Tuesday Evening, Cruft Laboratory, Harvard University

"The Automatic Sequence Calculator"[3] by H. H. AIKEN, and members of his staff.

Wednesday Morning, M.I.T.

"Simulative Techniques as an Aid to Analysis" by G. A. PHILBRICK.

"Application of Punched-Card Methods to Scientific Computation" by W. J. ECKERT, presented by J. C. MCPHERSON.

"The M.I.T. Center-of-Analysis Research Program" by S. H. CALDWELL.

Wednesday Afternoon, M.I.T.

"The Scientific Application of Commercial Calculating Machines" by L. J. COMRIE.

The Conference was most notably successful, and one heard on every side expressions of the hope that such a Conference might become an annual event.

The Committee on Mathematical Tables and Other Aids to Computation is indeed greatly indebted to officers of the administration and members of the staff of the Massachusetts Institute of Technology who did everything in their power to provide a setting for securing fruitful results from such an assembly of experts.

R. C. A.

[1] Beginning with Nov. 1, 1945, S. H. C. became *chairman*, and L. J. C. *vice-chairman*.
[2] See MAC **19**.
[3] For illustrated notes on this Calculator see MAC **20**.

RECENT MATHEMATICAL TABLES

275[A, E, F].—PEDER PEDERSON, (a) *Über die numerische Berechnung der Kettenbrüche nebst einer Berechnung der Grundzahl der natürlichen Logarithmen*, 36 p., 1940; (b) *Berechnung der Grundzahl e der natürlichen Logarithmen mit 606 Dezimalen*, 17 p., 1942; (c) *Fortsetzung der Berechnung der Grundzahl e der natürlichen Logarithmen bis zur 808. Dezimalstelle*, 21 p., 1944. Denmark, Geodætisk Institut, *Meddelelse*, nos. **14**, **16**, and **17** respectively. 14.3×22 cm.

These three papers are accounts of the author's calculations of the fundamental constant e. Values of e are given to 404D in (a), to 606D in (b) and to 808D in (c). The method of calculation was based on the continued fraction

$$\frac{e-1}{2}=\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{10+}\,\frac{1}{14+}\cdots$$

which is due to EULER,[1] but is applicable to any regular continued fraction. The general idea is set forth in (a) and may be roughly described as follows. Let x be the exact value of a continued fraction and let x_n be a given approximation to x which is correct to n decimals. Let also A_k/B_k be any convergent of the continued fraction in which k is large enough so that the value of this convergent would give x to more than n decimals. This means that, roughly speaking, B_k has about $n/2$ digits. Then x_kB_k will be a decimal whose first $n/2$ significant figures are the digits of A_k and whose next $n/2$ digits are zeros. Therefore if we compute denominators B_k only, and compute only the digits of the product x_kB_k which lie near the nth significant figure, the appearance of zeros (or nines) at this place serves as a check on the digits of x_n. Moreover, the method may be used to extend the value of x beyond the nth decimal place by simply filling in those digits which will produce more zeros until the accuracy implicit in a particular B_k is exceeded.

The author started with SHANKS' 137D value of e verified by GLAISHER,[2] and went on to check the value of Shanks[3] (205D, last 18 wrong) and of BOORMAN[4] (346D, last 123 wrong). The 223 correct decimals of Boorman was then the basis of an extension to

404D, and then to 606D. At this point the author became aware of a 707D value given in 1926 by the reviewer,[5] which he verified and used to extend the value of e to the impressive result of 808D. The suggested method of checking and extending the value of a continued fraction, given in 1926, is thus independently discovered and greatly elaborated by the author.

The determination of the digits of e, one at a time, seems to suggest that the author used no calculating machine to perform this immense calculation.

Besides the values of e mentioned above the author gives all the denominators of the convergents up to B_{170}, a 408 digit integer. These interesting numbers begin with $B_1 = 1$, $B_2 = 7$, $B_3 = 71$, $B_4 = 1001$ and were found by the recurrence formula

$$B_{k+1} = (4k + 2)B_k + B_{k-1}.$$

They have interesting divisibility properties discovered by D. N. LEHMER[6] and further studied by the reviewer.[7] The author gives no description of this part of the work and of the checks which are available to insure the accuracy of this, the most hazardous part of the calculation.

Since the periodical in which this value of e to 808D appears is highly inaccessible in this country, it seemed advisable to reproduce it below.

e = 2.71828	18284	59045	23536	02874	71352	66249	77572	47093	69995
95749	66967	62772	40766	30353	54759	45713	82178	52516	64274
27466	39193	20030	59921	81741	35966	29043	57290	03342	95260
59563	07381	32328	62794	34907	63233	82988	07531	95251	01901
15738	34187	93070	21540	89149	93488	41675	09244	76146	06680
82264	80016	84774	11853	74234	54424	37107	53907	77449	92069
55170	27618	38606	26133	13845	83000	75204	49338	26560	29760
67371	13200	70932	87091	27443	74704	72306	96977	20931	01416
92836	81902	55151	08657	46377	21112	52389	78442	50569	53696
77078	54499	69967	94686	44549	05987	93163	68892	30098	79312
77361	78215	42499	92295	76351	48220	82698	95193	66803	31825
28869	39849	64651	05820	93923	98294	88793	32036	25094	43117
30123	81970	68416	14039	70198	37679	32068	32823	76464	80429
53118	02328	78250	98194	55815	30175	67173	61332	06981	12509
96181	88159	30416	90351	59888	85193	45807	27386	67385	89422
87922	84998	92086	80582	57492	79610	48419	84443	63463	24496
84875	602...								

D. H. L.

[1] L. EULER, *Introductio in Analysin Infinitorum*, v. 1, Lausanne, 1748, p. 319; ***Opera Omnia***, s. 1, v. 8, Leipzig and Berlin, 1922, p. 388.

[2] J. W. L. GLAISHER, B.A.A.S., ***Report***, 1871, p. 16–18 (Sect. trans.), also ***Report***, 1873, p. 128, l. 12.

[3] W. SHANKS, Royal So. London, *Proc.*, v. 6, 1854, p. 397.

[4] J. M. BOORMAN, *Math. Mag.*, v. 1, 1884, p. 204. Another calculation was made by Fr. Tichâneck in 1893 to 225D (correct to within a unit in the last place), *Jahrbuch ü. d. Fortschritte d. Mathem.*, v. 25, p. 736; there is here a misprint in the 43rd decimal place, *for* 6 *read* 0.

[5] D. H. LEHMER, *Amer. J. Math.*, v. 48, 1926, p. 139–143.

[6] D. N. LEHMER, *Amer. J. Math.*, v. 40, 1918, p. 375–390. Also Nat. Acad. Sci., *Proc.*, v. 4, 1918, p. 214–218.

[7] D. H. LEHMER, *Annals Math.*, v. 23, 1932, p. 143–150.

EDITORIAL NOTE: For various values of e^n, see *MTAC*, v. 1, p. 54–55.

276[B, K, M].—A. N. KOLMOGOROV, (a) "Chislo popadaniĭ pri neskol'kikh vystrelakh i obshchie prin͡tsipy o͡tsenki effektivnosti sistemy strel'by" (The number of hits in several rounds, and the general principles of estimating effectiveness of a system of firing); (b) "Iskusstvennoe rasseivanie v sluchae porazheni͡ia odnim popadaniem i rasseivani͡ia v odnom izmerenii" (Intentional dispersions in case one hit is fatal and dispersions in one dimension), Akademiiâ Nauk SSSR, Leningrad, Matematicheskiĭ Institut imeni V. A. Steklova, *Trudy*, v. 12, June 1945, p. 7–45, 95–106.

This issue of the *Trudy* is devoted to a collection of articles on the theory of gunnery, edited by Kolmogorov. There are four tables (p. 96–106). **Table 1** gives values of

$$H(m, a) = 1 - e^{-a}\left(1 + \frac{a}{1!} + \frac{a^2}{2!} + \cdots + \frac{a^{m-1}}{(m-1)!}\right),$$

for a = [.1(.1)15; 5D], m = 1(1)11.

