

On a Scarce Factor Table

The table in question is by J. Ph. KULIK and has the following title: *Divisores numerorum decies centena millia non excedentium. Accedunt tabulae auxiliares ad calculandos numeri cujuscunque divisores destinatae. Tafeln der einfachen Factoren aller Zahlen unter Einer Million nebst Hülftafeln zur Bestimmung der Factoren jeder grösseren Zahl.* It was published at Graz (Austria), in 1825, and contains xxvi, +2+ 286 p. of size 13.4 × 20.7 cm. See *MTAC*, v. 2, p. 59f.

This table is not mentioned by GLAISHER in his list "On factor tables"¹; nor by CUNNINGHAM in his paper "Factor tables. Errata"²; nor even by LEHMER "On the history of factor tables".³ But the work is noted by HENDERSON in his bibliographic list,⁴ is mentioned by DICKSON⁵ and is listed by Kulik in "POGGENDORFF."⁶

This table appeared 14 years after the publication of the famous table of L. CHERNAC: *Cribrum arithmeticum*, giving the *complete* factorization of all the numbers not divisible by 2, 3 or 5 up to 1020000; and 8 years after the publication of *Table des diviseurs pour tous les nombres du premier million* of J. C. BURCKHARDT. Hence there was no need for a third table covering the same region of the natural numbers. In the Latin Preface K⁷ writes that the table of B is folio (?) and that it must be consulted more than twice to find the complete factorization of a number having more than two prime factors. Therefore K apparently planned a different tabular arrangement so that only one or two consultations should be necessary to get the complete factorization of a number.

In his book K first gives a nine-page table of the complete factorization of all *composite* numbers not divisible by 2, 3, 5 or 11, up to 21500. Then follows a 257-page table of factors of the numbers not divisible by 2, 3, 5 or 11 from 21500 up to one million; the last two figures of the numbers are printed on one separate page of somewhat larger size than the other pages. If a number has more than two prime factors this is indicated by a point following the printed factor, and K states that he always selected that factor, the residual factor of which lies below 21500, the upper limit of the nine-page table, so that *only one* division would suffice to have all the prime factors. In the Introduction, §11, we read that the second table gives *only one* factor, the *smallest* one, but (in §13) that in some cases the number has factors too small to make the residual factor <21500 and that in *most of these cases* the table gives two factors. I wish to point out: (i). In many cases the residual factor of the given factor is not below 21500; (ii). In many cases the table gives not the smallest factor, but a larger one or two prime factors. (iii). There are cases in which it is necessary to consult the table three times. I shall later confirm these three assertions by examples. In §12 we read that the point, mentioned above, following the factor, is not always clearly printed but that in this case the factor is printed as much to the left as would be necessary for the point; e.g. the fifth column and sixth row of page 180 where the factor 43 is printed to the left and without the point.

The size of the book is unusually small. To save space K uses (but not

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IX	X	XI	II	III	IV	V	VI	VII	VIII	IX	X	XI	II	III	IV	V	VI	VII	VIII	IX	X	XI	
6015	10	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98
41	59	83	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7	7
...

FIG. 2.

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Al pag. 10. 2. Tabula diviserum omnium numerorum inter 2:500 et 100000.

I	II	III	IV	V	VI	VII	VIII	IX	X	XI
03	01	03	07	13	19	25	31	37	43	49
...

FIG. 1.

consistently) literal symbols:

	a	b	c	d	e	f	g	h	i	to denote the factors					
	13	17	19	23	29	31	37	41	43	and					
a	b	c	d	e	f	g	h	i	k	m	n	o	p	q	for
10	11	12	13	14	15	16	17	18	19	21	22	23	24	25	
		r	s	t	u	v	w	y	z	A	B	C	for		
		26	27	28	29	30	31	33	34	35	36	37			

E.g.: $b3$ means 113; $r9$ means 269; $7c$ means $7 \cdot 19$; $17e$ means $17 \cdot 29$.

The page is divided into two parts by a heavy horizontal line. Each part of the page contains seven vertical sections headed by Roman numerals. Each section contains three or less columns and is divided into five equal boxes by horizontal lines. The digits, except the last two, of the numbers are to be found just above two heavy horizontal lines, one at the top of the page and the other in the middle.

Let us now see how to use the table. Take for instance 670177; 6701 is found on page 180 (Fig. 2) at the top of the first column of section VII. The last digits 77 are to be found in the same column of section VII (of either the upper or lower half) of the separate page (Fig. 1). Having found the place of 77 in this column, we find in the corresponding place in the first column of section VII the symbol . . , which means that 670177 is a prime.

