EDITORIAL NOTE: See V. BUSH, F. D. GAGE & H. R. STEWART, "A continuous integraph," Franklin Institute, *J.*, v. 203, 1927, p. 63–84. Among other things this instrument integrates the product of two functions, making use of the principle of electrical integrating watt hour-meter, combined with a moving table.

NOTES

59. ADMIRALTY COMPUTING SERVICE.—In *MTAC* we have recently had occasion more than once to refer to publications of this Service, for example, v. 2, p. 31, 35, 36, 39, 40, 80. "At the end of the War, the Admiralty Computing Service was providing a fairly comprehensive mathematical and computational service which was not only meeting all demands from Admiralty sources but was also able to offer informal assistance to the other Services, Government departments and contractors who had no comparable facilities at their disposal." This is a quotation from an article in *Nature*, v. 157, 4 May 1946, p. 571–573, entitled, "Mathematics in Government Service and industry. Some deductions from the war-time experience of the Admiralty Computing Service." Actual computations were carried out at the Nautical Almanac Office by a group under the direction of its Superintendent, D. H. SADLER, and "arrangements were made whereby scientific workers in the universities and elsewhere could be employed as consultants." Most of the members of the computing groups were recently transferred to the permanent National Mathematical Laboratory in the Mathematics Division of the National Physical Laboratory at Teddington. This Division is under the direction of Mr. J. R. WOMERSLEY. In view of our introductory article in the present issue of *MTAC* it is interesting to note in the *Nature* survey that "our experience fully confirms the statements made on many occasions by Dr. L. J. Comrie, . . . that full exploitation of the capabilities of the commercial calculating machines (including the National accounting machine and the Hollerith) is usually the most efficient way of dealing with problems, and specially designed calculating machines and instruments are necessary only for large-scale investigations of infrequent occurrence."

60. COEFFICIENTS IN AN ASYMPTOTIC EXPANSION FOR $\int_a^b e^{P(u)}du$.—The expansion for $\int_a^b e^{P(u)}du$ which is given below is well known in principle, since it is the result of merely continuing an integration by parts where at each successive step $e^{P(u)}du$ is replaced by $\frac{de^{P(u)}}{P'(u)}$. It has been employed extensively by several mathematicians of the NYMTP, especially for the purpose of extending the range of the usual asymptotic series by expressing the remainder $\int_a^b F(u)du$ as $\int_a^b e^{P(u)}du$. This method is often applicable to an integral of a real oscillatory function by considering it as either the real or imaginary part of a complex integral over the range a to b, provided that $P' \equiv P'(u)$ has no zeros in that range while $P^{(m)} \equiv P^{(m)}(u)$ are sufficiently small in comparison with P'.

The purpose of the expansion below is to enable one to obtain many terms of an asymptotic expansion, while avoiding the excessive labor of differentiating more and more complicated expressions that arise in repeatedly integrating by parts the various special functions that might occur in the integrand. It turns out to be much more expedient to work from this general expression in terms of the derivatives of the logarithm of the integrand. The following expansion was arranged as a series in $1/P'$, and it includes terms as far as the twelfth power[1] of $1/P'$:

$$\int_a^b e^{P(u)}du \sim |P_1^{-1}e^P[1 + P_1^{-2}P_2 - P_1^{-3}P_3 + P_1^{-4}(P_4 + 3P_2^2) - P_1^{-5}(P_5 + 10P_3P_2) + P_1^{-6}(P_6 + 15P_4P_2 + 10P_3^2 + 15P_2^3) - P_1^{-7}(P_7 + 21P_5P_2 + 35P_4P_3 + 105P_3P_2^2) + P_1^{-8}(P_8 + 28P_6P_2 + 56P_5P_3 + 35P_4^2 + 210P_4P_2^2 + 280P_3^2P_2 + 105P_2^4) - P_1^{-9}(P_9 + 36P_7P_2 + 84P_6P_3 + 126P_5P_4 + 378P_5P_2^2 + 1260P_4P_3P_2 + 280P_3^3 + 1260P_3P_2^3) + P_1^{-10}(P_{10} + 45P_8P_2 + 120P_7P_3 + 210P_6P_4 + 630P_6P_2^2 + 126P_5^2 + 2520P_5P_3P_2 + 1575P_4^2P_2 + 2100P_4P_3^2 + 3150P_4P_2^3 + 6300P_3^2P_2^2 + 945P_2^5) - P_1^{-11}(P_{11} + 55P_9P_2 + 165P_8P_3 + 330P_7P_4 + 990P_7P_2^2 + 462P_6P_5 + 4620P_6P_3P_2 + 6930P_5P_4P_2 + 4620P_5P_3^2 + 6930P_5P_2^3 + 5775P_4^2P_3 + 34650P_4P_3P_2^2 + 15400P_3^3P_2 + 17325P_3P_2^4)]|_a^b,$$

where P, $P'[= P_1]$, $P^{(2)}[= P_2]$, etc. are taken at the limits $u = b$ and $u = a$. It usually happens that $b = \infty$ and the expression vanishes at ∞.

