QUERIES—REPLIES

27. TABLES TO MANY PLACES OF DECIMALS (Q1, v. 1, p. 30; QR1, **2**, **3**, **4**, v. 1, p. 30–31, 67–68, 99–100; also p. 309–312).—In discussions of the work of the National Bureau of Standards Mathematical Tables Project (NBSMTP)[1], the question of the number of decimal places to which tables should be computed has frequently arisen. Certain persons interested in the Project have expressed the opinion that the number of decimal places should almost always be severely limited because of considerations relating to the applications, and have expressed the view-point that the Project has been open to criticism in the past because some of its tables have been computed to a large number of decimal places. The purpose of this note is to explain the present attitude of the Project toward the matter in question.

In the first place, it must be admitted that there is plenty of precedent for computing certain types of tables to many decimal places. To cite a few examples, the values of logarithms are given in the *Logarithmetica Britannica* to 20D; the values of the natural trigonometric functions are given by ANDOYER to 15D; the entries in most of the tables by VAN ORSTRAND are given to at least 23D, and in some cases to as many as 62D; the values in the tables of exponentials by F. W. NEWMAN are given to 12D; the probability functions are given by BURGESS to 15D; a short table of sin x and cos x was given by J. PETERS to 21D; the tables of the polygamma functions have been given by H. T. DAVIS to 12D, or more; the table of $\Gamma(x)$, log $\Gamma(x)$ and $\Psi(x) = \Gamma'(x)/\Gamma(x)$ are given by H. T. DAVIS to 12D, or more; GLAISHER tabulated values of the sine, cosine and exponential integrals to 18D; HAYASHI gives 12-place values of Bessel Functions of fractional order, 12-place values of $J_0(x)$ and $J_1(x)$, and 12-place values of elliptic integrals; The *Jn. Appl. Physics*, Feb. 1946, contains a short table by the late Professor HARRY BATEMAN to 15D.

Thus on the basis of such circumstantial evidence, there seems to have been a genuine need, at least in the past, for tables computed to a large number of D. Does this need still exist? To answer this question, consider first the following typical situations encountered by professional computers in day-to-day operations:

(a) *Use of recurrence relations:* It is frequently necessary to employ recurrence formulae several times in succession. In such cases it is vital to begin with an adequate number of D in order to offset the loss of S inherent in the process.

(b) *Computation of derivatives:* It is frequently necessary to resort to numerical evaluation of the derivatives of analytic functions, since the computation from the explicit expression of these derivatives would involve a prohibitive amount of labor. In such cases the accuracy of the derivatives (particularly those of high order) would be inadequate unless the functional values used in the evaluation of these derivatives are known to a sufficiently large number of D. To illustrate the need for numerical differentiation, suffice it to mention the evaluation of the derivatives of the spherical scattering functions in the Mie theory for purposes of evaluating these functions for complex indices of refraction. (See *MTAC*, v. 2, p. 140–143).

(c) *Tabulations from leading differences:* When the value of a function is built up from its leading differences, the accuracy of the computed values increases with the accuracy of the initial entries which give rise to the leading differences in question.

(d) *Sequences of arithmetic operations:* There is an inevitable loss of significant figures in any arithmetic process involving several operations. In such cases, the error arising from the accumulation of rounding-off errors can be considerable unless the data are given to a sufficiently large number of D. Very often loss of S is inherent in certain formulae even when the number of operations involved is relatively small. It is true that frequently the loss of S inherent in a particular method of computation may be obviated by choosing other methods. However such methods may involve more labor than the original method based upon the knowledge of certain functions to an adequate number of D.

Consider now the needs of the users of mathematical tables who are not professional computers. The requirements of this group are the determining factor.

Many of the situations, of the type outlined above which occur in connection with professional tabulation work, are frequently encountered by applied mathematicians in the everyday use of tables. Significant figures are lost in mathematical computations by the use of recurrence formulae to compute non-tabulated functions from tabulated ones. They are lost by chains of arithmetical processes. A particularly serious loss occurs when high powers of numbers are obtained by logarithms (unless the logarithms are known to a sufficient number of places), or when two nearly equal terms are subtracted from each other. In the evaluation of derivatives and turning points of functions, there is an inevitable loss in S. [Further discussions along these lines by HARRY BATEMAN, C. R. COSENS, and J. C. P. MILLER will be found at references given at the beginning of these notes.]

