NOTES

(1) Confusion here; both Hansen and Schlömilch give the table for $\frac{1}{2}x = 0(.05)10$, Lommel's table is for $x = 0(.1)20$, and is not an extension. See *MTAC*, v. 1, p. 194, MTE 29.
(2) These errors were noted in BAASMTC, *Bessel Functions, Part I*, p. xvii. All are corrected in the second edition, the correction to $Y_1(2.96)$ being trivial.
(3) All of Table IV has been compared with BAASMTC ms. tables, with the exception of the tables of $e^{-x}I_n(x)$ on page 736. All errors found are listed.
(4) The table of $J_{\pm(n+\frac{1}{2})}(x)$ for $x \leqslant 20$ was compared with Airey's table in the B.A.A.S., *Report for 1925*; for $x > 20$, the table is unchecked. All discrepancies of a unit or more in the sixth decimal have been listed; there are also 42 others less than a unit in amount. The error in $J_{-9/2}(16)$ was noted by Airey.
(5) Given by J. W. Wrench, Jr., *MTAC*, v. 1, p. 366, MTE 58; they were known to Airey for $x \leqslant 20$. Wrench also gives 26 further changes of a unit in the sixth decimal.
(6) These discrepancies were found by comparison with BAASMTC ms. tables, by S. Johnston and the writer. Eight other discrepancies of a unit were also noted. The ms. tables have not yet been fully checked, so that this list must be regarded as provisional.

J. C. P. MILLER

UNPUBLISHED MATHEMATICAL TABLES

Reference has been made to an unpublished table in RMT 262 (Great Britain), 263 (Carsten & McKerrow), 266 (Great Britain), 267 (Great Britain), QR 20 (Miller & Johnston).

39[B].—*Table of Powers of* $z = x + iy$. Manuscript prepared by, and in possession of, the NYMTP.

This ms. gives the exact values of z^n for $n = 1(1)25$, $z = x + iy$, where x and y each ranges from 0 through 10 at unit intervals.

A. N. LOWAN

40[L].—ENZO CAMBI, *Tables of* $J_n(x)$. Manuscript in the possession of the author, a doctor of engineering, Via Giovanni Antonelli 3, Rome, Italy.

These are tables that I have calculated in recent years. The first contains $J_n(x)$ for $x = [0(.001).5; 15D]$ and $n = 1(1)11$, that is to say, to that value of n where $J_n(0.5)$ is of the order of 10^{-15}. Such a table, in conjunction with the well-known addition formula for the Bessel functions, and, for instance, with Meissel's table of $J_n(x)$ for integral values of x up to 24 (see *MTAC*, v. 1, p. 216), makes it possible to calculate directly $J_n(x)$ for $x = [0(.001)24.5; 15D]$.

The second table gives $J_n(x)$ for $x = [0(.01)10.5; 10D]$ and $n = 1(1)29$. By the use of this table it is possible to compute easily the value of any function given in the form of a Neumann series of Bessel functions for $x = [0(.01)10.5; 9 \text{ or } 10D]$. This range of x covers the field of most applications in physics.

For the main table, values of $J_n(x)$ for $x = 0(.05)10.5$ were first computed to 12D, for even n up to $x = 1.5$, power series being used; for higher values of x, derivatives at unit interval of x formed from Meissel's table were used to get values at interval .05 with the aid of Taylor's series. These were checked by differencing, subtabulated to interval .01, and the final values checked by

$$J_0 + 2J_2 + 2J_4 + 2J_6 + \cdots = 1.$$

Values for odd n were formed by recurrence and checked by

$$J_1 + 3J_3 + 5J_5 + 7J_7 + \cdots = \tfrac{1}{2}x.$$

ENZO CAMBI

EDITORIAL NOTE: A great deal of these tables overlap the still unpublished tables of $J_n(x)$ that L. J. C. prepared for the British Association in 1935 and 1936, covering values of x up to 25 and n up to 20, part with 8D and part with 10D. See *MTAC*, v. 1, p. 283.