To illustrate the effectiveness and convenience of this method, consider the ordinary asymptotic expansion for the error function in the form $F(z) \equiv e^{z^2}\int_z^\infty e^{-u^2}du$. It is well known that

$$F(z) = \frac{1}{2z} - \frac{1}{4z^3} + \frac{1\cdot 3}{8z^5} - \frac{1\cdot 3\cdot 5}{16z^7} + \cdots + \frac{(-1)^n 1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}z^{2n+1}} + R_{n+1},$$

where

$$R_{n+1} = (-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}}e^{z^2}I_{n+1},$$

and where

$$I_{n+1} = \int_z^\infty \frac{e^{-u^2}du}{u^{2n+2}}.$$

Thus

$$I_{n+1} = \int_z^\infty e^{-u^2-(2n+2)\ln u}du, \quad P = -u^2 - (2n+2)\ln u,$$

$$P' = -2u - \frac{2n+2}{u}, \quad P^{(2)} = -2 + \frac{2n+2}{u^2},$$

and for $m > 2$,

$$P^{(m)} = \frac{(-1)^m(m-1)!(2n+2)}{u^m}.$$

Hence one obtains

$$I_{n+1} = \frac{e^{-z^2}}{2z^{2n+1}(z^2+n+1)}\left[1 + \frac{(-z^2+n+1)}{2(z^2+n+1)^2} - \frac{(n+1)}{2(z^2+n+1)^3} + \frac{3(z^4 - 2(n+1)z^2 + (n+1)(n+2))}{4(z^2+n+1)^4} + \frac{(n+1)(5z^2-5n-8)}{2(z^2+n+1)^5} + \cdots\right].$$

Suppose that $z = 3$ and $n = 9$. Then the ordinary asymptotic series will have a relative error of about $2 \cdot 10^{-4}$. But the first five terms of this expansion improve the accuracy by a factor whose relative error is about 10^{-5}, or in other words, about 5 additional places. For larger values of z and n, taking further terms for I_{n+1} would improve it by relatively much more. This expansion for I_{n+1} can be used for almost any real or complex value of z that is not too small, also provided that z is not "too near" either of the singular points $\pm i\sqrt{n+1}$.

HERBERT E. SALZER

NYMTP

[1] For convenience of printing, the order of the derivative has here been replaced by a subscript, so that the kth derivative raised to the lth degree, i.e. $P^{(k)l}$, is here denoted by P_k^l.

61. GUIDE (*MTAC*, no. 7), SUPPL. 4 (for Suppl. 1–3, see *MTAC*, v. 1, p. 403, v. 2, p. 59, 92).—The following entries are extracted from an unpublished ms. of 263 p., found in Darmstadt, Germany, *Verzeichnis berechneter Funktionen* by ERICH WILLI HERMANN KAMKE (1890–):

F. BORGNIS, "Die elektrische Grundschwingung des kreiszylindrischen Zweischichten-Hohlraums," *Hochfrequenztechnik u. Elektroakustik*, v. 59, 1942, p. 23.

Graphs for the least solutions y of

$$\frac{aJ_0(y)J_1(ay) - J_1(y)J_0(ay)}{aY_0(y)J_1(ay) - Y_1(y)J_0(ay)} = \frac{J_0(y/x)}{Y_0(y/x)},$$

for $x = .1, .25(.25)1; 0 \leqq a \leqq 9$.

K. FEDERHOFER, "Biegungschwingungen der in ihrer Mittelebene belasteten Kreisplatte," *Ingenieur-Archiv*, v. 6, 1935, p. 73. Solutions to 4D of the equations

$$ivJ_0(u)J_1(iv) - uJ_0(iv)J_1(u) = 0$$
$$v^2 - u^2 = m^2, \quad m = -4(1)0.$$

K. FEDERHOFER, "Berechnung der Auslenkung beim Ausbeulen dünner Kreisplatten," *Ingenieur-Archiv*, v. 11, 1940, p. 121–124.