Thus it would seem that there is still a definite need for mathematical tables computed to a large number of decimal places for at least some of the basic functions.

The Mathematical Tables Project does not specialize in tables to a large number of D. Of all the tables prepared by the Project only the exponentials, the natural logarithms and the probability functions are computed to very large number of D (18, 16, and 15 respectively). The tables of $\tan^{-1} x$ and of $\sin^{-1} x$ are given to 12D.

These tables constitute only a small fraction of the output of the Project, both in published and unpublished tables, as well as in tables of a specialized nature prepared for the war effort. In the last mentioned tables, the number of D or S is always less than 10 and sometimes as small as 2. It is interesting to note, however, that the first editions of all the basic tables computed to a large number of D are almost entirely sold out.

It seems entirely within the realm of possibility that the new electronic digital computing machines now being developed will render certain types of mathematical tabulation obsolete. It is certainly expected that these machines will render it possible to use a direct numerical approach to the mathematical formulations of many physical problems, rather than the classical approach involving reduction of the mathematical formulation to previously tabulated functions. But it will be some years before these machines become widely available, and even when they do, it would appear at present that function tables may play an important role in their most efficient use. If that is true, then of course the machines will lose significant figures in just the same way that the hand computers do. In particular, it is considered probable at present that recurrence relations will frequently be used in programming, so if function tables are to be used in connection with such machines, many S in these tables may be indispensable.

In the planning of the work on basic mathematical tables, the NBSMTP has had the advantage of a continual exchange of views with outstanding mathematicians, physicists and engineers both here and abroad. The final decisions regarding the scope of the tables in question, the range and interval of the arguments and the accuracy of the entries (number of D or S) reflect the judgment of these authorities.

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[1] Editorial Note: The Mathematical Tables Project has been a unit of the National Bureau of Standards since 1941. Beginning with 1947, along with other *MTAC* changes in notation, we shall refer to this Project by the symbol NBSMTP (like BAASMTC) and no longer use NYMTP.

CORRIGENDA

V. 1

P. 228, A_6, heading, *read* $J_n(x)I_{n+1}(x) + J_{n+1}(x)I_n(x) = J_n(x)I_{n-1}(x) - J_{n-1}(x)I_n(x) = J_n(x)I_n'(x) - J_n'(x)I_n(x)$.

P. 233, **B** 1, *for* 14, *read* 1.4.

P. 235, **E** 1, *for* 1½(½)10½, *read* 1½(1)10½.

P. 238, C_1 7, *for* $-4(.4)+4$, *read* $-4(.2)+4$.

P. 246, l. 2, *for* $G(x)$, *read* $G(x)/\pi$.

P. 252, l. 17, *for* i^n, *read* i^{-n}.

P. 254, **A** 3, *for* 10D, *read* 4D.

P. 255, B_4 heading, *for* $-\frac{1}{2}\pi$ her′ x, *read* $\frac{1}{2}\pi$ her′ x; B_4 11, *for* −her′ x, hei′ x, *read* her′ x, −hei′ x.

P. 257, **E**6, *for* $\theta_0(x)$, *read* $-\theta_0(x)$; **E**12, *for* $N_1(x)$, *read* $(2/\pi)N_1(x)$; **E** 13, 14, *add* values of $x = 0(.1)10$.

P. 261, A_1 6, *for* $(x/\pi)^{\frac{1}{2}}$, *read* $(2x/\pi)^{\frac{1}{2}}$.

V. 2

P. 60, l. 14, *for* 266, *read* 288.

P. 103, l. 2, *for* and, , *read* and

P. 106, l. 1, *for* open certain circuit lines, *read* open circuit certain lines.

P. 125, **306**(d), *for* T_n^2, *read* T_n^1.

P. 136, l. 22, *for* $C_1(u)$, *read* $c_1(u)$.

P. 139, l. 10, *for* Tallquist, *read* Tallqvist.

P. 149, l. 15, *for* stopped-wheel, *read* stepped-wheel.

P. 163, l. 2, *for* This edition was soon, *read* This edition, soon; l. 2, 4, *for* p. xxx, *read* p. XXX.

P. 167, RMT **323**, formula (2), *for* $f(n)$, read $f(x_n)$.

P. 168, formula (4), *for* $f(x_n)$, *read* $f(n)$.

P. 175, **334**, l. 2, *for* Development, *read* Experiment.

P. 194, **15**, *for* 306 (G), *read* **306**(b); **25** *for* 266, *read* 288.