For the same range **T. 2** gives the values of

$$\nabla^2 H(m, a) = H(m, a) - 2H(m-1, a) + H(m-2, a).$$

In **T. 3**, values are given for

$$\tau(u) = \frac{1}{2u}\int_{-\infty}^{+\infty} \ln \{1 - \tfrac{1}{2}[\Phi(z+u) - \Phi(z-u)]\} dz$$

for u = [0(.05)1(1)5; 3D], with Δ; $\Phi(z) = \frac{2\rho}{\sqrt{\pi}}\int_0^z e^{\rho^2 z^2} du$, $\rho = .476936\ldots$

T. 4 gives the values of $P(N) = \Phi(s_0) - 2s_0\phi(s_0)$, $\phi(z) = e^{\rho^2 z^2}/\pi^{\frac{1}{2}}$, $s_0(N) = \rho^{-1}(\frac{3}{4}\pi^{\frac{1}{2}})N^{\frac{1}{3}}$, and $d^2(N) = \frac{3}{5}s_0^2$, for N = [0(.1)1(.2)5(.5)10; 3S] with Δ.

On p. 104–106 are graphs of (a) $H(m, a)$, m = 1(1)10, a = 0(1)15; (b) $R(k, c, n)$, $k = \frac{1}{2}$, $c = 4/15$, n = 2(2)6, 12(6)24, $0<a<4.8$, the values of R being given in a table on p. 33 for given values of n and a; (c) R, $k = \frac{1}{2}$, $c = 4/15$, n = 3, 6(6)24, $0<a<2.4$, the values of R being given on p. 34 for values of n and a.

R. C. A.

277[D, J].—GREAT BRITAIN, H.M. NAUTICAL ALMANAC OFFICE, *Tables for the Summation of Trigonometrical Series*. Department of Scientific Research and Experiment, Admiralty Computing Service, November, 1945. No. SRE/ACS. 93. 24 leaves mimeographed on one side of the paper. 20.2×32.9 cm. This publication is available only to certain Government agencies and activities.

These tables give the values of sin nx and cos nx for n = 1(1)10, x = 0(1°)180°, and also for radian arguments x = 0(.01)3.14. The values corresponding to n = 1, 2, 3, and 4 are given to 5D; those corresponding to n = 5, 6, and 7 to 4D; and those corresponding to n = 8, 9, and 10 to 3D.

These tables were compiled for use in the summation of trigonometric series of the form $\Sigma(A_n \cos nx + B_n \sin nx)$ when the convergence is fast enough for the sum of the first ten terms to give adequate accuracy. In conjunction with a table of powers, the tables may also be used to convert polar into cartesian coordinates.

ARNOLD N. LOWAN

EDITORIAL NOTE: For radian argument the above-mentioned table ranges, at varying intervals, from 0 to 31.4 radians, 5–3D. The NYMTP *Tables of Sines and Cosines for Radian Arguments*, 1940, give, among other tables, sin x and cos x, for x = [0(.001)25; 8D], and x = [0(1)100; 8D].

278[E, M].—LOUIS DUNOYER, "Calculs sur l'émission en bout des tubes lumineux et notamment des tubes électro- ou photoluminescents," *Revue d'Optique*, v. 18, 1939, p. 235–241. 16×24.2 cm.

There are several tables and graphs. The tables are as follows:

(a) $y = e^{-a/\sin\gamma} \sin\gamma$, for a = .5, and $a/\sin\gamma$ = [.5(.01).53, .55(.5).75, .85, 1(.25)1.5(.5)3(1)6; 5D].

(b) $H(m) = \int_u^{\frac{1}{2}\pi} e^{-a/\sin\gamma} \sin\gamma d\gamma$, for $u = [0°, 2°(4°)90°; 5D]$, and $a = .5$, $m = \cot u$.

(c) $H(m) = \int_u^{\frac{1}{2}\pi} e^{-a/\sin\gamma} \sin\gamma d\gamma$, for $a = 0, .2, .5, 1, 2$, and $m = [0, .01(.02).99, 1.05(.1)1.95$, also $.1(.2)8.5, 9.5; 4D]$.

(d) $\int_0^M [H(m) + H(M-m)]\Lambda(m)dm, \quad \Lambda(m) = \int_0^u e^{-kz/\cos\alpha}\sin\alpha d\alpha,$

$H(M-m) = \int_{u'}^{\frac{1}{2}\pi} e^{-a/\sin\gamma}\sin\gamma d\gamma, \quad \cot u' = M - m$, for

$M = [0(.1)1(.2)4(1)10, \infty; 4D]$ and $a = k = 0, .2, .5, 1, 2$.

There are graphs of $H(m)$, $a = k = 0, .2, .5, 1, 2$, and $0 < m \leqslant 5$; of $Hm + H(M-m)$ for $M = .2(.2)1(.5)2(1)5$, $0 < m \leqslant 5$; and of $\int_0^M [H(m) + H(M-m)]\Lambda(m)dm$ for $a = k = 0$, $.2, .5, 1, 2$, and $0 < m \leqslant 4$.

R. C. A. & S. A. JOFFE

279[F].—N. G. W. H. BEEGER, "Extension of the table of least exponents ξ for which $2^\xi \equiv 1 \pmod p$." *Nieuw Archief v. Wiskunde*, s. 2, v. 20, 1940, p. 307–8. 15.7×23.6 cm.

The number ξ referred to in the title is a function of p familiar to number theorists and usually called the exponent to which 2 belongs (mod p). It is known that ξ is some divisor of $p - 1$. The other factor $\nu = (p-1)/\xi$ has been called by A. J. C. CUNNINGHAM the residue index of 2 (mod p) and this function, because it is nearly always much smaller than ν, is usually tabulated in lieu of ξ. The present note contains a one-page table giving ν for each prime p between 300000 and 310000. As a matter of fact those cases for which $\nu = 1$ and 2 are omitted in the table to save space. A missing p has $\nu = 1$ or 2 according as $(p^2 - 1)/8$ is odd or even. This table is an extension of the large table of ν for $p < 300000$ published by M. KRAITCHIK, in his *Recherches sur la Théorie des Nombres*, v. 1, Paris, 1924, p. 131–191. Tables of ν, like so many other difficult number-theoretic functions, must be constructed one entry at a time, and each entry must stand on its own merits, without hope of a smoothness check. Thus a small table like the present one represents a considerable amount of careful computation.

The writer has in his possession a photostat copy of a manuscript containing an extension by Beeger of the table here reviewed; it covers the primes between 310000 and 320000.

Arithmeticians will be glad to learn that Mr. Beeger not only survived the war but was able to continue his researches in the theory of numbers.

D. H. L.

280[F].—A. GLODEN, "Table des solutions de la congruence $X^4+1\equiv 0 \pmod p$ pour $2\cdot 10^5 < p < 3\cdot 10^5$," *Mathematica* (Rumania), v. 21, 1945, p. 45–65.

This table is an extension of two previous tables:
A. J. C. CUNNINGHAM, *Binomial Factorisations*, v. 1, London, 1923, p. 1–95, and v. 2, 1924, p. 1–104, for $p < 10^5$; and S. HOPPENOT, "Table des solutions de la congruence $x^4 \equiv -1 \pmod N$ pour $100000 < N < 200000$," Brussels, Librairie du *Sphinx*, 1935, 18 p.; see *MTAC*, RMT **48**, v. 1, p. 6.

In order that the congruence

$$X^4 + 1 \equiv 0 \pmod p \tag{1}$$

have a solution it is necessary and sufficient that the prime p be of the form $8n + 1$. Accordingly, the table is concerned with such primes only. These primes p are capable of being represented in essentially one way by the quadratic forms

$$p = x^2 + y^2 = 2z^2 + t^2.$$

If one solves the linear congruences

$$x \equiv uy \pmod p \quad \text{and} \quad z \equiv vt \pmod p$$

for u and v, then the four solutions of the quartic congruence (1) are given by

$$X = \pm u(v \pm 1).$$

The table gives for each $p = 8n + 1$ between 200000 and 300000 two values of X which are less than $p/2$; the other two values of X are of course negatives of these modulo p. There is one curious exception at $p = 200569$ where, for some reason, the second solution is given as 100317 instead of 100252. Since the product of all the roots of (1) is clearly congruent to unity modulo p, the product of any two roots is congruent to ± 1 modulo p. This fact was used as a check of the table.

At the end of the table is given a list of 25 numbers of the form $x^4 + 1$ and two of the form $x^8 + 1$ with their complete decompositions into primes made possible by this table. For example

$$1620^4 + 1 = 289889 \cdot 23759009.$$

The first factor follows from the entry

$$p = 289889, \quad X_1 = 1620, \quad X_2 = 15926$$

of the main table. The second factor is clearly a prime having no factors < 300000. A more extensive factor table of 54 entries (including only 8 of the above 27) based presumably upon a corresponding table of solutions of (1) up to $p < 500000$, appeared in *Mathesis*, v. 55, 1945, p. 81–82.

It is perhaps worth pointing out that the main table is useful as a list of primes of the form $8n + 1$ even if one is not concerned with the quartic congruence (1).

