routines), although the Dahlgren "Mark II" is approximately equal to it in this respect.

To sum up: This Bell Laboratories' machine and the ENIAC represent two extremes, with the other existing machines fitting in between. The ENIAC is adapted to very long problems, provided they are not too complicated and do not need too much storage capacity. The Bell Laboratories' machine will handle the most complicated problems, requiring considerable number storage, provided they are not too long. The ENIAC prefers continuous runs, the Bell Laboratories' machine does not mind "on-and-off" problems. None of the existing machines will handle problems which are long and require a great deal of number storage. Future machines are expected to fill this gap.

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[1] This machine was built and largely designed by the IBM.

RECENT MATHEMATICAL TABLES

476[A, B].—PETER BARLOW, *Tables des Carrés-Cubes, Racines Carrées, Racines Cubiques et Inverses de tous les nombres entiers de 1 jusqu'à 10.000.* Paris et Liége, Béranger, 1946, iv, 200 p. 10.7 × 17 cm. Bound in boards. 120 francs.

This is, apparently, a facsimile reproduction of one of the stereotyped editions of Barlow's tables edited by AUGUSTUS DE MORGAN, with headings of the columns in French instead of the English originals. See *MTAC*, v. 1, p. 16–17, 26, 100, 169, 356–357; v. 2, p. 85.

477[A, B, C].—EMILIO CAZZOLA, *Tavole Grafiche dei Logaritmi a 6 decimali con interpolazione ottica sulla 6ª cifra Mantisse dei Logaritmi e dei cologaritmi Immediate per numeri da 1 a 100.000 Antilogaritmi Immediati per mantisse da 00.000 a 99.999. (Brevetto N. 415.030.) Tavole Aritmetiche e Numeriche ad uso degli Ingegneri, Matematici, Fisici, Chimici, Geometri, Periti, ecc. nonchè delle Scuole Medie Superiori (Licei ed Istituti tecnici).* Milan, Hoepli, 1947, xix, 90 p. 18.5× 26.1 cm. Stiff paper covers, 1000 lire.

The principal table of the volume (p. 1–63) gives log $N, N = 1(1)100\,000$ to 6D. For $N = 1(1)999$ the mantissae are given in the usual manner. But for $N > 999$, numbers of more than 4 digits are found to correspond to a certain mark or point on a vertical scale; opposite this mark on an adjacent mantissa scale the six-place mantissa may then be read off, the first four digits being given and the last two read off from the scale. Adjacent to each printed four-figure mantissa is given the corresponding cologarithm decimal part. Thus the author claims that in his 63-page table, 8 columns to the page, one has a 6D logarithmic and antilogarithmic table, without interpolation, for 5-figure numbers. In the sixth digit, however, there is, at times, decided uncertainty, to the extent, perhaps, of 4 units. PIERO CALDIROLA, professor of physics at the University of Pavia, praises this work, in an introductory note, as one originally conceived by the author. It may be the first six-place table of the kind, but as long ago as 1925, a five-place table of exactly this same type (but without cologarithms) was published in New York, Macmillan, by ADRIEN LACROIX & CHARLES L. RAGOT. Their work was entitled *A Graphic Table combining Logarithms and Anti-logarithms Giving directly without Interpolation the Logarithm to five places of all five-place numbers and the numbers to five places corresponding to all five-place logarithms, also a* *graphic table as above reading to four places.* 48 p. 17.2 × 24.6 cm. The scales are arranged horizontally.

In Cazzola's book (p. 64–65) is a table giving 7D ln N, equivalents of log N = 1(.01)10.09; and on p. 66–67, 7D log N, equivalents of ln N = 1(.01)10.09. On p. 71–85 are tables of n^2, n^3, $n^{\frac{1}{2}}$, $n^{\frac{1}{3}}$, $e^{\pm n/100}$, ln n, $1000/n$, πn, $\frac{1}{4}\pi n^2$, n = 1(1)1000. On p. 85 are also 6D tables of π and e, their multiples, powers, and logarithms. On p. 86–87 are tables of n^p, n = 1(1)100, p = 4(1)9; also tables of $n!$, log $n!$, colog $n!$, n = 1(1)20. On p. 88–89 are the prime factors of the numbers not divisible by 2, 3, 5, 11, for 91 to 9373; and on p. 90 a table of primes from 2 to 11677. Two of five illustrations on p. "xi" are erroneous.