Values to 4S of $U_1 = \int_0^{x_1} tJ_1^4(t)dt$, and $U_2 = \int_0^{x_1} J_1^4(t)dt/t$, $x_1 = 3.8317$, being the approximate first zero of $J_1(x)$.

G. FRANKE, "Die Theorie der Resonanzmembran," Siemens-Konzern, *Wiss. Veröffentl.*, v. 9, part 2, 1930, p. 162 and 164.

(a) The two smallest solutions of the equations

$$\tfrac{1}{4}\epsilon xk[J_0(xk)Y_0(x) - Y_0(xk)J_0(x)] = J_1(xk)Y_0(x) - Y_1(xk)J_0(x), \text{ and}$$

(b) The smallest solutions of the equations

$$J_0(xk)Y_1(x) - Y_0(xk)J_1(x) = \left(\frac{1}{xk} - \frac{\epsilon}{4}xk\right)[J_1(xk)Y_1(x) - Y_1(xk)J_1(x)],$$

for $\epsilon = 0, 30, 50, 100$; $k = [0, \frac{1}{8}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}; 3D]$. Also figures for $\epsilon = 30, 50, 100, \frac{1}{6} \leqq k \leqq \frac{3}{4}$.

A. A. GERSHUN, "Berechnung des Volumleuchtens," *Physikalische Z. d. Sowjetunion*, v. 2, 1932, p. 180.

$$F(x) = 1 - \int_0^{\pi/2} e^{-x\cos t}\cos t dt = \tfrac{1}{2}\pi[\mathbf{H}_1(ix) - iJ_1(ix)]$$
$$= \tfrac{1}{2}\pi[I_1(x) - L_1(x)],$$

where

$$L_1(x) = -\mathbf{H}_1(ix) = \sum_{m=1}^{\infty} \frac{(x/2)^{2m}}{\Gamma(m+\frac{1}{2})\Gamma(m+\frac{3}{2})},$$

for $x = [0(.1)1(.2)2(.5)6(1)10; 4D]$.

K. KARAS, "Eigenschwingungen inhomogener Saiten," Akad. d. Wissen., Vienna, *Sitzungsb.*, Abt. IIa, v. 145, 1936, p. 797–826, contains results as follows:

I. The first two solutions of the equations

$J_{1/3}(\frac{2}{3}\sqrt{x}) = 0$, $J_{1/4}(\frac{1}{2}\sqrt{x}) = 0$, $J_{\pm 1/3}(\frac{1}{6}\sqrt{2x}) = 0$, $J_{-2/3}(\frac{1}{6}\sqrt{2x}) = 0$, $J_{\pm 1/4}(\frac{1}{8}\sqrt{x}) = 0$, $J_{-3/4}(\frac{1}{8}\sqrt{x}) = 0$, to 3D or 5–6S, (21), (27), (61), (62), (64), (68), (69), (71), p. 803, 820.

II. The first two solutions, to 3D, of the equations

$$J_{1/3}(\tfrac{2}{3}\sqrt{x})J_{-1/3}(\tfrac{4}{3}\sqrt{2x}) - J_{-1/3}(\tfrac{2}{3}\sqrt{x})J_{1/3}(\tfrac{4}{3}\sqrt{2x}) = 0,$$
$$J_{1/3}(\tfrac{2}{3}\sqrt{x})J_{2/3}(\tfrac{1}{6}\sqrt{2x}) + J_{-1/3}(\tfrac{2}{3}\sqrt{x})J_{-2/3}(\tfrac{1}{6}\sqrt{2x}) = 0,$$
$$J_{1/3}(\tfrac{2}{3}\sqrt{x})J_{-1/3}(2\sqrt{3x}) - J_{-1/3}(\tfrac{2}{3}\sqrt{x})J_{1/3}(2\sqrt{3x}) = 0,$$
$$J_{1/3}(-\tfrac{2}{3}\sqrt{x})J_{2/3}(-\tfrac{1}{6}\sqrt{2x}) - J_{-1/3}(-\tfrac{2}{3}\sqrt{x})J_{-2/3}(-\tfrac{1}{6}\sqrt{2x}) = 0,$$
$$J_{1/3}(-\tfrac{2}{3}\sqrt{x})J_{-1/3}(-\tfrac{1}{6}\sqrt{2x}) + J_{-1/3}(-\tfrac{2}{3}\sqrt{x})J_{1/3}(-\tfrac{1}{6}\sqrt{2x}) = 0,$$

(23), (65), (66), (67), p. 803, 820.