D. H. L.

281[F].—V. THÉBAULT, *Les Récréations Mathématiques* (*Parmi les nombres curieux*), Supplément à *Mathesis*, Gembloux, p. 1–80, v. 54, no. 10, Aug. 1943; p. 81–119, no. 11, Feb. 1945. 16.3×25.2 cm.

Here are numerous independent tables, not all published for the first time, including the following:

1. With the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken once, form two numbers of three digits whose sum is equal to numbers of (a) three digits chosen once; (b) four digits chosen once. On p. 8, for (a) a table of 21 solutions is given, and for (b) 48. For example, $286 + 173 = 459$; $789 + 246 = 1035$. For (a) see *L'Éducation Math.*, v. 23, 1920, p. 34–35.

2. With the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, taken once, form an integer which is a square. On p. 11, 30 solutions are given from $11826^2 = 139854276$, to $30384^2 = 923187456$; 29 of these were published earlier in *Mathesis*, v. 51, 1937, p. 279–281. On p. 12 are 87 squares formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken once, from $32043^2 = 1026753849$, to $99066^2 = 9814072356$.

3. P. 62–68 gives all 1-digit, 2-digit, 3-digit, and 4-digit square endings for the base 10, together with an inverse table showing the forms of those numbers whose squares end in a specified 1-digit, 2-digit or 3-digit number. The author has published this before in *Mathesis*, v. 48 Suppl. Oct. 1934, 22 p. A slightly more complete table is due to SCHADY, *J. f. d. reine u. angew. Math.*, v. 84, 1877, p. 86–88.

4. "Table des carrés des nombres entiers de 1 à 1000 dans les systèmes de numération de bases $B = 2, 3, 4, 5, 6, 7, 8, 9, 11, 12$," p. 89–119.

These four tables are of interest to "digitologers," a sect of which the author is a leading exponent. Table **3**, however, is of practical use in the rapid identification of non-squares.

D. H. L.

282[I].—HERBERT E. SALZER, "Formulas for direct and inverse interpolation of a complex function tabulated along equidistant circular arcs," *J. Math. Phys.*, v. 24, Nov. 1945, p. 141–143. 17.4×26.4 cm.

The purpose of this paper is to facilitate direct as well as inverse interpolation of an analytic function $f(z)$ whose numerical values are known in a sequence of points z_ν which are equally spaced along a circle in the complex plane. In terms of the new variable $P = (z - z_0)/(z_1 - z_0)$ the problem reduces to the case when the functional values are given at the points P_ν defined by

$$P_0 = 0,\ P_\nu = 1 + \omega + \omega^2 + \cdots + \omega^{\nu-1}\ (\nu \geqq 1)$$
$$P_\nu = -(\omega^{-1} + \omega^{-2} + \cdots + \omega^{\nu})\ (\nu \leqq -1),\ (\omega = e^{i\theta}).$$

Let

$$F(P) = \sum_{\nu=\mu-n+1}^{\mu} L_\nu^{(n)}(P)F(P_\nu),\ (\mu = [n/2]),$$

be the n-point Lagrange-Hermite interpolation formula based on our points P_ν. Setting $P^r = p_r + iq_r$ $(r = 0, 1, 2, \cdots)$, the paper furnishes explicit expressions, in terms of p_r, q_r and the angle θ, of the real and imaginary parts of the coefficients $L_\nu^{(n)}(P)$, for $n = 3$, 4, and 5. It is also shown how these expressions may be used to apply also in the present situation the author's recent formula for inverse interpolation (see "A new formula for inverse interpolation," Amer. Math. So., *Bull.*, v. 50, 1944, p. 513–516).

I. J. SCHOENBERG

Univ. of Pennsylvania

283[I].—HERBERT E. SALZER, "Inverse interpolation for eight-, nine-, ten-, and eleven-point direct interpolation," *J. Math. Phys.*, v. 24, May 1945, p. 106–108. 17.4×26.4 cm.

This paper extends a formula which was first published by the author as "A new formula for inverse interpolation," Amer. Math. So., *Bull.*, v. 50, 1944, p. 513–516. The expression in question is the inversion of the Lagrange interpolation formula.

If we write $f_x = f(a_0 + xh)$, where h is the tabular interval, and if p is desired when we are given f_p, then p can be expressed in terms of the tabulated values, $f_0, f_1, f_2, \cdots f_{-1}, f_{-2}$, etc. by means of the formula:

$$p = \sum_{m=1}^{k-1} \frac{(f_p - f_0)^m}{m!} \left[\frac{d^{m-1}}{da^{m-1}} \left(1 \Big/ \sum_{i=-\frac{1}{2}(n-1)}^{\frac{1}{2}n} \frac{L_i^{(n)}(a)}{a} (f_i - f_0) \right)^m \right]_{a=0} + R_k,$$

in which $L_i^{(n)}(a)$ is a polynomial of $(n - 1)$ degree in a.

The author reduced this expression to the form

$$p = r - r^2s + r^3(2s^2 - t) + r^4(-5s^3 + 5st - u) + \cdots$$

where the quantities r, s, t, u, etc. are expressed in terms of the tabulated values of the function.

In the present paper the explicit values of these variables are given for the case where eight-, nine-, ten-, or eleven-point direct interpolation is used.

It is difficult to imagine when an equation of this extent may be required in practical work, but the author remarks: "its full use can provide unusual accuracy in solving equations (both real and complex) up to the tenth degree when the values of the polynomials are tabulated near the root at equal intervals."

H. T. D.

284[I].—HERBERT E. SALZER, "Table of coefficients for double quadrature without differences, for integrating second order differential equations," *J. Math. Phys.*, v. 24, Nov. 1945, p. 135–140. 17.4×26.4 cm.

A function $f(x)$ may be expressed in terms of its second derivative as

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} f''(x_2)dx_2. \tag{1}$$

If numerical values of $f''(x)$ are given in the n points

$$x_p = x + ph \quad (p = \mu - n + 1, \cdots, \mu, \text{ where } \mu = [n/2]),$$

we may approximate $f''(x)$ by the n-point Lagrange formula. A double integration of the interpolating polynomial leads from (1) to the approximation formula

$$f(x_0 + ph) = f(x_0) + phf'(x_0) + h^2 \sum_{i=\mu-n+1}^{\mu} C_{i,n}(p)f''(x_0 + ih) + R_p, \tag{2}$$

where $C_{i,n}(p)$ are polynomials in p of degree $n + 1$, given by

$$C_{i,n}(p) = (-1)^{i+\mu} \int_0^p dp_1 \int_0^{p_1} \left\{ \prod_{\lambda=\mu-n+1}^{\mu}{}' (p_2 - \lambda) \right\} dp_2/\{(i + n - \mu - 1)!\,(\mu - i)!\}, \tag{3}$$

the prime on the product sign indicating that the factor $p_2 - i$ is omitted. The paper furnishes for $n = 3(1)11$, the numerical (rational) values of the coefficients (3) for $p = (\mu - n + 1)(1)\mu$, reduced to their least common denominator.

The author discusses briefly the application of the formula (2) to the numerical integration of a differential equation of the type

$$y'' + \phi(x, y) = 0,$$

to which any second order linear differential equation may be reduced by a proper change of unknown function. The solution once "started," the familiar procedure of the progressive refinement of a first approximation to a new value of y applies, by working back and forth between the columns of values of y and y''. At each step, however, the value of $hf'(x_0)$ must also be determined by (2) for any single value of p ($\neq 0$) for which $f(x_0 + ph)$ is already known. Thus a column of values of hy' is also determined. The author concludes by pointing out further useful applications to the integration of partial differential equations of the type

$$u_{xx} = u_t + \psi(u, x, t).$$

I. J. SCHOENBERG

285[K].—ØYVIND BURRAU, "Middelfejlen som Usikkerhedsmaal," *Matematisk Tidsskrift*, 1943B, p. 9–16. 15.1×24 cm.

Suppose $x_1, x_2, \cdots, x_n$ are values of x in a sample of n items from a normal population with mean a and standard deviation σ. W. S. GOSSETT, under the pseudonym "Student,"[1] in 1908 conjectured that the sampling distribution of

$$z = \frac{\bar{x} - a}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

was given by[2]

$$f(z)dz = C(1 + z^2)^{-\frac{1}{2}n}dz \tag{1}$$

where C is a constant, such that $\int_{-\infty}^{\infty} f(z)dz = 1$. This conjecture was verified in 1925 by R. A. FISHER.[3] Gossett originally tabulated values of $\int_{-\infty}^{z} f(v)dv$ for which $z = .1(.1)3$, and $n = 4(1)10$. Fisher[4] has tabulated values of $t = z\sqrt{(n - 1)}$ for which

$$\int_{-z\sqrt{(n-1)}}^{+z\sqrt{(n-1)}} f(v)dv = .1(.1).9, .95, .98, .99,$$

and for $n = 1(1)30$.