41[L].—*A*. OSWALD ALWIN WALTHER (1898–), who directed the preparation during 1944–45 of 9 Reports on *Tables of Bessel Functions*, at the Institute of Practical Mathematics, Technische Hochschule, Darmstadt, Germany. *B*. BAASMTC, Unpublished *Tables of Bessel Functions*. In some respects the summary differs from that given *MTAC*, v. 1, p. 282–284. Descriptions of *A–B* published in Great Britain, Admiralty Computing Service, Department of Scientific Research and Experiment, *Miscellaneous Information Sheet—III*, no. SRE/ACS 91, August, 1945, 5p. Mimeographed. 21.5 × 33 cm. Since this *Information Sheet* is available only to certain Government agencies and activities, the British Admiralty has given us permission to reprint the greater part of it in what now follows. It was O. A. Walther, and others, who computed the table of $\Lambda_n(x)$ in JAHNKE & EMDE, *Tables of Functions*, 1933+.

1. Introduction.

Certain unpublished tables of Bessel functions are now in the possession of Admiralty Computing Service. Though general circulation of these tables is impossible it is desirable that the values should be made available to any establishment having a requirement for them. The object of this report is to inform establishments what requests for values of Bessel functions can be met immediately by Admiralty Computing Service. Though, as a rule, particular values only can be supplied, in a few cases a complete table is available on loan, this fact being indicated in the text of this report.

2. List of unpublished German tables.

. . . .

Report No. 1.

$J_n(x)$ to 4S for $n = 9(1)15$, $x = 4.5, 4.8, 5.2, 5.5, 6.5, 7.2, 7.5$.

Report No. 2.

Zeros j'_{nm} of J'_n, together with values of $J_n(j'_{nm})$ to 7D for $n = 1(1)4$, $m = 1(1)10$. Also for $n = 1$, $m = 11$.

Reports Nos. 3, 5 and 8.

These three reports contain $J_n(x)$, for $n = 0(1)30$, $Y_0(x)$ and $Y_1(x)$ to 7D for $x = 0(.2)65$. For x less than n, values less than 10^{-7} are not tabulated, while values between 10^{-7} and 10^{-2} are given to 6S.

Report No. 4.

The coefficients $(n + \frac{1}{2}, m)/2^m$ of Hankel's asymptotic series for $J_{n+\frac{1}{2}}(x)$. These are tabulated to 8S for $n = 0(1)30$, $m = 0(1)n$.

Reports Nos. 6, 9.

These two reports contain $J_{n+\frac{1}{2}}(x)$ for $n = -31(1)+31$, $x = 0(.2)20$; 5S being given for negative values of n, 5D elsewhere.

Report No. 7.

Some numerical examples on the calculation of Bessel functions.

Particular values of the above quantities, or of any deducible from them, can be supplied when required.

3. General review of the German tables.

All the above reports were reproduced by a simple bromide process, with the result that the legibility varies considerably from page to page, according to the quality of the printing: in general the figures are clear but some of the badly printed pages are almost illegible.

A rough indication of the number of errors to be expected was obtained by examination and differencing of a dozen pages taken at random. In some thousand values one large error was found, one obvious typing error and half a dozen errors in sign (in the neighborhood of the zeros). There are errors of one or two units in the end-figure in many places but this is expected as, in some cases, the authors have deliberately printed all the figures available from the computations, rather than sacrifice some for the sake of last-figure accuracy. As a war-time policy this is quite justifiable.

Strong criticism must be made of the awkward, and inconsistent, notations employed to denote very large or very small numbers. This is illustrated by the following examples:

Report No. 3. $0.^{5}626446$ denotes $(6.26446)\ 10^{-6}$
Report No. 9. $2.^{4}2045$ denotes $(2.2045)\ 10^{4}$
2. 2045 denotes 2.2045.

In the second case the notation is explained in the introduction but this can hardly excuse its use. In the cases when a blank space appears between the decimal point and the following figure it is particularly confusing.

An alternative method of spacing the rows of the tables might be preferred but, since there are not more than four columns to the page, no difficulty of interpretation arises.

4. Detailed review of the German tables.

Report No. 1.

The first report contains only odd values and is not of great interest.

Report No. 2.

The zeros j'_{0m} of J'_0 are the same as the zeros j_{1m} of J_1. These are tabulated to 7D for $m = 1(1)40$ in Watson, *Bessel Functions*, p. 748, and are accordingly omitted from the report.

