UNPUBLISHED MATHEMATICAL TABLES

For other unpublished tabular numbers see MTE **87** (Lehmer and Wrench).

46[B, J].—ENZO CAMBI, Tables of $S_n = 1 + 2^{-n} + 3^{-n} + \cdots$, and of $(1 - x^2)^{\frac{1}{2}}$, mss. in possession of, and prepared by the author, Via Giovanni Antonelli 3, Rome, Italy.

I have computed S_n, for $n = [3(2)71; 42$ to $60D]$; also a table of $(1 - x^2)^{\frac{1}{2}}, x = [0(.001)1; 16D]$.

E. CAMBI

EDITORIAL NOTE: Compare MTE **87**, and *MTAC*, v. 1, p. 456–457. Of the function $(1 - x^2)^{\frac{1}{2}} = P_{1,1}(x)$, there are tables in A. H. H. TALLQUIST Finska Wetenskaps-Societeten, *Acta*, v. 33, 1906, p. 9, $x = [0(.01)1;10D]$; in T. L. KELLEY, *The Kelley Statistical Tables*, New York, 1938, p. 14f, $x = [0(.0001)1;8D]$; and in J. R. MINER, *Tables of* $\sqrt{1 - r^2}$ *and* $1 - r^2$ *for use in partial correlation and in Trigonometry*, Baltimore, Md., 1922, $x = r = [0(.0001)1;6D]$.

47[E].—K. S. JOHNSON & P. H. RICHARDSON, *Tables of Useful Exponential and Hyperbolic Functions*. 16 p. 22 × 25.5 cm. Ms. property of the Bell Telephone Laboratories, 463 West St., New York City.

In RMT **243**, reference was made to the first published mathematical tables, 1945, with arguments decibels (N), and the corresponding number of nepers (x). There, for $N = [.05, .1(.1)2(.2)4(.5)20(1)50(5)100; 5\text{–}6S]$, are given values of x, e^x, e^{-x}, e^{2x}, sinh x, cosh x, tanh x, tanh $(\frac{1}{2}x)$. Johnson & Richardson's tables, calculated in October, 1929 (with revisions at the beginning and end in August, 1935) are of x, e^x, e^{-x}, sinh x, cosh x, and tanh x, for $N = [.001(.001).01(.005)1(.1)30(.2)55; 5\text{–}7S]$. Near the beginning of the table are some entries of more than 7S.

For large values of N the corresponding values of e^x or e^{-x} may be obtained by moving the decimal point to the right or left one point for each 20 N. For example, corresponding to $N = 75$ (from $N = 55$), $e^x = 5623.416$ and $e^{-x} = .00017783$.

Sinh x and cosh x may then be obtained from the rigorous relations

$$\sinh x = \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x}, \quad \cosh x = \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x};$$

or from the approximate relationship

$$\sinh x \doteq \cosh x \doteq \tfrac{1}{2}e^x, \quad \tanh x = \sinh x/\cosh x \doteq 1 - 2e^{-2x}.$$

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48[F].—KULIK'S *Magnus Canon Divisorum* . . ., 8 ms. volumes deposited in the Library of the Academy of Sciences, Vienna, after Kulik's death in February 1863; see Akad. d. Wissen., Vienna, *Sitzungsb.*, v. 53, 1866, p. 460–462, and POGGENDORFF, *Biog.-Literar. Handwörterbuch*, v. 3, 1897, p. 757. Compare MTAC, v. 2, p. 30, 59–60.