Mr. Burrau tabulates values of the integral

$$(2) \qquad \int_{-z\sqrt{(n-1)}}^{z\sqrt{(n-1)}} f(v)dv$$

for which $z\sqrt{(n-1)} = 0(.2)5$, and for $n = 4(2)12$ and ∞. The case for $n = \infty$ is, of course, equivalent to finding the values of the Gaussian integral $\frac{1}{\sqrt{2\pi}}\int_{-t}^{t} e^{-\frac{1}{2}z^2}dz$ for $t = 0(.2)5$.

He then observes that since the mean value of z^2, i.e. $E(z^2)$, is $1/(n-3)$, thus making $E[(n-1)z^2] = 1$, one might consider the tabulation of the integral

$$(3) \qquad \int_{-z\sqrt{(n-3)}}^{z\sqrt{(n-3)}} f(v)dv$$

for $z\sqrt{(n-3)} = 0(.2)5$, and for $n = 4(2)12$ and ∞, rather than the analogous tabulation of (2), with the hope of obtaining values which change relatively little as n changes. The tabulation of (3) is made and the resulting values are considerably more constant with respect to changes in n than the analogous tabulations of (2). For example, the values of (2) and (3) for $z\sqrt{(n-1)} = 2$ and $z\sqrt{(n-3)} = 2$ are:

n	Value of (2) for $z\sqrt{(n-1)} = 2$	Value of (3) for $z\sqrt{(n-3)} = 2$
4	0.861	0.959
6	0.898	0.950
8	0.914	0.950
10	0.923	0.950
12	0.929	0.954
∞	0.954	0.954.

This indicates that the "significance level" corresponding to $\pm z\sqrt{(n-3)}$ is approximately 95 per cent for all values of $n \geqq 4$. The author does not discuss the problem for $n < 4$.

S. S. W.

[1] W. S. GOSSETT (Student), (a) "The probable error of a mean," *Biometrika*, v. 6, 1908, p. 1–25; (b) *Collected Papers*, Cambridge, 1942, p. 11–34.

[2] The z being used here is that originally used by "Student," and is being used throughout this review. Mr. Burrau, although actually writing about "Student's" problem, unfortunately denoted $z/n^{\frac{1}{2}}$ (in "Student's" sense) by the letter z.

[3] R. A. FISHER, "Applications of 'Student's' distribution," *Metron*, v. 5, no. 3, 1925, p. 90–104.

[4] R. A. FISHER, *ibid.*; the table has also been published in Fisher's *Statistical Methods for Research Workers*, eighth ed., Edinburgh, Oliver & Boyd, 1941, p. 167.

286[K, M].—A. H. R. GRIMSEY, "On the accumulation of chance effects and the Gaussian frequency distribution," *Phil. Mag.*, s. 7, v. 36, Apr. 1945, p. 295. 16.8×25.3 cm.

There are Tables of P_n, Q_n, U_n, for $n = [1(1)12; 8D]$, exact rational and decimal values being both given for P_n, where

$$P_n = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin v}{v}\right)^n dv, \quad Q_n = [6/(\pi n)]^{\frac{1}{2}}, \quad U_n = \frac{2}{\pi}\int_0^\infty \frac{(\sin v)^{n+2}}{v^n} dv.$$

Q_n are the values to which P_n approximates.

Mr. Grimsey's communication was inspired by misprints in L. SILBERSTEIN, "The accumulation of chance effects and the Gaussian frequency distribution," *Phil. Mag.*, s. 7, v. 35, 1944, p. 395–404, and in particular by his values of P_n for $n = 1(1)6$, accompanied by the statement "so far as we know, these integrals have not been previously determined," and by the suggestion that mechanical quadrature would give the values for larger n. Mr. Grimsey refers to detailed discussion bearing on this question in J. EDWARDS, *A Treatise on the Integral Calculus*, v. 2, London, 1922, p. 204–212, where the values of P_n

are incidentally given for $n = 1(1)7$, that for $P_6 = 11/20$ or 88/160, occurring in place of 83/160 given by Silberstein on p. 399.

R. C. A.

287[L].—Murlan S. Corrington, "Tables of Bessel function $J_n(1000)$," *J. Math. Phys.*, v. 24, Nov. 1945, p. 144–147. 17.3×26.3 cm. Compare MTE **74**, v. 2, p. 47–48; the Meissel errors are again noted here.

Quotations: A table of Bessel functions for $J_n(1000)$ is given for $n = [935(1)1035; 8D]$, also a graph of $J_n(1000)$ for $930 < n < 1040$. All computations were made to 11D and later rounded off to 8D.

When a frequency-modulated radio wave is analyzed into a spectrum, the carrier and side-frequency amplitudes are proportional to Bessel functions of the first kind, with order equal to the number of the side frequency and argument equal to the modulation index, in accord with the following equation:

$$\sin(w_0t + (D/\mu)\sin 2\pi\mu t) = \sum_{n=-\infty}^{\infty} J_n(D/\mu)\sin(w_0 + 2\pi n\mu)t$$

where

w_0 = carrier frequency, radians per second,
t = time in seconds,
D = deviation in cycles per second,
μ = audio frequency in cycles per second.

According to the present standards for commercial broadcasting, the deviation can be as much as 75,000 cycles per second and the audio frequency can be as low as 30 cycles per second, so the maximum value of the modulation index, D/μ, is 2500. The table in this article was computed to determine the relative side frequency amplitudes near the edge of the band for a modulation index of 1000. The graph shows that when the side frequency number n exceeds 1000, the amplitudes decrease very rapidly towards zero.

288[L].—J. Cossar & A. Erdélyi, *Dictionary of Laplace Transforms.* Admiralty Computing Service, Department of Scientific Research and Experiment, London; Part 3A, no. SRE/ACS. 102, Nov. 1945, 29 leaves. 20.2×33 cm. Mimeographed on one side of each leaf. These publications are available only to certain Government agencies and activities.

This penultimate part of the *Dictionary* is section VII. Section VI (Parts 2A, 1944; 2B, 1945) was a classification according to $f(t)$, the Laplace transform of $f(t)$ being defined by the relation

$$\phi(p) = \int_0^\infty e^{-pt}f(t)dt = L\{f(t); p\};$$

see *MTAC*, v. 1, p. 424–425. Part 3A is a Table of inverse Laplace Transforms, that is, a classification according to $\phi(p)$ for the Elementary Functions, Gamma and Related Functions; and Functions defined by Integrals—as in Part 2A. Presumably Part 3B will classify according to $\phi(p)$ the other functions listed on p. 425.

R. C. A.

289[L].—Rufus Isaacs, "Airfoil theory for flow of variable velocity," *J. Aeronautical Sci.*, v. 12, 1945, p. 117. 20.5×28.6 cm.

It is stated that Theodorsen's tabulation[1] of the function

$$C(x) = H_1^{(2)}(x)/[H_1^{(2)}(x) + iH_0^{(2)}(x)]$$

is incorrect for $x \leqslant .1$ and so a new table is given for the real and imaginary parts of $C(x) = F + iG$, for $x = 0(.02).1(.05)4$. There are 3D for F, 4D for G.

H. B.

[1] THEODORE THEODORSEN, "General theory of aerodynamic instability and the mechanism of flutter," NACA, *Techn. Reports*, no. 496, 1934, p. 418, 426; see *MTAC*, v. 1, p. 404.

IN MEMORIAM NOTE: This review was included in a letter written to me on January 20, 1946, the day on which Harry Bateman left Pasadena, California, by train for New York, where he was to have received notable honors. On the following morning his rare spirit took flight, as a result of a coronary thrombosis. The loss to scholarship, to our Committee, and to *MTAC*, of this extraordinarily able, profoundly learned, and ever helpful intellectual, is irreparable.