III. The first two solutions, to 3D, of the equations

$$J_{1/4}(\tfrac{1}{8}\sqrt{x})J_{-1/4}(\tfrac{9}{8}\sqrt{x}) - J_{-1/4}(\tfrac{1}{8}\sqrt{x})J_{1/4}(\tfrac{9}{8}\sqrt{x}) = 0,$$
$$J_{1/4}(\tfrac{1}{2}\sqrt{x})J_{-1/4}(2\sqrt{x}) - J_{-1/4}(\tfrac{1}{2}\sqrt{x})J_{1/4}(2\sqrt{x}) = 0,$$
$$J_{1/4}(\tfrac{1}{2}\sqrt{x})J_{-1/4}(\tfrac{9}{8}\sqrt{x}) - J_{-1/4}(\tfrac{1}{2}\sqrt{x})J_{1/4}(\tfrac{9}{8}\sqrt{x}) = 0,$$
$$J_{1/4}(\tfrac{9}{8}\sqrt{x})J_{3/4}(\tfrac{1}{2}\sqrt{x}) + J_{-1/4}(\tfrac{9}{8}\sqrt{x})J_{-3/4}(\tfrac{1}{2}\sqrt{x}) = 0,$$

(29), (72), (73), (74), p. 803, 820. Of the third equation only the first solution is given.

IWAO KOBAYASHI, "Das elektrostatische Potential um zwei auf derselben Ebene liegende und sich nicht schneidende gleichgrosse Kreisscheiben," Sendai, Tôhoku Teikoku Daigaku, *Science Reports*, s. I, v. 27, 1939, p. 387–391. There are tables of

$$g(\lambda,\mu,\nu,p) = \frac{1}{\sin\frac{1}{2}\pi(\mu+\nu-\lambda)}\int_0^{\infty} J_\lambda(pt)J_\mu(t)J_\nu(t)dt/t,$$

$p = [2(.2)3(.5)4, 5, 7, 10; 7D]$
$\lambda = 0(1)5$; $\mu = \frac{1}{2}(1)2\frac{1}{2}$; $\nu = \frac{1}{2}(1)5\frac{1}{2}$, $\mu \leqq \nu$, $\mu + \nu \leqq 6$; λ and $\mu + \nu$ not simultaneously even or uneven; otherwise all combinations of λ, μ, ν. Also

$$H_{mh}^{kn} = (2m + 4n + 1)g(h + m, m + 2n + \tfrac{1}{2}, h + 2k + \tfrac{1}{2}; p),$$

$$H_{oh}^{kn} = (4n + 1)g(h, 2n + \tfrac{1}{2}, h + 2k + \tfrac{1}{2}; p),$$

for $p = [2(.2)3(.5)4, 5, 7, 10; 5D]$, $k = 0(1)2$, $n = 0(1)2$, $m = 0(1)5$, $h = 0(1)5$.

B. van der Pol & H. Bremmer, "The propagation of radio waves over a finitely conducting spherical earth," *Phil. Mag.*, s. 7, v. 25, 1938, p. 823.
$H\nu^{(1)}[\frac{1}{3}(-2x)^{\frac{3}{2}}] = 0$, $\nu = \frac{1}{3}$ and $\frac{2}{3}$, the first 6 zeros to 4S.

S. Tomotika, "The instability of a cylindrical column of a perfect liquid surrounded by another perfect fluid," Nippon Suugaku-buturigakkwai Kizi, Tokyo, *Proc.*, s. 3, v. 18, 1936, p. 559–561.
$F(x) = (1 - x^2)x/[\{K_0(x)/K_1(x)\} + AI_0(x)/I_1(x)]$, to 5–6D, for $A = 5$, 1, .2, .1; $x = 0(.1)1$. Also $A = 5$ and 1, $x = .6(.02).8$; and $A = .2$ and .1, $x = .5(.02).7$.