Again let us take 670423; 6704 is at the top of the first column of section VIII (this VIII is omitted to save space). On the separate page in the first column of section VIII, we look for the last two digits 23. Then on page 180 in the corresponding place in the first column of section VIII we find the symbol 13a, that is to say 670423 has the factor $13 \cdot 13$. The residual factor is 3967. Entering the nine-page table we see that 3967 is omitted; hence it is a prime and we have the complete factorization $670423 = 13 \cdot 13 \cdot 3967$.

The printed factor is not always the smallest one. Take for example 668423; 6684 is at the top of the second column of section I, on page 180. On the separate page the second column of section I look for the number '23. In the corresponding place on page 180, second column of section I, we find 41. indicating that 668423 has the factor 41 and that it is divisible by more than two prime factors. The residual factor is 16303 and the nine-page table gives $16303 = 7b \cdot 137$. Hence the complete factorization $668423 = 41 \cdot 7 \cdot 17 \cdot 137$.

The residual factor is not always < 21500 . For the number 996659 the table gives 17., the residual factor is 58627. The second table gives the factor 23 not followed by a point, which means that $58627 = 23 \cdot 2549$ in primes. The same for 997441 for which the table gives 17., the residual factor being $58673 = 23 \cdot 2551$.

There are cases in which the table must be consulted three times. Thus for 986453 the table gives 13., the residual factor being 75881 above the upper limit of the nine-page table. The table gives for 75881 again 13., the residual factor is now 5837, for which the nine-page table gives $13 \cdot 449$. Again for 973271 the table gives 13., the residual factor being 74867, and for this number the table gives 13.; the residual factor is 5759 for which the nine-page table gives $13 \cdot 443$.

K gives no explanation as to the manner in which he obtained the entries. As we have pointed out, the main table does not always give the smallest prime factor, and it may give two prime factors. In view of the additional fact that some factors are followed by a point to indicate that the number has more than two prime factors, it seems clear that for the *calculation* of the entries it would have been necessary to *recalculate* the entire table of C. However we do not find anywhere in the book that K did this. Therefore it seems probable that he borrowed all his entries from C's table.

K gives a list of errors in C's table, containing all the errors detected 8 years earlier by B, but K added two new ones: $311909 \text{ loco } 13 \cdot 23 \cdot 993$ *lege* $13 \cdot 23993$ and $445193 \text{ loco } 59639$ *lege* 63599 (I saw in 2 copies of C's table $659 \cdot 9$, and in a third one 63599 which is correct). None of these errors is listed by D. H. Lehmer.⁹

K has not indicated any error in B's table.

There are additional tables as follows: a table of the squares up to 7500^2 in ten pages; a table of the linear forms of the divisors of $x^2 + ay^2$ up to $a = 106$ and of $x^2 - ay^2$ up to $a = 101$. K mentions (p. xxi) the table of LEGENDRE⁸ and gives a list of errors in this table, containing a part of the list of errors published by D. N. Lehmer.⁹ However the table of K contains numerous other errata. Further, the book contains a small table of the primes up to 3761, and a table of the powers of 2, 3, 5 up to 2^{71} , 3^{37} , 5^{37} .

In the Introduction K explains at length how the auxiliary tables may be used to factor a number beyond the range of the table, after having given rules on divisibility by 16, 9, 11, 101, 37, etc. He explains how to find quadratic forms for a given number and how to use them to get quadratic residues and linear forms of the divisors. For instance he factorizes $3^{24} - 1$ and identifies 10 091 401 as a prime.

K devotes only 30 lines (half a page) of the *Praefatio* to the description of the table. In the remaining $2\frac{1}{2}$ pages he explains in detail how he constructed another table giving a factor of every number not divisible by 2, 3 or 5 up to 30 030 000. As there are 80 numbers not divisible by 2, 3 or 5 among every 300 numbers $300v + 1, 2, 3, \dots, 300$, he made a sheet with 80 rows and 77 columns, headed 0, 3, 6, 9, \dots , denoting the hundreds. On this sheet he inserted the factors 7 and 11 in the proper places, and ordered copies printed (without the headings). These sheets are substantially the same as the sheet described by Glaisher.¹ K describes, with many technical details, the printing of more sheets until he possessed 1300 sheets upon which all the factors 7, 11, 13, 17, 19, 23 were *printed*. They covered the set of the numbers not divisible by 2, 3 or 5 up to 30 030 000. He tells us further that he inserted the factors 29, 31, \dots , 503 by the aid of what we now call "stencils" and the larger factors by the so-called "multiple method."

Comparing this description with that of the famous 100-million manuscript of K given by D. N. Lehmer² it seems that the 30 million table is *not* a part of the 100 million manuscript. For in the above description by K, literal symbols for numbers are not mentioned at all, while K did use them throughout in the *100 million* manuscript. Moreover D. N. Lehmer mentions that K used "stencils" as far as 997 in the *100 million* manuscript.

Hence as early as 1825 K was in possession of a manuscript factor table up to 30 million.