EDITORIAL NOTE: This great factor table of numbers to 100,330,200 (except for multiples of 2,3,5) has the following title: *Magnus Canon Divisorum pro omnibus numeris per* 2, 3, *et* 5 *non divisibilibus, et numerorum primorum interjacentium ad millies centena millia accuratius ad* 100 330 201 *usque*. Details of the notation and of other matters connected with the table were set forth by D. N. LEHMER in his 1. *Factor Table for the First Ten Millions* . . ., Washington, 1909, cols. IX, X, XIII, XIV, and 2. *List of Prime Numbers from* 1 *to* 10,006,721, Washington, 1914, cols. XI, XII. Lehmer mentions that volume 2 of Kulik's table was already missing 35 years ago. Mainly with information furnished by Lehmer, S. A. JOFFE figured out the exact contents of each of the 8 volumes. On submitting this survey to the Librarian of the Academy of Sciences, W. OBERHUMMER, we learned from him, by a letter dated 21 February 1946, that the analysis was accurate in every particular; that volume 2 was still missing; and that the other 7 volumes are intact in the Academy's Library. Mr. Joffe's report now follows.

As indicated in *MTAC*, v. 2, p. 30, D. N. Lehmer published information regarding the number of pages in each of the extant volumes, and the corresponding range of consecutive integers covered in each volume. A study of Lehmer's results, which are incorrect in several instances, enabled the writer to correct Lehmer's errors and to present the consecutive results in tabular form.

When Kulik commenced work on his factor table, there were no other factor tables but those of J. Ch. Burckhardt, *Table des Diviseurs*, v. 1, 1817, v. 2, 1814, v. 3, 1816, and Kulik began where Burckhardt left off. However, the last integer in Burckhardt's v. 3 is 3,036,000, but for some reason not yet ascertained the opening-number in Kulik's v. 1 is 3,033,001.

Disregarding Kulik's missing v. 2, we find that in all the following volumes the tables are written on both sides of the paper. On the assumption that the methodical and economical Kulik would cover each page in full, without blank spaces, and that therefore the right and left sides of each volume must contain the same number of pages, (and also of consecutive integers), it was established that each page covered 23,100 consecutive integers. Incidentally, from this we see that a million consecutive integers would require, roughly 43⅓ p., as stated in *MTAC*, v. 2, p. 30.

The following table gives in detail the contents of each v., both as to the number of pages and the number of consecutive integers.

Volume r=right l=left	Number of p. in vol.	Serial number of p. in vol.	Number of Consec. Integers in vol.	Cumulative Consecutive Integers in set of 8 vol.
I	416	1–416	9,609,600	3,033,001 – 12,642,600
II	442	417–858	10,210,200	12,642,601 – 22,852,800
III r	277	859[1]–1135	6,398,700	22,852,801 – 29,251,500
IV r	276	1136–1411	6,375,600	29,251,501 – 35,627,100[2]
V r	276	1412–1687	6,375,600	35,627,101 – 42,002,700
VI r	276	1688–1963	6,375,600	42,002,701 – 48,378,300
III l	277	1964–2240	6,398,700	48,378,301 – 54,777,000
IV l	276	2241–2516	6,375,600	54,777,001 – 61,152,600
V l	276	2517–2792	6,375,600	61,152,601 – 67,528,200
VI l	276	2793–3068	6,375,600	67,528,201 – 73,903,800
VII r	285	3069–3353	6,583,500	73,903,801[3] – 80,487,300
VIII r	287	3354–3640	6,629,700	80,487,301 – 87,117,000
VII l	285	3641–3925	6,583,500	87,117,001 – 93,700,500[4]
VIII l	287	3926–4212	6,629,700	93,700,501 – 100,330,200

[1] Lehmer has 828.
[2] Lehmer has 35,626,799(800).
[3] Lehmer has 79,903,801.
[4] Lehmer has 93,709,499(500).

S. A. Joffe

49[L].—NYMTP, *Tables of Intensity Functions*. Tables prepared by and in possession of the NYMTP.