R. C. A.

478[A, B, C, D].—Friedrich Gustav Gauss, (**a**) *Techniker-Tafel. Allgemeine Zahlentafeln und vierstellige trigonometrische und logarithmische Tafeln. Ausgabe für technische Schulen und Praxis.* Edited by Hans Heinrich Gobbin, grandson of the author. Sixth to tenth editions. Stuttgart, Wittwer, 1947, iv, 105 p. 15 × 23.2 cm. (**b**) *Fünfstellige vollständige logarithmische und trigonometrische Tafeln (sexagesimal unterteilter Altgrad). Zum Gebrauche für Schule und Praxis . . . Neu herausgegeben von Dr.-Ing.* Gobbin. 316–320. *Auflage*, Stuttgart, Wittwer, 1946, ii, 184, xxxiv p. 15.1 × 23.6 cm.

Many elementary tables prepared by Gauss, some of them in many editions, have been published during the past 77 years. Apart from brief reviews below we note some facts with regard to such editions and some other five-place tables. (**a**) Here are included tables of (i) n^2, n^3, $n^{\frac{1}{2}}$, $n^{\frac{1}{3}}$, $1000/n$, πn, $\frac{1}{4}\pi n^2$, n = 0(1)1000; (ii) natural trigonometric functions sin, tan, cot, cos for each sexagesimal minute, to 6D, 0 to 5° and 85° to 90°, and to 4D for the rest of the quadrant; (iii) log n, n = [1(1)10009; 4D] and n = [10000(1)11009; 7D]; (iv) log sin, and log tan, for [0(10″)2°; 4D], log sin, log tan, log cot, log cos, for each 1′ of the quadrant, to 4D; (v) ln N, N = [1(1)1000; 4D]; (vi) multiples of M and M^{-1}; small tables of $e^{\pm x}$. The first to fifth editions were in 1932. (**b**) The 5D tables include (p. 2–21) log N, N=1(1)-10009, and S and T for 1′(1′)3°4′; there is also a 7D table (p. 22–23) for N = 10000–11009; various functions of π (p. 24). Log sin [cos] and log tan [cot] for each 1″, 0 to 1° [89°–90°] (p. 25–37); for each 10″, 1° to 8° [82°–89°] (p. 38–50); log sin, log tan, log cot, log cos, for 0(1′)45° and S and T for 0(1′)3° (p. 51–96). Addition and subtraction logarithms (p. 97–108). Ln N, N = 1(1)1109 (p. 109–111). Multiples of M and M^{-1}, arc sin, arc cos, arc tan, arc cot (p. 113). Natural trigonometric functions, chords and heights and lengths of arcs (p. 114–124). N^2, N = [.001(.001)10.009; 4D] (p. 125–145). T. IX (p. 146–151) Interpolation; T. X–XI, Weights and measures (p. 151–154); T. XII (p. 154–161), The Earth spheroid; T. XIV (p. 164–171) Constants of nature; T. XV (p. 172–181) Astronomy; T. XVI (p. 182–183) Barometric measurements (p. 184). In the 261st edition, p. 151–184 were very thoroughly revised.

The first edition of (**b**) was in 1870, vii, 38, 142, [1] p.; the second in 1871; the third in 1872; the fourth in 1873; 14, 1881; 17, 1882; 21 and 22, 1884; 25, 1886; 35, 1891; 44, 1894; 47, 1895; 54–57, 1898; 60–67, 1900; 71, 1902; 80–83, 1905; 84–87, 1905; 101–105, 1909; 111–115, Halle, 1911; 116–125, 1912; 176–190, 1920; 260, 1931; 261, 1935; 271–280, 1937; 281–290, 1939; 301–310, 1943. Of (**b**) there was a *Kleine Ausgabe* in 1873; third ed. 1891; fifth 1893; sixth 1895; seventh 1896; twelfth 1901; 24, 1906; 29–33, 1910; 43, 1913; 59–63, 1920; 90, 1931.