H. B. was born in Manchester, England, in 1882. He graduated A.B., as senior wrangler, at Trinity College, Cambridge, in 1903, and was a Smith's Prizeman in 1905. During 1905–1911 he was a fellow of Trinity and received his A.M. in 1906. He was a student at Göttingen and Paris 1905–06, a lecturer at the University of Liverpool 1906–07, and a reader of mathematics and physics at the University of Manchester 1907–10. Coming to the United States in 1910 with 50 publications already to his credit (they included the notable report on Integral Equations), he was for two years a lecturer at Bryn Mawr College. During 1912–15 he was a Johnston scholar at The Johns Hopkins University (Ph.D. 1913), and was lecturer there from 1915 until 1917, when he was appointed professor of mathematics, physics, and aeronautics at the California Inst. Tech. He was elected a fellow of the American Philosophical Society in 1924, of the Royal Society of London in 1928, and of the National Academy of Sciences in 1930, and a vice-president (1935) and Gibbs lecturer (1943) of the American Mathematical Society. His published books were *The Mathematical Analysis of Electrical and Optical Wave-Motion on the Basis of Maxwell's Equations*, Cambridge, 1915; *Differential Equations*, London, 1918; *Partial Differential Equations of Mathematical Physics*, Cambridge, 1932; new edition, New York, 1944. He edited the lectures of H. A. LORENTZ, *Problems of Modern Physics*, Boston, 1927. He was also a collaborator on three published reports of the National Academy of Sciences: (a) Committee on Electrodynamics of Moving Media, 1922; (b) Committee on Hydrodynamics, 1932; (c) Committee on Numerical Integration. His report on tables of Bessel functions was published in *MTAC*, no. 7, 1944, and our Committee had hoped soon to receive his manuscripts of reports on tables of Higher Functions and Integrals, and of a great work on Definite Integrals, replacing the antiquated volume of BIERENS DE HAAN; compare *MTAC*, v. 1, p. 322. About 1900 H. B. participated in a chess tournament between England and America; and prior to 1911 he beat Steinmetz in a tournament. For other details see *Who's Who 1945*, London; *Who's Who in America 1944–45; Amer. Men of Science*, seventh ed., 1944; *Who's Who in Engineering, 1941; Who's Who in Aviation . . . 1942–43; Pan Pacific Who's Who, . . . 1940–41 edition*, Honolulu, Hawaii, 1941; *Leaders in Education*, second ed., 1941; C. M. NEALE, *The Senior Wranglers of the University of Cambridge*, Bury St. Edmunds, 1907; "Poggendorff," v. 5, 1925, and v. 6, 1936.

R. C. A.

290[L].—NYMTP, *Table of Characteristic Values of Mathieu's Differential Equation*. Prepared for the Applied Mathematics Panel of the National Defense Research Committee. No. AMP Report 165.1 R. New York, September, 1945. xxiv, 39 p. 20×28 cm. Produced by photo-offset from typescript. These tables are available only to certain Government agencies and activities.

The aim of the volume is to provide an 8D table of the characteristic values of Mathieu's differential equation, for orders up to 15, interpolable in the parameter s (defined below), within the range given.

An account of previous tables of these values has been given by the reviewer in *MTAC*, v. 1, p. 409–419.

Mathieu's differential equation is written in the form

$$d^2y/dt^2 + (b - s\cos^2 t)y = 0. \tag{1}$$

For a prescribed value of s this equation has a solution admitting the period 2π in t only when b has one of a discrete but infinite sequence of "characteristic values." The periodic solutions fall naturally into two classes, according as they are even or odd in t, and the corresponding characteristic values are denoted respectively by be_r and bo_r, the suffix denoting the order in the sequence arranged according to increasing numerical magnitude. The ranges included are $r = 0(1)15$, $0 \leqslant s \leqslant 100$; the interval varies, being chosen so that the maximum attainable accuracy is obtained by the use of modified second differences, and these differences are provided. In detail

	be_r		bo_r
r	s	r	s
0	0(.2)20(.5)37(1)100	1(1)4	0(.5)100
1	0(.2)15(.5)50(1)100	5(1)7	0(1)100
2	0(.2)40(.5)77(1)100	8, 9	0(2)100
3	0(.2)13(.5)56(1)100	10(1)15	0(4)100 and 10(20)90
4, 5, 6	0(.5)100		
7, 8	0(1)100		
9, 10	0(2)100		
11(1)15	0(4)100 and 10(20)90		

A table of the Everett interpolation coefficients, $E_2(p)$ and $F_2(p)$, for p = [0(.001)1; 6D], is given.

A graph showing the characteristic values as functions of s is reproduced on p. xxiv of the introduction.

It should suffice to say that the NYMTP has carried out the task assigned to it in the efficient manner we have come to expect from this Project. It is clear from the introduction that considerable advances have been made in the technique for the routine computation of these characteristic values, although the new methods do not become really powerful until several decimals of the characteristic value have been determined. In this respect the computer derives, from the small interval, advantages in extrapolation, commensurate with those in interpolation which he confers on the user. Despite some adverse criticism to be given below, the Table is to be welcomed, since any table is infinitely better than no table, and because this seems to be the first table of any magnitude which is easily interpolable in the parameter, and in which provision for interpolation is made. It is also fundamental, since a knowledge of the characteristic values is necessary for the calculation of the Fourier coefficients, and hence of the periodic functions themselves.

On the evidence available to us, however, it does appear that the NYMTP was upon some points ill advised (or instructed). Through ignorance or perversity the judgment of MATHIEU, HEINE, WHITTAKER, GOLDSTEIN, INCE, and nearly every other worker in the field, has been ignored or disregarded by choosing the form (1) of the differential equations, a form which is inconvenient in so many ways both theoretically and practically, and which seems to have only one trivial circumstance in its favor. Indeed, the explanation of the methods of computation and the formulae given in the introduction, and the computations themselves, are all relative to the equation

$$d^2y/dt^2 + (\alpha - 2\theta \cos 2t)y = 0 \tag{2}$$

in which, of course, the parameters are related to those in (1) by

$$\alpha = b - \tfrac{1}{2}s, \quad \theta = \tfrac{1}{4}s. \tag{3}$$

The superiority of (2) over (1) as a canonical form seems to be almost universally recognized.

Fundamentally this rests upon the simplicity and symmetry of the relations between solutions for positive and negative values of θ (or s). For this, reference may be made to *MTAC*, v. 2, p. 1–11. Changing the sign of θ in (2) leaves α unchanged, and is equivalent to replacing t by $\pm \frac{1}{2}\pi \pm t$. Had α and not b been tabulated, the table would have served equally well for negative and positive values of s, but as it is, it is necessary, in order to derive the characteristic number corresponding to a negative value of s, to subtract $|s|$ from the value corresponding to $|s|$.

This point can be seen more immediately and strikingly if one compares the elegant symmetry of the graph of characteristic values against θ (or some equivalent parameter) as given by INCE,[1] STRUTT,[2] LUBKIN & STOKER,[3] MCLACHLAN,[4] and others, with the diagram on p. xxiv of the present volume, imagined carried into the region of negative s.

Moreover, we are so far given only characteristic values. If the Fourier coefficients of the periodic solutions are needed—essential before the functions themselves can be calculated—then all the formulae are most conveniently expressed (as in the introduction!) in terms of α. All the indications go to show that NYMTP did, in fact, compute α, and then converted to b—and that in computing the tables of the corresponding Fourier coefficients (which are promised) it will be α and not b that NYMTP will actually use.

It is fervently to be desired that information equivalent to that given in these tables may before long be made more generally available, but we urge, with all the force at our command, that before this is done the tables should be reconverted so as to give α and not b.

In one respect, perhaps, equation (1) gives a lead that was no longer followed. It indicates more clearly than does (2) that the parameter s is generally connected with the square of some significant physical quantity. This is an argument for the retention of the parameter c ($s = c^2$) of some earlier tables.[5]

W. G. BICKLEY

Imperial College of Science and Technology, London

[1] E. L. INCE, "Researches into the characteristic numbers of the Mathieu equation," R. So. Edinburgh, *Proc.*, v. 46, 1925, p. 28.
[2] M. J. O. STRUTT, *Lamésche, Mathieusche, und verwandte Funktionen in Physik und Technik*, Berlin, Springer, 1932, p. 24.
[3] S. LUBKIN & J. J. STOKER, "Stability of columns and strings under periodically varying forces," *Quarterly Appl. Math.*, v. 1, 1943, p. 221.
[4] N. W. MCLACHLAN, "Computation of the solution of Mathieu's equation," *Phil. Mag.*, s. 7, v. 36, 1945, p. 404.
[5] STRATTON, MORSE, CHU & HUTNER, *Elliptic Cylinder and Spheroidal Wave Functions*, New York, Wiley, 1941; compare *MTAC*, v. 1, p. 157f, 414.

291[L].—NYMTP, *Tables of Associated Legendre Functions.* New York, Columbia University Press, 1945, xlvi, 303 p. 18.6×26.5 cm. Reproduced by a photo-offset process. $5.00.