The zeros were calculated using Newton's approximation with two correcting terms. The first approximations to the roots were obtained for m greater than or equal to 5 from McMahon's formula. For m less than 5, $n = 1$, they were found by inverse interpolation in Watson's table of J_1. For m less than 5, $n = 2$, 3 or 4, the formula $2\ J'_n = J_{n-1} - J_{n+1}$ was used in conjunction with Jahnke & Emde's table of J_n.

The turning values were calculated from Taylor series. The last figures of both zeros and turning values are doubtful.

Reports Nos. 3, 5 and 8.

J_0, J_1, Y_0, Y_1 (which are denoted by N_0, N_1 as is usual in Germany), were taken from Watson's tables for values of the argument x up to 16. For x greater than 16 they were calculated from Hankel's asymptotic expansion, and the relation $J_1N_0 - J_0N_1 = 2/\pi x$ was used as a check.

For values of n less than x the recurrence formula was used with increasing n. For n greater than x the recurrence formula for the ratio J_{n-1}/J_n was used with decreasing n. Checks were made by comparison with Watson's values of $J_n(n)$. Since Watson's tables of J_0, J_1, which form the basis of these tables, are to 7D the values given, also to 7D, are uncertain in the last figure. Subject to this uncertainty, seven figure accuracy can be obtained by using fourth differences, while linear interpolation gives three figure accuracy.

Report No. 8 also contains graphs of $J_n(x)$ and its first six derivatives for $n = 2$, $x = 0$–28; $n = 9$, $x = 5$–33; $n = 16$, $x = 10$–38; $n = 23$, $x = 20$–48; $n = 30$, $x = 25$–53. These were obtained from the identity between 2^m times the mth derivative of J_n and the mth difference, at unit interval, in the n direction.

Report No. 4.

The quantity $(n + \frac{1}{2}, m)$ is written in a form very convenient for computation for the range of values m to be covered;

$$(n + \tfrac{1}{2}, m) = N(N - 1)(N - 3) \cdots \quad (N - M)/m!$$

where

$$N = \tfrac{1}{2}n(n + 1), \quad M = \tfrac{1}{2}m(m - 1).$$

A check is provided by the relation $(n + \tfrac{1}{2}, n) = (2n)!/n!$

Reports Nos. 6 and 9.

The initial values of $J_{\frac{1}{2}}$ and $J_{-\frac{1}{2}}$ were taken from Hayashi's tables of Bessel functions to six and more figures.

For positive values of the order, as in tables 3, 5 and 8, the two forms of the recurrence relation were used according as n exceeded or was less than x, using six decimals throughout. Checks were provided by differencing, and by values calculated from Hankel's and Debye's asymptotic series.

For negative values of the order the recurrence relation was used with increasing $|n|$ throughout the whole range, eight figures being retained. In the worst case the building up error is 300 times the rounding off error and it is accordingly possible to give the final values to 5S. Checks were provided by differencing, and by values calculated from Debye's asymptotic series.

Report No. 7.

This report contains remarks on various methods of computing Bessel functions and is based on the experience gained in the preparation of the tables given in the other reports. Comparisons are given of the results of various methods of computation, in particular of different forms of asymptotic series.

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5. Unpublished British tables.

Some of the British Association Tables Committee's as yet unpublished work on Bessel functions is at present in the keeping of Admiralty Computing Service. Through the courtesy of the committee, values obtainable from these tables can be made available to establishments requiring them.

The tables consist of the four principal functions J, Y, I and K, or of auxiliary functions from which they can be deduced, roughly to 10 or 12 figures, for all integral values of the order up to 20, and at intervals of .1 in the argument up to 20. The list appended below, while not complete in every detail, is a sufficient indication of what values are available on request. Except where it is specifically stated otherwise, the interval of tabulation in the argument is .1 while the interval in the order is unity.