Two other five-place tables of Gauss were the following:

Fünfstellige logarithmisch-trigonometrische Tafeln für Dezimalteilung des Quadranten, first published in Berlin, by Rauh, 1873; second ed. 1898; third ed. 1904; fifth-sixth, 1926; seventh–eighth, 1937, of which there was a new ninth–tenth ed. with *dezimal unterteilter Neugrad*, ed. by Gobbin, 1940.

Fünfstellige vollständige trigonometrische und polygonometrische Tafeln für Maschinenrechnen. Halle, 1901, 100, xviii p.; Stuttgart, second ed. 1912; fourth-fifth 1925; sixth–seventh 1934; eighth–tenth 1938.

Friedrich Gustav Gauss, often called Kataster-Gauss [Surveyor-Gauss] was born 20 June 1829 in Bielefeld, Germany, and died, 26 June 1915, in Berlin, as Generalinspektor des Preussischen Katasters (E. HARBERT, *Vermessungskunde*, v. 1, third ed., Berlin, 1943, p. 276). He has been confused with the Gymnasium mathematics teacher ALEXANDER FRIEDRICH GUSTAV THEODOR GAUSS (b. 1831, see POGGENDORFF, v. 4) to whom (1) all of the books of F. G. Gauss are credited (Dec. 1947) in the Catalogue of the Library of Columbia University; (2) F. G. Gauss's *Polygonometrische Tafeln*, Halle, 1893, is credited on p. 385 of E. WÖLFFING, *Mathematischer Bücherschatz*, Leipzig, 1903.

In the Library Catalogue of Brown University (Nov. 1947) he has also been confused with C. F. GAUSS, in cataloguing the Russian tables, B. NUMEROV, *Tables for Calculation of Geographic and Rectangular Coordinates of Gauss-Krüger*, Leningrad, 1933. C. F. Gauss's work prior to 1820 was in this connection developed by J. H. L. KRÜGER (1857–1923); see E. HARBERT, *ibid.*, p. 92. In our review of tables by PETERS, *MTAC*, v. 2, p. 299, we referred to his table for the improvement of the "Gauss-Krüger" projection. See also elsewhere in this issue, OAC, Bibliography Z–III 13.

R. C. A.

479[A, B, C, D, E].—L. J. COMRIE, *Chambers's Four-Figure Mathematical Tables.* Edinburgh & London, W. & R. Chambers Ltd., 1947, iv, 64 p. 19.2 × 25.5 cm. Limp linen cover. 5 shillings.

"These tables represent a deliberate attempt to raise the standard of the four-figure mathematical tables used in the highest school classes, and in technical colleges and universities, as well as in industrial practice. Most of the existing compilations have shown a tendency to follow conventional paths and thus lag behind the progress made by applied mathematicians, physicists, and engineers." In this way Dr. Comrie commences his Preface. Innovations of his presentation are so numerous in the admirably clear, closely packed pages, with excellent typographical displays throughout, that it is not easy in brief space to give an adequate idea of the contents. They are evidently arranged by a master of computational methods in practice. The presentation is naturally somewhat reminiscent of the contents of *Standard Four-Figure Mathematical Tables* by L. M. MILNE-THOMSON & L. J. COMRIE, London, 1931.

In connection with tables of (a) trigonometric and (b) circular functions only two types of tables are used, one being for (a) at interval 0°.1 or 6′, with proportional parts for 0°.01 and 1′, and the other for (b), with radian argument. In the tabulation of the trigonometric functions the method of complementary arguments has been used so that no separate tabulation is given of co-functions.

We do not recall any other table where quite the same care and clarity are displayed in guiding the reader in the use of tables of S, T, $\sigma = x \csc x$, $\tau = x \cot x$, $Sh = \log(\sinh x/x)$, $Th = \log(\tanh x/x)$, σh, τh, critical table(s) (C. T.). On consecutive pairs of pages (10–35) are the following:

Log sin, 0(1′)17°; C.T. 81°–90°; S for minutes. Log sin, 0(0°.01)11°; C.T. 81°–90°; S for degrees. Log sin, 10°(0°.1)90°. Log tan, 0(1′)18°; T for minutes. Log tan, 0(0°.01)11°; T for degrees. Log tan, 10°(0°.1)90°. Nat. sin, 0(0°.1)90°; C.T. near 0 and 90°. Nat. tan, 0(0°.1)80°.