A. Several years ago the NYMTP, upon the recommendation of V. K. LaMer, professor of chemistry at Columbia University, then a member of Division 10 of the National Defense Research Council (of the Office of Scientific Research and Development), computed extensive tables of Intensity Functions for both real and complex values of the index of refraction. The pertinent tabular material relating to real indices of refraction has since been incorporated in a joint report by LaMer & Sinclair, entitled *Progress Report on the Verification of Mie's Theory—Calculations and Measurements of Light Scattering by Dielectric Spherical Particles*, OSRD Report 1851, September 29, 1943. A limited number of mimeographed copies of this report have been made available to the OSRD for distribution to those working on problems related to scattering of light by spherical particles. A detailed description of the complete manuscript on Intensity Functions, prepared by the NYMTP for both real and complex indices of refraction, is given below. The physical significance of the tabulated functions is discussed in **B**.

(1) *First Part*

Definitions:

$$i_1 = \left| \sum_{n=1}^{\infty} \{A_n\pi_n + P_n[x\pi_n - (1 - x^2)\pi_n']\} \right|^2$$

$$i_2 = \left| \sum_{n=1}^{\infty} \{A_n[x\pi_n - (1 - x^2)\pi_n'] + P_n\pi_n\} \right|^2$$

$$\tfrac{1}{2}\alpha^2 K(m;\alpha) = \tfrac{1}{2}\int_0^{\pi} (i_1 + i_2)\sin\gamma d\gamma = \sum_{n=1}^{\infty} \frac{R^2(A_n) + I^2(A_n) + R^2(P_n) + I^2(P_n)}{(2n+1)/n^2(n+1)^2}$$

$$A_n = a_n/n(n+1); \qquad P_n = p_n/n(n+1); \qquad x = \cos\gamma,$$

$$a_n = (-1)^{n+\frac{1}{2}}(2n+1)\frac{S_n'(\beta)S_n(\alpha) - mS_n'(\alpha)S_n(\beta)}{S_n'(\beta)\varphi_n(\alpha) - m\varphi_n'(\alpha)S_n(\beta)},$$

$$p_n = (-1)^{n+\frac{1}{2}}(2n+1)\frac{mS_n(\alpha)S_n'(\beta) - S_n(\beta)S_n'(\alpha)}{m\varphi_n(\alpha)S_n'(\beta) - S_n(\beta)\varphi_n'(\alpha)};$$

and

$$S_n(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x); \qquad C_n(x) = (-1)^n(\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_{-n-\frac{1}{2}}(x);$$

$$\pi_n = \pi_n(x) = \frac{\partial P_n(x)}{\partial x}; \qquad \pi_n' = \pi_n'(x) = \frac{\partial^2 P_n(x)}{\partial x^2};$$

$$\varphi_n(x) = S_n(x) + iC_n(x); \qquad \beta = m\alpha; \qquad \alpha = 2\pi r/\lambda.$$

$R(A_n)$, $I(A_n)$, $R(P_n)$ and $I(P_n)$ stand for the real and imaginary parts of A_n and P_n respectively.

$J_{n+\frac{1}{2}}(x)$ and $J_{-n-\frac{1}{2}}(x)$ are Bessel Functions of half-integral order. $P_n(x)$ is the Legendre Polynomial.

The tabulated material consists of values of i_1, i_2, $R(A_n)$, $I(A_n)$, $R(P_n)$ and $I(P_n)$ (the number of decimal places and significant figures varying), for $m = 1.33$, 1.44, 1.55, 2.0; $\alpha = .5$, .6, 1, 1.2, 1.5, 1.8, 2, 2.4, 2.5, 3, 3.6, 4, 4.8, 5, 6; $\gamma = 0(10°)180°$.

The integral $\tfrac{1}{2}\int_0^{\pi} (i_1 + i_2)\sin\gamma d\gamma$ is given to 3S for the above values of m and α.

(2) *Second Part*

$$K(m;\alpha) = \frac{2}{\alpha^2}\,\text{Real}\left\{\sum_{n=1}^{\infty} (-1)^n(2n+1)(C_n^1 + C_n^2)\right\},$$

where

$$C_n^1 = -ia_n/(2n+1), \qquad C_n^2 = ip_n/(2n+1),$$

a_n and p_n having the same meaning as in (1).