P. F. Ward, "The transverse vibrations of a rod of varying cross-section," *Phil. Mag.*, s. 6, v. 25, 1913, p. 89, 94, 97, 100, 101, 103. We here find zeros, to 5S, of the following functions:

$$\frac{d}{dx}[J_0(2\sqrt{x})I_0(2\sqrt{x})], \text{ first } 4;$$

$$\frac{d}{dx}\left[\frac{1}{x}J_1(2\sqrt{x})I_1(2\sqrt{x})\right], \text{ first } 3;$$

$$\frac{d}{dx}\left[\frac{d}{dx}\left\{\frac{1}{\sqrt{x}}J_1(2\sqrt{x})\right\}\frac{d}{dx}\left\{\frac{1}{\sqrt{x}}I_1(2\sqrt{x})\right\}\right], \text{ first } 4;$$

$$\frac{d}{dx}\left[\frac{d}{dx}\left\{\frac{1}{x}J_2(2\sqrt{x})\right\}\frac{d}{dx}\left\{\frac{1}{x}I_2(2\sqrt{x})\right\}\right], \text{ first } 3;$$

$$I_1(2\sqrt{x})\frac{d^2}{dx^2}\left[\frac{1}{\sqrt{x}}J_1(2\sqrt{x})\right] - J_1(2\sqrt{x})\frac{d^2}{dx^2}\left[\frac{1}{\sqrt{x}}I_1(2\sqrt{x})\right], \text{ first } 3;$$

$$I_2(2\sqrt{x})\frac{d^2}{dx^2}\left[\frac{1}{x}J_2(2\sqrt{x})\right] - J_2(2\sqrt{x})\frac{d^2}{dx^2}\left[\frac{1}{x}I_2(2\sqrt{x})\right], \text{ first } 3.$$

K. Yosikata, "Values of the functions $\bar{K}_0$ and $\bar{K}_1$," Sendai, Tôhoku Teikoku Daigaku, *Technology Reports*, v. 9, 1929, p. 347 f.

$kI_0(x) - K_0(x)$ and $kI_1(x) + K_1(x)$,
$k = \ln 2 - C = .11593\cdots$, $x = [.1(.1)5; 10D]$,
$x = [5(.1)6(1)11; 8\text{–}9S]$, $x = [.02(.02)1; 7D]$.

62. Tables of $(\sin x)/x$, and of some of its Functions.—Such tables are desirable for certain applications in diffraction phenomena and in a

variety of other fields as the following references suggest. Augmentation of the list would be welcomed.

1. K. Hayashi, *Tafeln der Besselschen, Theta-, Kugel- und anderer Funktionen*, Berlin, 1930, p. 30–47.
[0(.01)10(.1)20(1)100; 8D].

2. *Comisión Impulsora y Coordinadora de la Investigación Científica, Anuario 1944*. Mexico, 1945, p. 223–247. See *MTAC*, v. 2, p. 121.
[0(.001)1; 7–8S].

3. N. R. Jørgensen, *Undersøgelser over Frekvensflader og Korrelation*, Diss., Copenhagen, 1916, p. 159–165. The form of statement of the tabulation was suggested by FMR *Index*.
[0(2^g)2000^g; 7D], Δ.

4. BAASMTC, *Mathematical Tables*, v. 2, *Emden Functions* . . . , London, 1932, p. 1, $y = (\sin x)/x$. Suggested by FMR *Index*.
[0(.1)3.1; 7D]. $y^{(n)}$, $n = 1(1)5$ are also tabulated for the same x.

5. E. C. J. v. Lommel, "Die Beugungserscheinungen geradlinig begrenzter Schirme," Bayer. Akad. d. Wissen., *math. natw. Abt., Abh.*, v. 15, 1886, p. 651.
[0(1)50; 6D]. $(\sin^2 x)/x^2$ is also tabulated for the same ranges. There are also 17 6D values of maxima and minima of $(\sin x)/x$ and $(\sin^2 x)/x^2$.

6. S. H. Bauer, "The function of sin x/x," Optical So. Amer., *J.*, v. 22, 1932, p. 537.
[0(.1)1.6, $\frac{1}{2}\pi$, .5(.5)50; 5D]. This table was "checked by a large scale plot, all points having been found to fall on a perfectly smooth curve."

7. A. Schuster, *Introduction to the Theory of Optics*, London, 1904, p. 101; third ed., 1924, p. 105.
[0(15°)540°; 4D]. $(\sin^2 x)/x^2$ is also tabulated for [0(15°)180°; 4D], [180°(15°)540°; 5D].