Nat. tan, 79°(0°.01)90°; τ for degrees. Nat. tan, 72°(1′)90°; τ for minutes. Nat. sec, 0(0°.1)80°. Nat. sec 79°(0°.01)90°; C.T. to 5°; σ for degrees. Nat. sec, 72°(1′)90°; C.T. to 5°; σ for minutes. It is pointed out that the advantages of critical tables are (1) they do not need interpolation, and (2) they always give results that are correct to within half a unit of the last decimal.

Then on p. 50–62, are tables of circular functions $0(0^r.01)1^r.57$ with auxiliary tables; τ and σ for circular cotangents and cosecants; circular and hyperbolic functions 0(.001).2; exponential and hyperbolic functions 0(.01)3(.1)5.4; τh, σh, etc.

All that is necessary for exact interpolation is carefully provided. On pages ii, iii of the cover are proportional parts for differences 1(1)201. Differences greater than 201 may be dealt with by compounding, or by using a slide rule.

Inverse circular and hyperbolic functions are not explicitly tabulated, but it is suggested that they may be obtained by inverse interpolation, except in the case of large functions, when the same method would be applied to the reciprocals of the functions. Interpolation with second differences is introduced with an example on p. 49.

Other tables are of squares, square roots, reciprocals, natural logarithms, prime numbers, factors, binomial coefficients, and probability integral.

The volume has also many footnotes connected with the tables given on the same page. For example: Derivatives and integrals involving powers (p. 39); Common and natural logarithms (p. 41); Evaluation of $x = a^b$ (p. 43–45); Series for (i) circular functions, (ii) inverse circular functions, (iii) hyperbolic functions, also Series, derivatives and integrals for exponential functions, Derivatives, integrals and log series for hyperbolic functions, and Formulae and series for inverse hyperbolic functions (p. 50–60).

R. C. A.

480[A, R].—J. J. LEVALLOIS, "Compensation des réseaux géodesiques par la méthode des gisements," L'Assoc. Intern. de Géodésie, *Bull. Géodésique*, 1947, no. 3, Jan., p. 49–82; tables p. 77–80. 16.6 × 24.9 cm.

Table I is of $2(K + x - 1)/(K + x)$, for $K = 1(1)20$, $x = [.92(.002)1; 4D]$. Table II is for $6(K + x - 1)/[4(K + x - 1) + 3]$ and of $[8(K + x - 1) + 6]/[8(K + x - 1) + 7]$, for $K = 3(1)20$, and $x = [.92(.01)1; 4D]$.

481[B].—**A.** ALBERT GLODEN & J. BONNEAU, *Table des Bicarrés des entiers de 5 001–10 000*. Luxembourg, 30 October 1947, i, 21 leaves. 28.5 × 28.5 cm. Typescript one side of each leaf. Edition of three copies made from mss. Other copies will be made, at 30 Belgian francs each, for those applying to A. Gloden, rue Jean Jaurès 11, Luxembourg. There is a copy in the Brown University Library. **B.** A. GLODEN, *Table des Bicarrés* N^4 *pour* $3\,001 < N \leqslant 5\,000$. Luxembourg, the author, 30 October 1947. 19 leaves. 20.2 × 29.5 cm. Mimeographed one side of each leaf. For Gloden's table of N^4, $1000 < N \leqslant 3000$, see *MTAC*, v. 2, p. 250. Thus there are now tables of N^4, $N = 1(1)10^4$.

A. Here is given a table of N^4, for $5000 < N \leqslant 10\,000$. On each leaf after the first are nine columns: N, N^4, $N + 1250$, $(N + 1250)^4$, $N + 2500$, $(N + 2500)^4$, $N + 3750$, $(N + 3750)^4$, the ninth column containing the last four digits for each of the fourth powers in the line. The table was prepared for verifying factorizations of numbers of the form $N^4 + 1$. Each result was calculated twice, directly, and by the aid of finite differences.

