ and $Q$ of the total force are given by the equation

$$P - iQ = F'(z)/F(z). \tag{6.1}$$

Roots of the derived function $f(z) = F'(z)$ are equilibrium positions for $N$. The orthogonal trajectories of the isodynamic lines are curves of degree $2p - 1$ (called "lignes halysiques") passing alternately through the $p$ roots of $F(z)$ and the $n = p - 1$ roots of $F'(z)$, and are used in Lucas' generalization of Rolle's theorem.

A second paper[35] considers placing electrodes instead of masses at points $Z_1, Z_2, \cdots, Z_{n+1}$. The equipotential curves are Cassinian ovals whose nodal points occur at the $n$ roots of the derivative $F'(z)$, which we denote by $f(z)$. In the experimental method due to GUÉBHARD,[41] a large circular conductor is used in place of a Cassinian oval at the outer boundary of a salt solution in which a thin plate of polished metal is immersed. The points of a bundle of wires are then placed as electrodes representing as nearly as possible the points $Z_1, Z_2, \cdots, Z_{n+1}$. Colored rings deposited by electrolytic action furnish a diagram of equipotential lines whose nodal points are the $n$ roots $z_1, z_2, \cdots, z_n$ of the derivative $F'(z)$. Such figures, obtained experimentally for the second degree binomial and the fourth degree trinomial, are displayed in the above quoted work by Guébhard.[41]

A third paper of Lucas[36] gives an electric resolution of algebraic equations with numerically given real or complex coefficients, by reducing the given equation to an equation of lower degree. An equation $f(z) = 0$ of degree $n$, can be solved electrically, as described above, if we know the $n + 1$ roots of the function $F(z)$, which is an integral of $f(z)$. In applying the first reduction method of Lucas we suppose some integral of $f(z)$ to be separated into even and odd functions $\phi(z^2) + z\psi(z^2)$, and we then calculate one of the $[n/2]$ roots of the polynomial $\psi$ considered as a function of $z^2$. If this root be $\lambda^2$ then a particular integral of $f(z)$, namely $F(z) = \phi(z^2) - \phi(\lambda^2) + z\psi(z^2)$, is divisible by $z^2 - \lambda^2$, and the quotient is a polynomial of lower degree, $n - 1$, whose roots together with $+\lambda$ and $-\lambda$ are the required electrode points for solving $f(z)$. For example, in solving the biquadratic equation $f(z) = 5z^4 + 10z^3 + 3z^2 - 2z - 6 = 0$, we have $\psi(z^2) = z^4 + z^2 - 6$, which has the factor $z^2 - 2$. Then choosing $F(z) = (z^2 - 2)(z^3 + 3z + 2.5z^2 + 4)$ we next find three roots of the cubic factor. An integral of the cubic is $[(5/6)z^2 + 4] \times (.3z^2 + z + .36)$, of which the first factor was the "$\psi$" function for this cubic. Using the four roots of this factored biquadratic as electrode points, we solve the cubic electrically; then we use these three nodal points and the two points $\pm\sqrt{2}$ as five electrode points to solve the given equation electrically. An alternative method of reduction is also given, which has the disadvantage of introducing extraneous roots, and which we shall not describe here.

A fourth paper [37] discussing the electric determination of the isodynamic lines of any polynomial—which are the equimodular lines of its logarithmic derivative—describes the field of force due to a system of repulsive centers at the roots of $F(z)$, combined with a system of attractive centers at the roots of the derivative $F'(z)$.

The climax of these investigations is reached in a fifth paper,[38] in which the use of partial fractions provides Lucas with a simple and direct procedure for locating the complex roots of a polynomial with real coefficients. An auxiliary polynomial $p(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_{n+1})$ is chosen with distinct roots at convenient points $L$ on the $x$-axis. Instead of working with $f(z)$ and using the roots of its integral as electrode points, Lucas now works with the function $f(z)/p(z) = \sum \mu_j/(z - \lambda_j)$ and its exponential integral

$$F(z) = (z - \lambda_1)^{\mu_1}(z - \lambda_2)^{\mu_2} \cdots (z - \lambda_{n+1})^{\mu_{n+1}}. \tag{6.2}$$