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¹ JAMES GLAISHER, *Factor Table for the Fourth Million*. London, 1879; "Mode of construction of the table," p. 9-17, including a description of sieves; "On factor tables," p. 17-28.

² A. J. C. CUNNINGHAM, *Mess. Math.*, v. 34, 1904, p. 24-31.

³ D. N. LEHMER, *Factor Table for the First Ten Millions*. Washington, 1909, p. V; X-XIV (See *MTAC*, v. 2, p. 139-140).

⁴ J. HENDERSON, in *Factor Table* by J. PETERS, A. LODGE & E. J. TERNOUTH, & E. GIFFORD. London, 1935.

⁵ L. E. DICKSON, *History of the Theory of Numbers*. Washington, v. 1, 1919, p. 351.

⁶ *Biographisches-Literarisches Handwörterbuch z. Gesch. d. exacten Wissen.*, Leipzig, v. 1, 1863.

⁷ To abbreviate, K = Kulik, B = Burckhardt, C = Chernac.

⁸ A. M. LEGENDRE, *Essai sur la Théorie des Nombres*. Paris, 1798, T.III, $x^3 - ay^3$, up to $a = 79$, p. [477-483] and T.IV-V, $x^3 + ay^3$, up to $a = 105$, p. [484-489].

⁹ D. N. LEHMER, *Amer. Math. Soc., Bull.*, v. 8, 1902, p. 401-402. See also D. H. LEHMER, *Guide to Tables in the Theory of Numbers*. Washington, 1941, p. 159-160.

EDITORIAL NOTES: Some parts of Dr. Beeger's paper are here somewhat amplified. The main points of Kulik's plan to avoid a large-sized book, such as Burckhardt's, may be outlined as follows:

(1) If the number N is prime, $N = p$, then the corresponding space (in the meeting of the horizontal and vertical lines) is marked with two dots $\cdot\cdot$.

(2) If the composite N has only two prime factors, $N = p_1 p_2$, $p_1 < p_2$, then $p_1 < \sqrt{1000000} < 1000$, and has therefore at most 3 digits, for which there is sufficient space; so it is entered in full (see 988027 = 991·997).

(3) If N has three factors, $N = p_1 p_2 p_3$, then even the smallest p_1 and p_2 , i.e., $p_1 = 7$, $p_2 = 7$, make $p_3 < 1000000/49$ or < 20408 ; therefore p_3 is within the range of T. 1 (goes to 21500). Hence p_1 and p_2 are entered completely, but when necessary for saving space, Roman letters, a to i, are used.

(3a) However sometimes only p_3 is entered (although not the smallest) when $N/p_3 < 21500$, and after p_3 a dot is inserted in order to indicate that N/p_3 is composite and the other factors should be looked up in T. 1.

(3b) Moreover, the smallest p which gives $1000000/p < 21500$ is $p = 47$. Hence when we have a factor ≥ 47 , we can enter that p , although it may not be the smallest, and write a dot after that p , to indicate that N/p is not prime, to be looked up in T. 1. But if all the p 's < 47 we may have to write two factors, and in such cases, we utilize the Roman letters $a \dots i = 13 \dots 43$.

(3c) Again if we have one factor of 3 digits to be followed by a dot and as there is no space for 4 marks, italic letters $a \dots C$ are used.

(4) If N has 4 or more p 's the procedure is the same as for 3 p 's, except that in these cases the two factors entered will *always* be followed by a dot, since $N/p_1 p_2$ is *always* composite.

Since in a publication of this kind errors may easily be made it should be noted that the four examples of failures in Kulik's system, according to Dr. Beeger, may be justified by the following corrigenda:

In 996659 and 997441, for 17., read 17d

In 986453 and 973271, for 13., read 13a.

In his account of the mss. of JOHN THOMSON (1782-1855), see *MTAC*, v. 1, p. 368, J. W. L. GLAISHER notes that "it is a coincidence" that H. G. Köhler's *Logarithmisch-trigonometrisches Handbuch*, Leipzig, 1848 (first ed. 1847, see RMT 430) contained a 9-page factor table of numbers not divisible by 2, 3, 5, 11, up to 21,524, while Thomson gave a factor table "differing very little" from it, up to 21,460. Both of these tables are practically identical with Kulik's T. 1, up to 21,500, referred to above. Did Thomson copy from Kulik or Köhler? Did Köhler copy from Kulik? Or were all three tables independently original? Of course Lambert (1770) gave a table of the least factors of all numbers not divisible by 2, 3, 5, up to 102000 (see RMT 432).

Even after extensive inquiries we have been unable to find in the United States any copy of Kulik's work here discussed, and the only copy in Great Britain seems to be the one in the Graves Library of the University of London. For editorial checking Dr. Beeger kindly loaned us his personal copy, of which a film reproduction was made for the Library of Brown University.