Function tabulated	Range of the order n	Range of the argument x	Number of decimals or significant figures
$J_n(x)$	2–14	0(.01)10	8D*
	0–20	0–25	10D*
	$-\tfrac{3}{4}, -\tfrac{1}{4}, +\tfrac{1}{4}, +\tfrac{3}{4}$	5–20	7D
$Y_n(x)$	0–21	0–21–25	13D and 19D
$y_n = x^n Y_n$	0–21	0–21	14S or 15S
$I_n(x)$	0–22	6–20	13D
$i_n = x^{-n} I_n$	0–22	0–6	15S
$e^{-x} I_n$	2–5	5–20	10S
	6–15	10–20	10S
	16–20	15–20	10S
$K_n(x)$	0–20	6–20	12S or 13S
$k_n = x^n K_n$	2–11	0(.01)5	8S*
	12–20	0–5	8S*
	6–15	5–10	10S
	16–20	5–15	10S
$e^x K_n$	0–7	5–20	9S or 10S
	8–15	10–20	9S or 10S
	16–20	15–20	9S or 10S

The tables of $J_n(x)$ and of $k_n(x)$ marked with asterisks are available in the form of bound photostats for loan for a limited period. It should be emphasized that these are not the Committee's final tables and that they are only made available, by courtesy of the Committee, in order not to hold up the work of establishments having a requirement for them.

Two other tables from which values can be supplied are those of the zeros of J_n and J'_n. j_{ns} is tabulated to 4D for $n = 0(1)19$, $s = 1(1)8$, and j'_{ns} to 4D for $n = 0(1)22$, $s = 1(1)9$.

MECHANICAL AIDS TO COMPUTATION

17[Z].—H. A. PETERSON & C. CONCORDIA, "Analyzers for use in engineering and scientific problems," *General Electric Review*, v. 48, September, 1945, p. 29–37. 20.7 × 28.5 cm.

This paper describes each of the following four analyzers in use by the General Electric Company:

(1) The direct current network analyzer,
(2) The alternating current network analyzer,
(3) The transient network analyzer,
(4) The differential analyzer.

A short account of the engineering applications of each analyzer is given, together with illustrations, and a bibliography of 63 titles, of which 14 of the 18 on Differential Analyzers were given in *MTAC*, v. 1, p. 452–454.

D. H. L.

NOTES

45. DAWSON'S OR POISSON'S INTEGRAL.—Various tables involving $\int_0^x e^{t^2}dt$ have been already noted, *MTAC*, v. 1, p. 322–323, 422–423. H. G. Dawson's table, appearing originally in London Math. So., *Proc.*, v. 29, 1898, p. 521–522, was recomputed and reprinted with corrections by H. M. TERRILL & LUCILE SWEENY, in Franklin Institute, *J.*, v. 238, Sept. 1944, p. 220f. An abridgement of Dawson's table is given in JAHNKE & EMDE, *Tables of Functions*, 1933 and later eds., $x = [0(.01)2; 4S]$. H. B. drew my attention to the fact that the integral here discussed occurs in Poisson's paper of 1815, "Sur la théorie des ondes," Acad. d. Sci., *Mémoires*, n.s., v. 1, 1818, p. 128–130. Two more references to tables involving the integral are as follows:

1. H. LAMB, "On water waves due to disturbance beneath the surface," London Math. So., *Proc.*, s. 2, v. 21, 1922, p. 367; table of $F(x) = x + (1 - 2x^2)e^{-x^2}\int_0^x e^{t^2}dt$, and of $F'(x)$, for $x = [0(.1)2; 3D]$. Also, p. 368, graph of $F(x)$, $0 < x \leqslant 2$.
2. Y. NOMURA, "On the waves of water of finite depth due to disturbance beneath the surface," Japan, Tôhoku teikoku daigaku, *Science Reports*, s. 1, *math. phys. chem.*, v. 25, 1936, p. 1082; table of $\psi_0(x) = 2\int_0^\infty t^2e^{-t^2}\sin(2xt)dt = x + (1 - 2x^2)e^{-x^2}\int_0^x e^{t^2}dt$, $x = [0(.1)2(1)6(2)10; 4D]$. Also, for the same range, tables of

$$\psi_s(x) = \sum_{r=0}^{s} \frac{x^{2r+1}}{(s-r)!r!}\frac{d^s}{d(x^2)^s}[x^{2s-2r-1}\psi_0(x)],\ s = 1, 2, 3.$$

R. C. A.