The constants (residues) $\mu_j = f(\lambda_j)/P'(\lambda_j)$ are easily computed and are introduced into the machine as charges on the fixed electrodes at the points $z = \lambda_j$. The roots of the logarithmic derivative of $F(z)$ are the required roots of the given polynomial $f(z)$, and appear as nodal points of the isodynamic lines. Experimentally these can be deposited electrolytically on a polished metal plate, or they can be traced out by the Kirchhoff and Carey Foster method. In the perfected model of the Lucas electric equation-solver, given in a sixth paper,[39] the need for an outside conductor is eliminated by choosing $n + 2$ instead of $n + 1$ sources and sinks on the axis. The algebraic sum of the charges on the $n + 2$ electrodes is zero, and the apparatus is somewhat simpler to construct. Still another paper published two years later[40] suggests a method of introducing the coefficients into the machine by means of resistances proportional to $1/\mu_j$.

An electric device proposed by KANN[42] in 1902 was the result of his inventing a balance similar to that of Lalanne,[19] without having been at first aware of Lalanne's result. When Mehmke called this to his attention, he conceived the idea of carrying out the same principle electrically, replacing the moments of weight by variable resistances. In his mechanical balance are slits or wires in the shape of curves $\pm x$, $\pm x^2/10$, $\pm x^5/100$, etc., fastened in a horizontal plane from which are suspended weights proportional to the coefficients. These weights are constrained to lie under a movable guide slot perpendicular to the $x$-axis on which the plane is balanced. Values of $x$ for which the moments are in equilibrium are the required roots. In Kann's first electrical model, each of the functions $\pm x$, $\pm x^2/10$, etc., is represented by a curve of heavy wire fastened in a removable template, which is to be mounted vertically by sliding it into one of several pairs of slots at the top and bottom of a frame. For positive terms the templates are slid in right side up, and for negative terms up side down. A sliding vertical frame in a plane perpendicular to all these templates, and perpendicular to the horizontal axis in each, serves as a root finder. The coefficients $a_m$ are introduced by placing in the root finder, where its plane intersects the $m^{\text{th}}$ template, a fine wire whose resistance per unit length is proportional to $|a_m|$ (multiplied if necessary by the power of 10 used in the reduction of the template graph). The current then flows through a length of this wire proportional to the ordinate, and thus through a resistance proportional to the given term. These fine wires of considerable resistance make contact with the heavy wire curves of the templates on the one hand, and with leads in the root finder frame on the other hand, and the successive templates are connected in series in such a way that the connecting resistances are either negligible or may be assumed constant and be balanced out by the bridge resistance which controls the constant term in the equation. As the root finder frame is moved in the direction of the $x$-axes of the templates, the roots are found at points where a bridge galvanometer which balances the equation reads zero. At these points an adjusted resistance (equal to the fixed resistances in the circuit, plus the constant resistances introduced by inverting the templates for negative variable terms) just balances a resistance for the given constant term plus all the fixed and variable resistances connected in series in the main circuit.

To avoid having to use wires of different resistances for each new set of coefficients, Kann describes in a final paragraph of the same paper a gear mechanism, one to be associated with each of the templates in the frame, whereby a resistance proportional to a given term may be unwound from a drum in a continuous manner as the root finder is moved along all the given power curves simultaneously. For each template there is in the root finder frame a straight vertical shaft with a notch on top which presses against a rigid metal curve in the template (such as the curve $y = x^2/10$). The lower part of each of these shafts is provided with gear teeth which mesh into an exchangeable gear having a number of teeth proportional to the coefficient of the corresponding term in the equation. The motion of the shaft, thus multiplied by the coefficient $|a_m|$, is made to turn a drum mounted on a horizontal axis, around which wire is wound whose specific resistance takes care of the reduction factor of a power of 10 in the template graph. A weight hanging on the wire which unwinds from the drum maintains the upward pressure on the shaft, and as the root finder moves, a resistance is unwound which is jointly proportional to the coefficient (number of teeth) and to the power of $x$ (ordinate of shaft top). These resistances are connected in series as in the other model, and the roots are found at the zero readings of the galvanometer. The method is limited to the determination of real roots, however.

RUSSELL & WRIGHT,[43] in an electric device constructed in 1909, combine the principle of slide rule multiplication with addition and subtraction. Multiplication is obtained from a thin insulating template in the shape of the curve $\log (y/k) = -x/n$, about which a hundred terms of no. 36 insulated manganin wire is wound, so that the wires are nearly parallel. The area in the interval $x_1 \leq x \leq x_2$ is proportional to $y_1 - y_2$. By adding a fixed resistance in series, the total variable resistance is made proportional to $y_1$, and the number $y_1$ is placed on a logarithmic scale under $x_1$ on the axis of abscissas. Powers of a variable $x$ can be multiplied into the coefficient resistances by setting them on a parallel logarithmic scale. Contact fingers using a tangent scale are used to adjust separately the powers of the unknown. Then terms are added electrically by combining currents in series (or in parallel, using reciprocal logarithmic scales). Finally the real root is obtained when combined currents vanish.