8. J. Sherman & L. Brockway, "A four place table of $(\sin x)/x$," *Z. f. Kristallographie*, v. 85A, 1933, p. 404–419.
[0(.01)20(.02)40(.05)100; 4D].
For $x < .25^r$, $y = (\sin x)/x$ was calculated by means of a Taylor's expansion about the origin. For $.25 < x < 6$, Peters' six-place table of natural sines was employed, and for x between 6 and 100 radians Lohse's five-place table of natural sines was basic. The table was checked by computing first and second differences throughout. When the second difference indicated probability of an error in y greater than a unit in the last place, the value of y was recomputed.

9. G. Bruhat, *Cours d'Optique*. Paris, Masson, 1930, p. 216; also second ed. 1935, p. 216.
[0(π/12)4π; 4D].

10. E. Jahnke & F. Emde, *Tables of Functions with Formulae and Curves*, fourth ed., New York, 1945, Addenda, p. 32–35. There's a graph of the function, p. 35, $0 < x < \pi$; also a table of max. and min. values on p. 30. See *MTAC*, v. 1, p. 203, 235, E, 6, 7, 8.
[0(.01)3.14; 4–5S].

11. R. Heger, *Fünfstellige logarithmische und goniometrische Tafeln sowie Hülfstafeln zur Auflösung höherer numerischer Gleichungen*, second ed. Leipzig and Berlin, 1913. T. 26, p. 85.
[20°(5°)50°(2°)90°; 4D].

12. K. STUMPFF, *Tafeln und Aufgaben zur harmonischen Analyse und Periodogrammrechnung*. Ann Arbor, 1944, p. 122–126; see *MTAC*, v. 2, p. 32. Also $x \csc x$ and $x^2 \csc^2 x$, $x \leqslant 100^g$; and $x \csc x$, $101 \leqslant x \leqslant 200$. [$1^g(1^g)200^g(2^g)2000^g(10^g)8000^g$; 4D]. Also an auxiliary table (p. 126) for calculating the function, $8000^g \leqslant x \leqslant 12000^g$.

13. °J. ENBERG & E. LÅNGSTRÖM, *Tables à six Décimales des Valeurs Naturelles des Fonctions Trigonométriques*. Stockholm, 1925.
$(\sin x)/x^g$ and $x^g \csc x$.
[$0(0^g.01)15^g$; 6S].

14. E. WEBER, "Die magnetischen Felder in leerlaufenden Synchronmaschinen," *Archiv f. Elektrotechnik*, v. 19, 1927, p. 197. There are graphs of $[\sin \frac{1}{2}\nu\pi x]/\frac{1}{2}\nu\pi x$, $\nu = 1(2)19$, $0 \leqslant x \leqslant .55$.

15. We have already surveyed various tables for $S = \log [(\sin x)/x]$, *MTAC*, v. 1, p. 83–85. See also v. 2, p. 125, RMT **306** (G).

16. FRIEDRICH RISTENPART & WILHELM EBERT, tables in P. HARZER, *Ueber die Bestimmung und Verbesserung der Bahnen von Himmelskörpern nach drei Beobachtungen*. Kiel, Sternwarte, *Publ.*, v. 11, 1901, p. 109–116. There is here a table of $\log [x \csc x]$ for $\log x = [\bar{2}.36(.001) + .04$; 7D], Δ.

17. S. GRADSTEIN, "Erzwungene Torsionsschwingungen von Kurbelwellen," *Ingenieur-Archiv*, v. 3, 1932, p. 212–214; $x \csc x$,
$x = [0(.05)10$; 5D].

18. J. T. PETERS, *Sechsstellige Tafel der trigonometrischen Funktionen . . . von zehn zu zehn Bogensekunden* . . . Berlin, 1929 and 1939, p. 24–31; Russian eds. 1937 and 1938; $x'' \csc x$
[$0(10'')1°20'$; 6S]; also, p. 22, critical values up to $1°20'10''$.

19. J. T. PETERS, *Sechsstellige trigonometrische Tafel für neue Teilung*, Berlin, 1930, and 1939, p. 44.
$x \csc x^g$, to 4D, up to 2^g.

20. L. J. COMRIE, *Four-Figure Tables . . . with Argument in Time*. London, 1931, p. 32, $x \csc x$ to $1055^s.5$.

21. W. O. LOHSE, *Tafeln für numerisches Rechnen mit Maschinen*, second ed. by P. V. Neugebauer. Leipzig, 1935, p. 15. $x \csc x°$, [$0(0°.1)3°$; 6S].