The tabulated material consists of

$K(m;\alpha)$, $(-1)^n(2n+1)$ Real (C_n^1) and $(-1)^n(2n+1)$ Imag (C_n^1)

for $m = 1.5$; $\alpha = .5$, .6, 1(.5)3, 1.2(1.2)12, 1.8, 3.2, 3.4, 3.8, 4(.5)5, 5.3, 7(1)10; while n varies from a maximum of 1 for $\alpha = .5$ to a maximum of 16 for $\alpha = 12$.

$K(m;\alpha)$ is given everywhere to 4S while the other functions are given to a constant number of D (predominantly 4).

(3) *Third Part*

$$F(m;\alpha) = K(m;\alpha) + iL(m;\alpha)$$

where $F(m;\alpha)$ is the complex function defined under (2) i.e. $K(m;\alpha) = \text{Real}\{F(m;\alpha)\}$.

If in above expression m is replaced by $z = m - imk$ then

$$F(z;\alpha) = F(m - imk;\alpha) = K(m, -mk;\alpha) + iL(m, -mk;\alpha).$$

The function desired was $K(m, -mk;\alpha)$ for $m = 1.44, 1.55$, the same values of α listed under (1) in addition to $\alpha = 3.2, 4.5$, and $k \gtreqless 10^{-1}$. Instead of tabular values of $K(m, -mk;\alpha)$ this section contains

(a) Approximation polynomials for $K(m;\alpha)$, $L(m;\alpha)$, $K(1.5, -1.5k;\alpha)$ and $K(m, -mk;\alpha)$
(b) Tables of $K(m;\alpha)$ and $L(m;\alpha)$ for the above values of α and m = [1.44(.01)1.55; 4D or 5D]

A. N. Lowan

B. These Intensity Functions give the angular distribution of intensity and the total light scattered by a small, spherical particle (such as a fog droplet) as a function of the parameter, $\alpha = 2\pi r/\lambda$, when the particle radius, r, is roughly equal to the wavelength, λ, of the incident light. The angular distribution and total light scattered by a suspension of particles of uniform size can be used to determine the particle size and concentration as described below.

In Part 1, the angular distribution functions i_1 and i_2 are proportional to the intensities of the two incoherent, plane polarized components scattered by a transparent particle illuminated with natural light. i_1 is the component whose electric vector is perpendicular to the plane of observation (the plane containing the direction of observation and the direction of propagation of the incident light), and i_2 is polarized perpendicular to i_1. When the particle is illuminated by two (2) units of energy per unit cross-sectional area, the actual intensities scattered per unit solid angle in the direction γ are $\lambda^2 i_1/(4\pi^2)$ and $\lambda^2 i_2/(4\pi^2)$. γ is the angle between the direction of propagation of the scattered light and the reversed direction of propagation of the incident light.

The polarization, i.e., the relative values of i_1 and i_2 (at a fixed angle γ), measured with a polarization photometer, provide a measure of particle size from .05 to .2 micron radius. Also, the angle of maximum polarization may be used.

When the particle is illuminated by polarized light of unit intensity, $\lambda^2 i_1/(4\pi^2)$ is the intensity scattered in a direction perpendicular to the incident electric vector, and $\lambda^2 i_2/(4\pi^2)$ is the intensity scattered in the plane containing the incident electric vector and its direction of propagation. The light scattered in any other directions by a particle illuminated by plane polarized light is elliptically polarized and i_1 and i_2 do *not* give its components.

Pairs of values of α, whose ratio is 1.2, were chosen for each value of the refractive index m. 1.2 is the wavelength ratio of red light ($\lambda = .629\mu$) to that of green light ($\lambda = .524\mu$), the most distinctive colors observed. Consequently, each pair of α values yields the intensities scattered at these two wavelengths (or any other pair of wavelengths bearing the same ratio) by a particle of a given radius.