Another important volume has been added to the impressive list of publications of the NYMTP. This work, devoted to the associated Legendre functions of first and second kind, is a very useful addition to the already extensive number of tables which have appeared in this area of functional computation as described earlier in *MTAC*, v. 1, p. 116–119. The present table "was begun in 1940, to meet urgent needs for a table to about six significant figures at intervals of 0.1. At this interval in the argument, interpolation in these functions is not everywhere satisfactory; a very considerable amount of further work would be required, in order to make all these functions reasonably interpolable over the entire range covered." The Project has on its agenda the subtabulation of the functions to a more satisfactory interval of the argument.

By the associated Legendre functions of mth order and nth degree are meant the functions $P_n^m(x)$ and $Q_n^m(x)$, the former called the first kind and the latter the second kind. In the range $x^2 \leq 1$, it is customary to define these functions as follows:

$$P_n^m(x) = (1 - x^2)^{\frac{1}{2}m} \frac{d^m P_n(x)}{dx^m}, \quad Q_n^m(x) = (1 - x^2)^{\frac{1}{2}m} \frac{d^m Q_n(x)}{dx^m},$$

where $P_n(x)$ is the nth Legendrean polynomial, and $Q_n(x)$ the nth Legendrean function of second kind, defined as follows:

$$Q_n(x) = \tfrac{1}{2}P_n(x) \log \frac{1+x}{1-x} - Z_n(x),$$

$$Z_n(x) = \frac{2n-1}{1\cdot n} P_{n-1}(x) + \frac{2n-5}{3(n-1)} P_{n-3}(x) + \frac{2n-9}{5(n-2)} P_{n-5}(x) + \cdots.$$

It should be observed, however, that H. TALLQVIST, in his tables, defines $Q_n(x)$ as twice the value of the one given.

In the range $x^2 > 1$, it is customary to change the multipliers of the derivatives of $P_n(x)$ and $Q_n(x)$ to $(x^2 - 1)^{\frac{1}{2}m}$ and to replace in the definition of $Q_n(x)$, the function $\log[(1 + x)/(1 - x)]$ by $\log[(x + 1)/(x - 1)]$.

In all previous computations of the Legendrean functions, the range of the independent variable has been restricted to the interval $-1 \leqq x \leqq +1$. The present work thus makes a fundamental contribution, since most of the tables are computed over ranges which lie outside the unit interval.

The present volume contains 19 tables which we may describe as follows:

In the first ten tables the values of m and n are taken over the range $n = 1(1)10$; $m = 0(1)4$, $m \leqq n$, and the computation is carried to 6S. **I.** $P_n^m(\cos\theta)$, $\theta = 0(1°)90°$. **II.** $dP_n^m(\cos\theta)/d\theta$, $\theta = 0(1°)90°$. **III.** $P_n^m(x)$, $x = 1(.1)10$. **IV.** $dP_n^m(x)/dx$, $x = 1(.1)10$. **V.** $(-1)^m Q_n^m(x)$, $x = 1(.1)10$. **VI.** $(-1)^{m+1} dQ_n^m(x)/dx$, $x = 1(.1)10$. **VII.** $i^{-n}P_n^m(ix)$, $x = 0(.1)10$. **VIII.** $i^{-n}dP_n^m(ix)/dx$, $x = 0(.1)10$. **IX.** $i^{n+2m+1}Q_n^m(ix)$, $x = 0(.1)10$. **X.** $i^{n+2m-1}dQ_n^m(ix)/dx$, $x = 0(.1)10$.

Of the remaining nine tables in the volume, the first four give the values of the functions and their first derivatives at fractional half orders, that is to say, for the subscript $n + \frac{1}{2}$, $n = -1(1) + 4$. In these tables $m = 0(1)4$. The tables may then be described as follows:

XI. $P^m_{n+\frac{1}{2}}(x)$, $x = [1(.1)10;$ 4S–6S]. **XII.** $dP^m_{n+\frac{1}{2}}(x)/dx$, $x = [1(.1)10;$ mainly to 6S]. **XIII.** $(-1)^m Q^m_{n+\frac{1}{2}}(x)$, $x = [1(.1)10;$ 6S]. **XIV.** $(-1)^{m+1}dQ^m_{n+\frac{1}{2}}(x)/dx$, $x = [1(.1)10;$ 6S].

The remaining five tables are supplementary and are contained in 10 pages. They may be described as follows:

XV. Exact values of $P_n^m(x)$ and $dP_n^m(x)/dx$, $m = 0, 2, 4$; $n = 4$ or $5(1)10$, $x = 1(.1)2$. **XVI.** Exact values of $P_n^m(x)/(x^2-1)^{\frac{1}{2}m}$ and of $\left[\frac{d}{dx}P_n^m(x)\right]\Big/(x^2-1)^{\frac{1}{2}m-1}$, $m = 1, 3$, $n = 2$ or $3(1)10$, $x = 1(.1)2$. **XVII.** Values of $(x^2-1)^{\frac{1}{2}}P^m_{n+\frac{1}{2}}(x)$ and of $(x^2-1)^{\frac{1}{2}}dP^m_{n+\frac{1}{2}}(x)/dx$, $m = 1, 3$, $n = -1(1)4$, $x = [1(.1)2;$ 4D–7D]. **XVIII.** Values of Legendre's normalizing factor, $N_n^m = \{2(n+m)!/[(2n+1)(n-m)!]\}^{\frac{1}{2}}$, $n = 0(1)10$, $m \leqq n$, 10S. **XIX.** First eleven coefficients in the expansion of $CF(m+\frac{1}{2};\ \frac{1}{2}-m;\ n+\frac{3}{2};\ -t)$, where C is equal to $\{\Gamma(\frac{1}{2})\Gamma(n+m+1)/[\sqrt{2}\Gamma(n+\frac{3}{2})]\}$.

Except in the case of $P_n^m(\cos\theta)$ and its derivative, which were computed directly from their expansions in terms of sin $p\theta$ and cos $p\theta$, the other tables were evaluated by means of standard recurrence formulae. In the final checking of the computation the following formula was used:

$$P_n^m(x)\frac{d}{dx}Q_n^m(x) - Q_n^m(x)\frac{d}{dx}P_n^m(x) = \frac{-(-4)^m\Gamma[\frac{1}{2}(n+m+1)]\Gamma[\frac{1}{2}(n+m+2)]}{(x^2-1)\Gamma[\frac{1}{2}(n-m+1)]\Gamma[\frac{1}{2}(n-m+2)]}.$$

In the case of $P_n^m(\cos\theta)$, when the values were checked against those published by Tallqvist, no errors were discovered.

In addition to the tables, a number of formulae are developed which were found useful in the computation of preliminary values. A bibliography of 47 references is also given, which provides an adequate survey of the present status of computation in this active field.

H. T. D.

292[L, M].—PIERO GIORGIO BORDONI, "The conical sound source," Acoustical Soc. Amer., *J.*, v. 17, Oct. 1945, p. 125.

$$\int_0^{2\pi} e^{iz\cos\phi}d\phi = \int_0^{2\pi} e^{-iz\cos\phi}d\phi = 2\pi T_0(z),\ G_n(z) = z^{-(n+2)}\int_0^z t^{n+1}T_0(t)dt.$$

There is a table of $G_n(z)$, for $n = 0(1)6$, $z = [0(1)5;$ 3D–4D].

293[L, M].—GREAT BRITAIN, NAUTICAL ALMANAC OFFICE, *Tables of the Integrals* $C(t) = k^{\frac{1}{2}}\int_0^\infty e^{-u}\cos(2tk^{\frac{1}{2}}u - ku^2)du$, $S(t) = k^{\frac{1}{2}}\int_0^\infty e^{-u}\sin(2tk^{\frac{1}{2}}u - ku^2)du$, Department of Scientific Research and Experiment, Admiralty Computing Service, September 1945. No. SRE/ACS. 90. 11 p. 20.1 × 30.8 cm. This publication is available only to certain Government agencies and activities.

Tables to 3D are given for C, S, and $(C^2 + S^2)^{\frac{1}{2}}$, for the following values of k and t:

$$
\begin{aligned}
k &= \tfrac{1}{4}, & t &= -10, -3(.1) + 3, 10;\\
k &= \tfrac{1}{2}, & t &= -8(1) - 5(.5) - 3(.1) + 8;\\
k &= 1, & t &= -6, -5(.5) - 3(.1) + 6;\\
k &= 2, 4, & t &= -5(.5) - 3(.1) + 7;\\
k &= 8, & t &= -5(.5) - 3(.1) + 9.
\end{aligned}
$$

"Graphical accuracy only was required and the method of computation was such that the last of the three decimals tabulated may be in error by a few units." There is a graph of $(S^2 + C^2)^{\frac{1}{2}}$ for $-5 < t < +6$.