Extracts from text

482[C, D].—ALFRED GEORGE CRACKNELL, *Clive's Mathematical Tables containing two-page Tables of Logarithms, Antilogarithms, Natural and Logarithmic Trigonometrical Functions, and Circular Measure.* London, University Tutorial Press, eleventh impression, 1946, ii, 50 p. 14 × 21.4 cm.

The nine tables (p. 31–49) include 5D tables of the six natural and logarithmic trigonometric functions. The first edition of these tables appeared in 1906 and the second impression in 1909.

483[D].—JOHANN ROHRER, *Tachymetrische Hilfstafel für zentesimale Kreisteilung*. Berlin-Grunewald, Wichmann, 1942, 12 p., stiff paper. 16.9 × 24.3 cm.

Pages 3–12 give 4D tables of the values of $\cos^2 \alpha$ and of $\sin \alpha \cos \alpha$, for every 2 centesimal minutes of the quadrant. A more substantial table of the first of these functions was given by E. A. SLOSSE (see *MTAC*, v. 1, p. 38; compare RMT **453**). More extensive tables of each of the functions are given by MARCHISIO (RMT **487**).

484[D, E].—NBSCL, *Table of Circular and Hyperbolic Tangents and Cotangents for Radian Arguments*. "Second printing," New York, Columbia University Press, [1947], xxxviii, 410 p. 19.6 × 26.5 cm. $7.50. See MTAC, v. 1, p. 178–179 (review by D.H.L.).

The second printing (edition of 700 copies) within four years not only of this volume but also of *Table of the Bessel Functions $J_0(z)$ and $J_1(z)$ for Complex Arguments* (edition of 845 copies), reviewed in our last issue, is a significant indication of the great need which has been met by these admirable publications. The changes from the first edition of the present volume, are as follows: The volume is slighter because the paper is thinner, and doubtless not so durable; the title-page date, 1943, has been removed; on p. ii has been added, "First printing 1943, Second printing 1947"; on p. vi, previously blank, appears "Acknowledgment," a paragraph formerly in the Preface, no longer on p. xi; on p. xi now is a Note stating that the present volume was prepared from the negatives used for the earlier printing except that for p. 316, where the error we noted *MTAC*, v. 2, p. 279, is now corrected; for the two-page list of publications of the NBSCL (following p. 410) are now substituted four new pages, although the incorrect reference to the National Bureau of Standards, rather than to the Superintendent of Documents, for the distribution of items, (1)–(12) in the list, remains. The same criticism may be made of the three footnote items p. xxi. The price of the second printing is fifty percent higher than that of the first.

As in *MTAC*, v. 3, p. 25, we find lack of careful revision of the Bibliography with its inaccuracies, omissions, infelicities, and lack of reference to readily available American editions.