22. J. T. PETERS, *Sechsstellige Werte der Kreis- und Evolventen-Funktionen* . . . Berlin and Bonn, 1937, p. 182. $x° \csc x°$ up to $1°.011$. Reprinted in W. F. VOGEL, *Involutometry and Trigonometry*, Detroit, 1945, p. 182.

23. C. K. SMOLEY, *Segmental Functions. Text and Tables*. Scranton, Pa., 1943, p. 250–255, $\frac{1}{2}x \csc \frac{1}{2}x$, $x = [10°(1')30°(10')180°$; 7D].

24. Kempe's *The Engineer's Year-Book . . . for 1929*, v. 36. London, 1929, p. 58, 60, 62, $y = \frac{1}{2}x \csc \frac{1}{2}x$, and $\log y$, for $x = 1°(1°)180°$; 6 and 7D]. These tables are not in the later volumes.

25. In *Nature*, v. 120, 1927, p. 478, 770, 918, there are discussions by V. NAYLOR and A. E. LEWIN, of methods of solution of the equation $(\sin x)/x = c$. From tables indicated above, it is clear that we can readily read off the approximate real solutions. The case $c = 1$ has been discussed by A. P. HILLMAN & H. E. SALZER in *Phil. Mag.*, s. 7, v. 34, 1943, p. 575, where the first ten solutions with positive real and imaginary parts are given to 6D. For the same authors' discussion of the case $c = -1$ and also for a table by J. FADLE, $c = \pm 1$, see *MTAC*, v. 2, p. 60–61.

26. Of the *sine integral,* $\int_0^x \sin t\,dt/t$, there are many tables, including NYMTP, *Tables of Sine, Cosine and Exponential Integrals,* 2 v., 1940; *Table of Sine and Cosine Integrals for Arguments from* 10 *to* 100, 1942, x = [0(.001)10(.01)100; 10D], [0(.0001)1.9999; 9D].

GIBBS CONSTANT. The Fourier series $\sum_{n=1}^{\infty} \frac{\sin nt}{n}$ converges for each t, and has the values $\frac{1}{2}(\pi - t)$ for $0 < t < \pi$, $\frac{1}{2}(t - \pi)$ for $-\pi < t < 0$, 0 for $t = 0$ or π. Thus convergence is non-uniform in any interval including the point $t = 0$. The approximating curves $y_n = \sum_{\nu=1}^{n} \frac{\sin \nu t}{\nu} = f_n(t)$ behave in a strange way, namely near zero the nth curve stays above the highest point $\frac{1}{2}\pi$ of the straight line by a definite ratio, $K = \frac{2}{\pi}\int_0^{\pi} \frac{\sin t}{t}\,dt$, observed by the physicist Gibbs and known as Gibbs constant; see *Nature,* v. 59, 1898, p. 200 and 1899, p. 606, or *Collected Works,* v. 2, 1928, part 2, p. 258–260. If the maximal value of $f_n(t_n)$ is taken at the point t_n so that

$$\max_{t<\pi} \sum_{\nu=1}^{n} \frac{\sin \nu t}{\nu} = \sum_{\nu=1}^{n} \frac{\sin \nu t_n}{\nu},$$

then for $t_n \downarrow 0$,

$$\lim_{n\to\infty} \sum_{\nu=1}^{n} \frac{\sin \nu t_n}{\nu} = \int_0^{\pi} \frac{\sin t}{t}\,dt = k,$$

which is the highest limiting point of the approximating curves and lies on the y-axis. This value is given as 1.85, approximately, in C. A. STEWART, *Advanced Calculus,* London, 1940, p. 431. Gibbs noted only that K existed but did not derive a numerical value. On the suggestion of OTTO SZÁSZ, in the Spring of 1944, the Harvard Automatic Sequence Controlled Calculator was used to find $K = 1.17897975\cdots$,[1] and this value is recorded in his paper "The generalized jump of a function and Gibbs' Phenomenon," *Duke Math. J.,* v. 11, 1944, p. 824. In two earlier papers he had given the incorrect value $1.08949\cdots$; Amer. Math. So., *Trans.,* v. 53, 1943, p. 440, and v. 54, 1943, p. 497, this incorrect value being taken from A. ZYGMUND, *Trigonometrical Series,* Warsaw-Lemberg, 1935, p. 180, as $1.089490\cdots$. The approximate value of k is given by S. A. COREY, *Amer. Math. Mo.,* v. 13, 1906, p. 13, as 1.851936—, which checks with Stewart's value noted above, but exhibits the notable inaccuracy of the value given in G. H. HARDY & W. W. ROGOSINSKI, *Fourier Series,* Cambridge, 1944, p. 36, $1.71\cdots$. This inaccurate value is that, $1.089\cdots \times \frac{1}{2}\pi$, given by Z. ZALCWASSER, *Fundamenta Mathem.,* v. 12, 1928, p. 127. The value of K given in P. FRANKLIN, *A Treatise on Advanced Calculus,* New York, 1940, p. 511, being $1.17\cdots$, is not far wrong. Anyone inclined to state that Gibbs was the first to observe his phenomenon should consult a little paper by H. WILBRAHAM, "On a certain periodic function," *Camb. and Dublin Math. J.,* v. 3, 1848, p. 198–201. For assistance in collecting material for this note **26** I am indebted to OTTO SZÁSZ, professor of mathematics at the Univ. of Cincinnati, and to PHILIP FRANKLIN, professor of mathematics at the Mass. Inst. of Technology. The first general and scientific discussion of the phenomenon was by M. BÔCHER, *Annals Math.,* s. 2, v. 7, 1906, p. 123f. See also H. S.