Shortly after the paper of Russell and Wright, RUSSELL & ALTY[44] (1909) brought magnetism across the electric root-finding trail blazed by Lucas; inventing another machine for determining complex roots of equations, and stating that the error in its readings would not exceed 1%. In their electromagnetic method "the horizontal field due to the earth's magnetism is used in an analogous manner to the conducting sheet in Mr. Lucas' method. A drawing board with a slit cut in it, a few pieces of bell-wire, any form of 'charm' compass, ordinary ammeters and rheostats or lamp resistance boards, such as are found in every physical laboratory, can be utilized at once for the experiment."

Using our previous notation, and assuming real coefficients, let

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0, \tag{6.3}$$

and let $p(z) = (z - \lambda_1) \cdots (z - \lambda_n)$, where the $n$ real numbers $\lambda_j$ are subject to the condition $\sum \lambda_j = -a_{n-1}/a_n$. Then let the partial fraction expansion be $f(z)/p(z) = a_n + \sum \mu_j/(z - \lambda_j)$.

In the horizontal complex plane let the earth's magnetic flux $H$ be directed parallel to the imaginary $y$-axis, so that the real $x$-axis is perpendicular to the magnetic meridian. Let vertical wires through the points $(\lambda_j, 0)$ carry currents $C_j$, producing a magnetic force $C_j/5r_j$ at the point $z = x + iy$, where $r_j = |z - \lambda_j|$. The components of resultant magnetic force at the point $z$ are

$$(6.4)\quad -X = \frac{C_j}{5r_j}\frac{y}{r_j}, \quad Y = H + \frac{C_j}{5r_j}\frac{x - \lambda_j}{r_j}; \quad Y + Xi = H + \sum \frac{C_j/5}{z - \lambda_j}$$

By adjusting the currents $C_j$ to equal $5H\cdot\mu_j/a_n$, respectively, the neutral points of zero force will be precisely those for which $f(z) = 0$. Only $n - 1$ ammeters and $n - 1$ rheostats are required for the apparatus.

An electric calculating machine for solving simultaneous linear equations was described by MALLOCK[45] in 1933. Improving upon Mallock's experimental model of 1931, this machine for solving ten equations in ten unknowns, to within an error less than 0.1% of the largest root, was constructed by the Cambridge Instrument Co. Ltd. The machine will also give a direct solution, by least squares, of a set of equations of condition, without forming the normal equations. One closed circuit represents each equation, which is first transformed algebraically so that all coefficients are less than unity. The coefficients are then represented by the relative number of turns of the variable coils in a given circuit, each wound about the transformer corresponding to one of the unknowns. Negative signs are obtained by reversing the current in a given coil, and the answers may be read from voltmeters, attached to unit coils about the several transformers. Compensators are introduced to balance out the effect of resistances in the circuits. Repeated approximations may be used to give greater accuracy.

A recent development in the mechanical solution of algebraic equations is the use of rotating and logarithmically expanding parts to represent arguments and absolute values of complex roots of equations. In an electric machine described by HART & TRAVIS,[46] a set of $n + 1$ coaxially mounted generators $G_0$ to $G_n$ have their rotors rigidly connected together, but the stators of $n$ of them are constrained by gears to rotate through $\theta, 2\theta, \cdots, n\theta$ with respect to the one on $G_0$ which is fixed. The alternating voltage $E \cos (\omega t + k\theta)$ on the $k^{\text{th}}$ generator is then multipled by $a_k$ in a coefficient potentiometer, and by $Z^k$ in a modulus potentiometer. The latter is activated by steel tapes wrapped around a spiral cam whose arc is proportional to the $k^{\text{th}}$ power of the angle of rotation of the cam shaft. (Only roots of modulus $<1$ are read directly, the others being obtained from the reciprocal equation.) The voltages $Ea_kZ^k \cos (\omega t + k\theta)$ are then added in series and connected to an indicator, the voltage on which may be written:

$$(6.5)\qquad (E/2)\sum_k[(a_kZ^ke^{ik\theta})e^{i\omega t} + (a_kZ^ke^{-ik\theta})e^{-i\omega t}].$$