The relative intensity of red to that of green in either i_1 or i_2 alone is a function of the angle γ. The number and angular position of the maxima of this function provide a measure of particle size from about .1 to 1.2 micron radius. The particle radius in microns is given roughly by dividing the number of maxima in i_1 by 10.

The integral or the sum when multiplied by $\lambda^2/(2\pi)$ gives the total light scattered by a particle when illuminated at unit intensity.

In Part 2, the total scattering coefficient, K, defined as the total energy scattered per second per unit cross-sectional area of particle (illuminated at unit intensity), is given for a transparent material of refractive index 1.5, over a greater range of α values, chosen as described in Part 1. (K can be obtained for the refractive indices in Part 1 by multiplying the integral or sum in Part 1 by $2/\alpha^2$).

In Part 3, the total scattering (and absorbing) coefficient, K, may be obtained for absorbing materials of extinction coefficient, k, ($e^{-4\pi k}$ is the fraction of light absorbed in travelling a distance equal to one wavelength through the bulk material) varying from zero to .1, and of real refractive index varying from 1.44 to 1.55. For the higher values of the extinction coefficient, the total scattering coefficient may be in error by 1%. When the extinction coefficient is zero, approximate values of the scattering coefficient will be obtained for transparent materials of refractive index as low as 1.33 and as high as 1.65 by substituting into the polynominal expression for $K(m;\alpha)$.

The number concentration, n, or the particle size, can be obtained when either is known by measuring the transmission at a known wavelength; or both can be obtained by measuring the transmission at two or more wavelengths. The transmission, $T = e^{-K\pi r^2 nl}$, where $K\pi r^2 n$ is the "absorption" coefficient as usually defined, i.e., $e^{-K\pi r^2 nl}$ is the fraction of light scattered by transparent particles or scattered and absorbed by absorbing particles per unit distance in the suspension.

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MECHANICAL AIDS TO COMPUTATION

See also the two introductory articles of this issue, as well as RMT **305**.

22. DE FOREST ON ELECTRICAL COMPUTERS.—To the *Yale Scientific Monthly*, published by the Senior class of the Sheffield Scientific School, Yale University, LEE DE FOREST (1873–) contributed an article ("A Wheatstone bridge for solving numerical equations," v. 3, 1897, p. 200–206),[1] explaining an idea of his for the electrical solution of equations. After describing a scheme in which the operator achieved his solution by adjusting the sliders on resistance potentiometers so as to balance a Wheatstone bridge, he pointed to the possibility of an "automatic balancer." He believed that "a relay type galvanometer driving or reversing some electrically governed mechanism might be devised, which would keep this length shifting until balance was attained."

If not the first, this is at least a very early description of the self-centering servomechanism as a computing device, the basic unit of some of the most important military computers of the recent war, including the electrical gun director for the control of antiaircraft fire.

DeForest may be pardoned for his qualifying remark—"it could not be very accurately done"—since, presumably, he did not yet know that he was going to invent the three-element thermionic valve, which did so much to make high precision possible.

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[1] EDITORIAL NOTE: This machine is one which escaped the attention of J. S. FRAME in his survey article "Machines for solving algebraic equations," *MTAC*, v. 1, p. 337–353. He did, however, deal with the machines of G. B. GRANT, C. V. BOYS, L. L. C. LALANNE referred to by DeForest in his introductory paragraphs. DeForest adds the new remark, that A. W. PHILLIPS (1844–1915) of Yale University was in 1879 an independent inventor of Lalanne's machine. Phillips was dean of the Graduate School 1895–1911, and the joint author of a number of texts in elementary mathematics.

NOTES

56. APPROXIMATIONS TO π.—In R. So. London, *Trans.*, 1841, p. 281–283, WILLIAM RUTHERFORD (1798?–1871) gave a value of π to 208D, which was correct to 152D, and derived from the Euler formula (1764)

$$(1) \qquad \frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}.$$