For the work of ADAMS & HIPPISLEY the date of the corrected reprint of 1939 would have been preferable to that of the original edition, 1922—Besides the *siebenstellige* volume of BRANDENBURG there should have been an elimination of the last-line comment, and a reference to the *sechsstellige* work, and also to errors listed in *MTAC*, v. 1, p. 162, 388; v. 2, p. 46–47, 277—The date 1912 instead of 1870 for the English edition of the work of BRUHNS is decidedly misleading (see *MTAC*, v. 2, p. 338–339)—So also for the *Tables Portatives* of CALLET with the remark "There are similar tables by W. GARDINER." Some understanding of the relation of Callet's 1783 work to Gardiner and of what was in Callet's 1795 volume (1860 is the apparently intended as a definitive date in the Bibliography) may be gleaned from HENDERSON—The German edition of *Tafeln elementarer Funktionen* by EMDE is listed, but not the American edition, 1945—To list mils tables of the Fort Sill, Field Artillery School, and to omit reference to the fine volume (1943) prepared under the direction of M. M. FLOOD (see *MTAC*, v. 1, p. 146) seems inexcusable—Since L. J. COMRIE is the well known author of the 1939 *Seven-Figure Trigonometrical Tables, for Every Second of Time* (see *MTAC*, v. 1, p. 43), why was his name suppressed?—On p. xxix, l. 16, *for* 62, *read* 62–64.—The third edition of the *Recueil* of HOÜEL, 1885, is referred to as if it were the first edition—The 1918 edition *Siebenstellige Werte der trigonometrischen Funktionen von Tausendstel zu Tausendstel des Grades* by PETERS, always almost impossible to procure, is listed, but not the public editions of 1930 and 1938, nor the American edition of 1942 (see *MTAC*, v. 1, p. 12–13)—We are told that the second edition of *Sechsstellige Tafel der trigonometrischen Funktionen* was published in 1929 (really the date of the first edition); the second edition did not appear until 1939—The inaccessible 1939 edition of *Achtstellige Tafel der trigonometrischen Funktionen für jede Sexagesimalsekunde des Quadranten* is listed, but there's no reference to the American edition of 1943 in English—The third edition of *Trigonometriae* by PITISCUS, 1612, is listed as if it were the first edition—The place of publication is lacking in connection with the 1624 (not 1625) table of URSINUS—There are no references to the following tables: M. TAYLOR, *Tables of Logarithms of all Numbers, from 1 to 101 000, and of the Sines and Tangents to every Second of the Quadrant*, London, 1792 (see *MTAC*, v. 1, p. 42); G. STEINBRENNER, *Fünfstellige Trigonometrische Tafeln* . . ., Brunswick, 1914 (see *MTAC*, v. 1, p. 38); and E. A. SLOSSE, *Tables des Valeurs Naturelles des Expressions Trigonométriques* . . ., Tientsin, China, 1923 (*MTAC*, v. 1, p. 38).

R. C. A.

485[D, E, L].—GEORGE WELLINGTON SPENCELEY & RHEBA MURRAY SPENCELEY, *Smithsonian Elliptic Functions Tables* (*Smithsonian Miscellaneous Collections*, v. 109), Publication no. 3863. Washington, Smithsonian Institution, 1947, iv, 366 p. 16 × 23½ cm. See *MTAC*, v. 1, p. 325, 331. $4.50

In these days of large computing installations, one hesitates to describe as monumental even so splendid an achievement as that under review. These tables may be more cautiously described as the most important elliptic tables published up to now. Their nearest rivals are LEGENDRE's classic tables of elliptic integrals (1816, 1826) and the well-known GREENHILL-HIPPISLEY tables published by the Smithsonian Institution in 1922.

Legendre's tables gave values to 9–10D of the elliptic integrals

$$F(\theta, \phi) = \int_0^\phi (1 - \sin^2\theta \sin^2\phi)^{-\frac{1}{2}} d\phi, \qquad E(\theta, \phi) = \int_0^\phi (1 - \sin^2\theta \sin^2\phi)^{\frac{1}{2}} d\phi$$

for $\theta = 0(1°)90°$, $\phi = 0(1°)90°$; there were also various single-entry tables, for instance, tables of the complete integrals $F(\theta, \frac{1}{2}\pi)$ and $E(\theta, \frac{1}{2}\pi)$, now usually called K and E respectively. In relation to their time, Legendre's tables were a tremendous achievement, involving a colossal amount of computation of new functions. Moreover, three photographic reprints, by POTIN, EMDE and PEARSON (see *MTAC*, v. 2, p. 136, 137, 181), within the last quarter of a century are testimony to their enduring value. It may be added in passing that elliptic *integrals*, in the Legendre form, are perhaps the most frequently needed elliptic quantities in practical work aiming at numerical results. Legendre's work is, thus, far from superseded; a considerable transfer of popularity to the use of $k^2 = \sin^2\theta$ instead of θ as modular argument has affected its position, but few would care to dispense with it; it should also be noted that the only tables of *incomplete* integrals with k^2 as the modular argument (namely those of SAMOĬLOVA-ÎAKHONTOVA, 1935) give only 5D and were computed merely by interpolation in Legendre. It cannot, however, be contended that Legendre gives more than a fraction of the information given in the volume under review; all of Legendre's values could be obtained by interpolation in Spenceley, while the reverse is far from true.

With the Greenhill-Hippisley tables the present volume is more closely connected; it may be described as a bigger and better version, tabulating more functions for more arguments to more decimals, but inspired by and closely patterned on the former, as the authors acknowledge in their opening words.