This vanishes, independently of $t$, if and only if

$$(6.6)\qquad \sum a_kZ^ke^{ik\theta} = 0, \qquad \text{or} \qquad f(Ze^{i\theta}) = 0.$$

To find the roots $Ze^{i\theta}$ the angle $\theta$ is turned fairly rapidly and the modulus cam shaft slowly so that $Ze^{i\theta}$ traces a spiral in the unit circle of the complex plane. Points of zero reading are the roots, and they can be read within 2% in modulus, and 1% in argument.

The use of a cam to generate a power of a variable appears also in a later article by GREEN,[47] whose square root extractor is used to change over graphically recorded pressure differences into a graph of rates of flow. One indicator contacts the edge of a cam whose contour is $r = k\theta^2$, while the other is driven by a pulley around the same shaft, producing a square root mechanically.

**7. Methods of harmonic analysis.** The isograph,[48, 49, 50] designed by T. C. FRY of the Bell Telephone Laboratories following a suggestion of A. J. KEMPNER in 1928, and the S. L. BROWN[51, 52] harmonic synthesizer-analyzer, are two modern machines for solving algebraic equations. Both solve the equation $f(re^{i\theta}) = 0$ by summing separately the sine and cosine terms in the function $f(re^{i\theta}) - c_0$, and mapping this function for fixed $r$ as a curve in a complex plane. The isograph can handle only real coefficients, whereas the Brown analyzer as modified by Brown and Wheeler[52] can also solve equations with complex coefficients. For further details on these machines the reader is referred to the reviews which have appeared in *MTAC*.[51, 52]

**8. Calculating machines.** It should not be forgotten that algebraic equations of higher degree in one unknown and simultaneous linear equations in several unknowns can be solved numerically by standard methods of algebra (such as Horner's or Newton's method), using a computing machine to save labor in the processes of addition, subtraction, multiplication and division. An excellent review of the history of Calculating Machines was written by BAXANDALL.[53] One such machine, the Hamann-Automat of the Deutsche Telephonwerke und Kabelsindustrie, Aktiengesellschaft, Berlin, described by WERKMEISTER,[54] has the advantage, in solving simultaneous linear equation such as $\sum a_i x_i = 0$, $\sum b_i x_i = 0$, $\cdots$, that a quotient $b_1/a_1$ obtained by a division can be automatically transferred from the result register to the multiplicand register without being copied, and can then be multiplied again by each of the successive coefficients $a_2, a_3, \cdots$ in the elimination of $x_1$ between the first two equations.

A survey article by LILLEY,[55] previously reviewed in *MTAC*, p. 61–62, mentions not only calculating machines, but such machines as the BUSH differential analyzer.[56] Interesting as this machine is, its primary purpose is the solution of differential equations rather than algebraic equations, so we shall not describe it here.

In conclusion the author wishes to express his thanks to R. C. A. for his suggestions and help in compiling the literature, and to B. H. Bissinger for his assistance in looking up several of the references. In the footnotes which follow, those papers which have not been examined are marked in the usual manner.

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# Zeros of Certain Bessel Functions of Fractional Order

The following tables contain the zeros of $J_\nu(x)$ for $x \leqslant 25$, where $\nu = \pm 1/3$, $\pm 2/3$, $\pm 1/4$, $\pm 3/4$. These zeros were obtained by inverse interpolation in a thirteen-place manuscript of these functions, computed by the NYMTP. The accuracy of the zeros to 10D is guaranteed, and the two additional places have a high probability of being correct.

| $s$ | $j_{+1/4,s}$ | $j_{-1/4,s}$ |
|---|---|---|
| 1 | 2.78088 77239 95 | 2.00629 96717 90 |
| 2 | 5.90614 26988 43 | 5.12306 27427 46 |
| 3 | 9.04238 36635 83 | 8.25795 11756 42 |
| 4 | 12.18134 15289 55 | 11.39646 76969 87 |
| 5 | 15.32136 98260 12 | 14.53629 98843 38 |
| 6 | 18.46192 72456 89 | 17.67675 35868 47 |
| 7 | 21.60278 44489 13 | 20.81754 94222 32 |
| 8 | 24.74382 77961 28 (24.740) | 23.95855 34952 86 (23.955) |