CARSLAW, (a) *Introd. to the Theory of Fourier's Series and Integrals*, 3rd ed. rev. and enl., London, 1930, p. 293–296; (b) "A historical note on Gibbs' phenomenon in Fourier's series and integrals," Amer. Math. So., *Bull.*, v. 31, 1925, p. 420–424; and also *Encycl. d. math. Wissen.*, v. II.3.2, p. 1203f.

R.C.A.

[1] D. H. L. writes as follows: This value has a last-figure error; in fact $K = 1.1789\ 79744\ 47216\ 72702\ 32029$. It is interesting to note that Zygmund's value is $\frac{1}{2}(K + 1)$. A value of $k = Si(\pi)$ is given to 16*S* in NYMTP, *Table of Sine, Cosine and Exponential Integrals*, v. 2, 1940, p. 206. From this it may be seen that Corey's value, referred to later, is also in error in the last figure, *for* 6 −, *read* 7. $k = 1.851\ 937051\ 98\bar{2}466$.

QUERY

19. THE INTEGRAL $\int_0^x e^{-A \sec \theta} d\theta$.—This integral arises in radium therapy discussion, and since tables of the function are so important for calculating the intensity of rod-shaped preparations, ROLF M. SIEVERT published such tables in his memoir, "Die *v*-Strahlungsintensität an der Oberfläche und in der nächsten Umgebung von Radiumnadeln," *Acta Radiologica*, Copenhagen, v. 11, 1930, p. 249–301. The tables on p. 271–280 are for $x = 30°(1°)90°$, $A = [.1(.01).5; 3D]$. In the recent work, C. W. WILSON, *Radium Therapy, its Physical Aspects*, London, Chapman & Hall, 1945, p. 213–214, there is an abridgment of these tables for $x = 30°(1°)90°$, $A = [0(.05).4; 3D]$. Current work connected with integrated radiation from a line source of radioactive material suggests the great desirability of extension of Sievert's table for $x < 30°$, and for $A > .5$. Have other tables of the integral been published?

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QUERIES—REPLIES

25. BRIGGS' ARITHMETICA LOGARITHMICA (Q7, v. 1, p. 170; QR21, v. 2, p. 94).—In the library of the University of Michigan is a copy of this volume with the extra 12 pages described in the query.

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26. SCARCE MATHEMATICAL TABLES (Q2, v. 1, p. 66; QR5, p. 100; 6, p. 132).—Four libraries have already been noted where HENRY GOODWYN, *A Table of the Circles. . .*, 1823, may be consulted. We may now add that copies are also available in the libraries of Brown University and of L. J. C.

CORRIGENDA

V. 1

P. 215, B_6 1, *for* $s = 1(1)50$, *read* $s = 1(1)150$.
P. 220, A_1 1, *for* 15D, *read* 15–20D; *for* 21.5, *read* 25.5. A_2 1, *delete* δ^2.
P. 221, A_3 5, *for* $x/8$, *read* $1/(2x)$.
P. 223, B_2 5, *for* .5(.1)1, *read* .5, .6, .8(.1)1.
P. 226, A_1 4, *delete* Δ; A_1 8, *for* 0(.01)1, *read* 0(.01)5.1.
P. 229, B_3 10, *for* $(9 + x^2)$, *read* $(9 + x^2)^{\frac{1}{2}}$.