The differential equations satisfied by $S(t)$ and $C(t)$ are

$$dS/dt = 2tC - S/k^{\frac{1}{2}}, \quad dC/dt = 1 - 2tS - C/k^{\frac{1}{2}},$$

for the range of values $-3 < t < +3$; $S(t)$ and $C(t)$ were obtained by numerical integration of these equations. For larger values of t, $S(t)$ and $C(t)$ were computed from certain appropriate asymptotic trigonometric series.

R. C. A.

294[M].—W. Heisenberg, "Die beobachtbaren Grössen in der Theorie der Elementarteilchen. II.," *Z. f. Physik*, v. 120, 1943, p. 702.

There is a short table of

$$p_n = \frac{1}{\pi}\int_0^\infty dx(2x)^n \frac{d}{dx}\arctan\left(\frac{4}{\sqrt{\pi}}\int_0^{x^{\frac{1}{2}}} ds\, s^2 e^{s^2}\right)$$

$p_1 = 1$, $p_2 = 3$, $p_3 = 13$, $p_4 = 79$, $p_5 = 641$, $p_6 = 6579$, etc. It is, of course, remarkable that these integrals are integers.

H. B.

295[U].—Hamburg, Deutsche Seewarte, publication no. 2154, *F-Tafel. Tafel zur vereinfachten Berechnung von Höhenstandlinien.* 3 Aufiage. Hamburg, August, 1941. xxiii, 88 p. 19.6 × 29.2 cm. In the third edition there were extensions and corrections of the introductory material, and of 8 of the 11 tables.

The method and principal table of this volume are similar in many respects to those of H. O. 208 (Dreisonstok, see *MTAC*, v. 1, p. 79f). The astronomical triangle is divided into two right spherical triangles by a perpendicular from the zenith upon the hour circle of the star; U is the co-declination of the foot of the perpendicular, and V is log cos B, where B is the angle subtended at the zenith by U. By Napier's rules,

$$\tan U = \cos t \cot L$$

and

$$\sin B = \sin t \cos L,$$

where t, L, and d are the local hour angle, latitude and declination respectively. By applying another of Napier's rules to the right triangle of which the star is one vertex, the altitude, h, may be found by

$$\sin h = \cos B \sin (d + U)$$

or

$$\log \sin h = V + \log \sin (d + U).$$

For the determination of azimuth, Z, two more auxiliary quantities are introduced, P which is the great circle distance from the star to the east- or west-point of the horizon, and Gr. δ which is the declination of the intersection of the hour circle of the star with the prime vertical. Thus, $\sin t \cos d = \cos P$ and $\sin Z = \cos P \sec h$. Also, $\tan \text{Gr. } \delta = \tan L \cos t$.

In **Table F I**, with vertical argument, latitude $0(1°)70°$, and horizontal argument, local hour angle $0(4^m)6^h$, three values per page, there are tabulated four quantities, U to the nearest $0'.1$, V to 5D, Gr. δ and P, each to the nearest $0°.1$. In the second part of Table F I, the vertical argument is latitude, $70°(1°)90°$, and the horizontal argument is local hour angle $0(4^m)6^h$, nine values per page, three in each horizontal section.

At the bottom of the vertical columns in Table F I are azimuths; entering the left hand column with altitude as argument, and moving across the pages horizontally until one finds under P, the value already copied out, one can drop to the bottom of the column and read off the azimuth angle. Since the tabulated values of the azimuth angle go up to 90° only, it is necessary to have another device to determine the quadrant. When the hour angle is greater than 6^h, the azimuth is measured from the elevated pole; when the local hour angle is less than 6^h and L and d are of opposite name, the azimuth is measured from the depressed pole. If the local hour angle is less than 6^h and L and d are of the same name, the azimuth is measured from the elevated or depressed pole according as the declination is greater or less than the quantity Gr. δ.

In case the altitude is great, or the azimuth near 90°, the value of the azimuth may be poorly determined by the use of Table F I. In such a case, it will be noted that the value of P lies below a dotted line running across the page. One must then use instead **Table F XI**, which gives P to the nearest minute of arc and the variation in P corresponding to $1'$ change in d or h.

Table F II is a table of $\log \sin x$, $x = [0(0' \cdot 1)6°(1')90°;\ 5D]$, with generous tables of proportional parts.

Tables F III and **F IV** represent the principal advantages this volume possesses over other similar tables; they permit one to determine the corrections (to the nearest $0'.1$) to the computed altitude corresponding to slight changes in time (up to 2^m by 10^s steps) or latitude (up to $30'$ by $1'$ steps) respectively. In both cases, one can interpolate very easily by a shift of the decimal point. **Table F III** is a well-designed triple-entry table occupying only five pages; one starts down the column at the left headed by the value nearest the assumed latitude, stops at the value nearest the computed azimuth and moves to the right to the column headed by the number of seconds change in time. **Table F IV** is a small double-entry table on a single page; the vertical argument is azimuth $0(5°)20°(2°)90°$, and the horizontal argument is change in latitude, $1'(1')10'(10')30'$. These two tables allow one to work either with an assumed position or with a dead reckoning position.

Table F V is for changing time into angular measure and conversely. **Table F VI** gives the corrections for refraction, semi-diameter and parallax to be applied to the altitude (3°–90°) of the lower- or upper-limb of the moon; there is a supplementary table for height of eye. **Table F VII** gives the combined correction for refraction and height of eye (0–30 meters) to be applied to the altitudes (3°–90°) of fixed stars or planets. **Table F VIII** yields the correction for refraction, semi-diameter and height of eye (0–30 meters) to be applied to altitudes (3°–90°) of the sun's lower limb; there are also two auxiliary tables to provide corrections to the altitudes to take care of the varying semi-diameter of the sun through the year, and for the case where the sun's upper limb was observed. The latter takes only a very small amount of space and would seem to be quite worthwhile. **Tables F IX** and **F X** provide similar corrections for use with the bubble sextant.

The tables are well printed on a good grade of paper. In a number of cases, the rules needed to make decisions as to quadrants, etc. are printed on each page. As for the accuracy of the tabulated values, only a few rounding off errors of a unit in the last place were discovered in a brief examination.

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296[U].—HYDROGRAPHIC OFFICE, *Publication no.* 351, *v.* 7, *Secret. Celestial Air Navigation Table. Volume* 7, *Latitudes* 60°–70°, Tokyo, November 1942. iii, 320 p. 18 × 25 cm.

This is the seventh volume of a series originally intended to include only six and to cover latitudes 60°S to 60°N. Since its general design and content are similar to those of the first six volumes which have already been reviewed (RMT **273**), it will suffice to point out the only change which has been made. The auxiliary table for changing time measure into angular measure has been replaced by another which has been pasted over the old one. Only the degrees were given in the main body of the old table and one had to move to the right hand edge of the page to pick up the minutes associated with the degrees; in the new table, the degrees and the minutes are given together in each entry.

It was earlier stated, *MTAC*, v. 2, p. 44, that v. 4 was the first of the series to be issued. Actually v. 3 appeared in August, 1940; v. 4 in May, 1941; v. 1 in September, 1941; v. 5 in December, 1941; and v. 6 in February, 1942. A copy of v. 2 has not yet reached the reviewer's hands.

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297[U].—HYDROGRAPHIC OFFICE, *Publication No.* 9582. *Secret. Altitude-Azimuth Tables for September* 1945 (*Sun, Moon, Planets and Fixed Stars*) *computed for Kisarazu, Naha, Iwo Jima and Kanoya and for Japanese Central Standard Time.* Tokyo, Japanese Hydrographic Office, August 1945, 42, vi, viii p., 18.2 × 25.7 cm.

These tables were designed for celestial air navigation over a limited region in the western Pacific. All descriptive material, explanations, star names, etc., are in Japanese characters; tabular material is in Arabic numerals. One volume is prepared for each month and contains all the material needed for that month. For each 20 minutes throughout the 24 hours of Japanese Central Standard Time (time zone −9), altitudes and azimuths of the celestial bodies, most useful to navigators, as computed for four locations, namely, Kisarazu near Tokyo, Naha on Okinawa, Iwo Jima, and Kanoya near Kagoshima, are given to the nearest minute of arc and to the nearest degree respectively. For the sun, the altitude (> 4°) is given with refraction for 4000 meters elevation added. In the case of the moon, the altitude (also > 4°) with corrections added for refraction at 4000 meters and parallax, is given. For bright stars and planets, the altitudes (occasionally down to 2° and always with refraction for 4000 meters), are also given.

Ten pages are used in presenting material for each of the four geographical positions listed above. Each page carries three vertical sections, each of which contains three double columns and includes data for a single day. The double column to the left gives the altitude and azimuth (say, 2956 050 which means 29° 56′ altitude, 50° azimuth) for a bright star until sunrise, then for the sun until sunset, and finally for another bright star until midnight. The bright stars in this column are chosen so as to be useful in determining latitude.

The second double column in the vertical section gives the altitude and azimuth of the moon when it is available for observation and for bright stars or planets for the remaining time. The third double column presents the altitude and azimuth of bright stars and planets.

Since only the sun and moon are easily observed in the daytime; one or two double columns would ordinarily be blank, opposite the daylight hours as listed on the left side of the page. These spaces have been filled with data for bright stars and planets, well located for observation in the ***night hours;*** this material is printed in heavy black type and the time argument is printed on the right side of the page in heavy black type also. Thus there are available for observation, one or two celestial objects during the day and five or four during the night. The planets included in this volume are Venus and Saturn; the bright stars used are Capella, Vega, Alpheratz, Arcturus, Regulus, Altair, Procyon, Rigel, Spica, Sirius, Antares, Fomalhaut and Canopus.

These tables differ considerably from common navigation tables in that much greater differences between the assumed and real positions of the observer are used here, and hence larger differences between computed and observed altitudes are encountered. In one of the examples worked out in the explanation in the back of the book, altitude intercepts of +2° 07′ and −37′ are used in plotting two lines of position and determining a fix. The inaccuracies introduced in this way would be too large for surface navigation, but may be tolerated in air navigation.

In addition to the principal tables just described, there is one page on which the altitude of Polaris is given for each of the four geographical positions named and for each 20 minutes of the night for every fifth day of the month. There follows a one-page interpolation table, giving the corrections to be applied to tabulated altitudes. The vertical argument is tabular difference in altitude $0(4')5°$, and the horizontal argument is: time interval $0^m\!.5(0^m\!.5)10^m$. There follow four pages of tables, one for each of the geographical positions giving corrections to the tabulated altitudes corresponding to other positions near the chosen ones. There are 10 such points near Tokyo, 8 near Iwo Jima, 9 near Kagoshima, and 6 near Okinawa. These positions vary as much as approximately 6° in latitude and 4° in longitude from the central position. It will be realized that these corrections will be most useful in the case of a body observed at low altitude. At the top of a column is the name of a geographical location for which the altitude correction applies. The left-hand column of the page contains values of the argument Z (azimuth of the celestial body) $0(3°)180°$; the right-hand column contains values of the argument Z $180°(3°)360°$. No provision is made for determining the change in azimuth in going from the standard position to the nearby one.

The remainder of the volume is devoted to explanations; an expert navigator may follow these, even though he does not read Japanese, since the numerals are Arabic. One notes that the page of tabular values reproduced for use in working the examples covers April 22–24, 1945, and that several improvements have been made in the arrangement of the tables since that time, particularly in the manner in which heavy black type is used to set apart night-time values printed opposite day-time hour arguments.

Since much of the air travel over oceans will in the future be over a limited number of well-defined paths, it may be that the type of table described here may come into use over such paths. The advantage of dispensing with the need of an almanac and the rapidity with which a fix can be determined may be considered as outweighing in *air* navigation, the inaccuracies introduced by the necessarily greater differences between dead reckoning position and assumed position. There remains the question as to whether or not it will be economically profitable to prepare such tables, no matter how convenient and rapid they may be, if their useful life is only one month.

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298[U].—P. V. H. WEEMS, *Line of Position Book*, fourth ed. Annapolis, Md. Weems System of Navigation, 1943, viii, 56 p. 14.5 × 24.3 cm. $2.00.

This book, in its first edition issued in 1927 by the U. S. Naval Institute, contained tables useful for latitudes 65° South to 65° North and was intended for surface navigation. The second edition appeared in 1928 and extended its field to include all latitudes; it also contained simplified procedures for aerial navigation. The third edition followed after an interval of 12 years. The author is now a Captain (retired) of the U. S. Navy and has long been well known for his contributions to the art and science of navigation. The tables in this volume are said to provide altitudes accurate to the nearest minute of arc without interpolation; greater accuracy can be had by interpolating.

There are three principal tables, Table A, Table B, and Table A-Polar. Table A has as vertical argument, latitude $0(1°)65°$, and as horizontal argument, local hour angle, $0(1°)180°$ by integral degrees. The values, 0 to 90° of local hour angle are found at the tops of pages, 5° per page except the first page which has 6°, and the tabulated quantities are A and K; local hour angles 90° to 180° are found at the bottoms of pages, and A and $(180° - K)$ are tabulated. In the notation of RMT **103**, A is $10^5 \log \sec a$ and is given to one decimal place for local hour angles 0 to 20°, and 160° to 180° to the nearest integer for local hour angles 20° to 160°. K is the declination of the foot of the perpendicular from the zenith upon the hour circle through the celestial body and is given to the nearest tenth of a minute of arc throughout.

Table B is a table of $10^5 \log \sec x$ for $x = 0(1')90°$ when the argument at the top and left of the page is used, or a table of $10^5 \log \csc x$ for the same interval when the argument at the right and bottom of the page is used. Values less than 665 are given to one decimal; other values to the nearest integer.

Tables A and B are credited to S. OGURA, hydrographic engineer of the Japanese Navy.

Table A-Polar is similar to Table A, except that latitudes 60° to 90° are covered and values of A less than 665 are given to one decimal, other values of A to the nearest integer.

Weems suggests that the formula $B(h) = A + B(K \sim d)$ be used for the determination of the altitude and that the azimuth diagram of ARMISTEAD RUST, USN Captain (retired), be used to determine the azimuth. This diagram is reproduced on a scale 1.5 times the original on two pages, one of which is folded. A diagram enabling one to determine the hour angle of a celestial body on the prime vertical takes care of the ambiguous case in the determination of the quadrant of the azimuth angle arising when the latitude is of the same name and greater than the declination.

Auxiliary tables given in the volume are those providing corrections for observed altitudes obtained with ordinary sextants and with bubble sextants. A brief but adequate explanation of the use of the tables is given and several examples are worked out.

In the preface to the third edition, the author says, "The Line of Position Book has stood the test of time. No errors have been noted in the Altitude Tables. Except for the method of H.O. 214, it remains the shortest, easiest method for working the line of position." In this statement, the author fails to mention one of the strong points of his method, namely that it works in all latitudes and for all altitudes. All of the modern tables with which a sight can be reduced in appreciably less time than with the *Line of Position Book* are limited methods, usually limited in both latitude and altitude.

Perhaps the outstanding criticism to be made of this book of tables is that local hour angle changes from page to page rather than latitude. There is a definite advantage in finding all of the data to be used at one time on a single page rather than scattered through the book. By the same argument, the polar table should be amalgamated with Table A. It may be noted here that these changes have been made by the author in his *New Line of Position Tables*, Annapolis, Md., 1944.

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MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **286** (Silberstein); N **53** (Furnas, Wrinch); **54** (McIntyre); Q **17** (?).

79. *Barlow's Tables of Squares, Cubes, Square Roots, Cube Roots, and Reciprocals, of all Integer Numbers up to* 12500, edited by L. J. COMRIE. Fourth ed., 1941. See *MTAC*, v. 1, p. 16–17.

As a part of the checks applied in the transcription of these tables to punch-cards, the tabular values of $f(n) = \sqrt{n}$, $\sqrt{10n}$, $\sqrt[3]{n}$, and $1/n$ for $n = 1000(1)12500$ were inverted; i.e., $f^{-1}[f(n)]$ was found. Differential corrections accurate to a few additional decimal places were computed. Doubtful cases were further examined. All values were found to be correct, except two which need a correction of a unit in the last place:

1) For $n = 3195$, $\sqrt{10n}$ is tabulated as 178.745630. The correct value is 178.7456293, which rounds off to 178.745629. Also the corresponding differences must be changed to 27975, 27971.

2) For $n = 4575$, $\sqrt[3]{n}$ is tabulated as 16.600851. The correct value is 16.6008516, which rounds off to 16.600852. Also the corrected differences, 1210, 1209.

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EDITORIAL NOTE: These errors are also to be found in the third ed. 1930, and in the 1935 pirated ed.

80. JAMES BURGESS, "On the definite integral $(2/\pi^{\frac{1}{2}})\int_0^t e^{-t^2}dt \cdots$," See *MTAC*, v. 1, p. 449.