The first to engage in extensive tabulation of theta functions appears to have been J. W. L. GLAISHER[1], under whose superintendence in 1872–75 tables were computed for the Tables Committee of the British Association for the Advancement of Science. The tables were completely set up in type (360 pages), but were not published, and no printed copy appears to have survived; the manuscript, however, still exists. Even had these tables been published, they would have been less important than those of the Spenceleys. They gave no Jacobian elliptic functions or integrals, but simply theta functions, differing from Jacobi's by simple factors. These factors made Glaisher's four theta functions constitute three numerators and a common denominator of the Jacobian elliptic functions sn u, cn u, dn u, so that they were excellent intermediate quantities for the computation of the latter, but as theta functions they were much less convenient than Jacobi's. The Glaisher tables[1] did, however, have arguments $\theta = 0(1°)89°$, $x = \pi u/2K = 0(1°)90°$, and it is only now, with the publication of the present tables with the same arguments, that one can safely assert that Glaisher's tables have little further value.

In 1911 Greenhill brought forward a scheme for the rearrangement of the British Association elliptic tables. In terms of ϑ's as in the standard text of Whittaker & Watson, his theta functions may be defined by

$$A(r) = \vartheta_1(x)/\vartheta_2, \qquad B(r) = \vartheta_2(x)/\vartheta_2, \qquad C(r) = \vartheta_3(x)/\vartheta_4, \qquad D(r) = \vartheta_4(x)/\vartheta_4,$$

where $x = \pi u/2K = r°$. These have the advantage, for what it may be worth, that while simple relationships of the kind $B(r) = A(90 - r)$, $C(r) = D(90 - r)$ are retained, the division-values arising at bisection, trisection, etc. of the quadrant (for x) or of the quarter-period K (for u) are algebraic functions of the modulus. For each modulus and for $r = 0(1)90$, Greenhill proposed to tabulate the four theta functions $A(r), \cdots, D(r)$, together with the Jacobian amplitude ϕ defined by $u = F(\theta, \phi)$ and the Jacobian zeta function (or periodic part of the second integral $\int_0^u \mathrm{dn}^2 u du$), which unfortunately he called $E(r)$. After some specimen tables had been published in the British Association *Reports* for 1911, 1912, 1913 and 1919, tables of this kind calculated by Hippisley were published by the Smithsonian Institution in 1922, with an introduction by Greenhill. These tables give one opening (two pages) to each of the modular angles $\theta = 5°(5°)80°(1°)89°$; the column headings on left pages are r, $F\phi$, ϕ, $E(r)$, $D(r)$, $A(r)$; the arrangement is semi-quadrantal, and other functions appearing in column headings on right pages are merely the complementary forms of the above, for example, $B(r)$ means $A(90 - r)$. The argument r takes values $0(1)90$; $F\phi$ means simply u, $= rK/90$, and is a subsidiary argument. The values are to 10D, except that ϕ, regrettably, is given only to the nearest minute. In the headings for each opening are given the values of $K, K', E, E', q, \Theta 0[= \vartheta_4], HK[= \vartheta_2]$ for the corresponding θ.

Little space is occupied in the Spenceley volume by a laconic preface and an equally laconic appendix; one would be interested to know what machines, presumably desk calculators, were used. The magnificent main table occupies p. 2–357. There are four pages to each of the modular angles $\theta = 1°(1°)89°$. The tabulation is quadrantal. The column headings are

r, $u = (r/90)K = F(\phi, k)$, sn u, cn u, dn u on left pages,
r, ϕ, $E(\phi, k)$, $A(r)$, $D(r)$ on right pages.

On alternate openings, $r = 0(1)45$ and $45(1)90$. All function values are to 12D. u is a subsidiary argument, as in Hippisley. The Jacobian elliptic functions sn u, cn u, dn u, not tabulated by Hippisley, nor to anything approaching the present extent by anyone else, were apparently computed as quotients of theta functions. ϕ is given to 12D in *radian* measure. The number of decimals in ϕ fills a great need, though anyone interested in checking Legendre by interpolating in Spenceley will view the radian measure with mixed feelings. $E(\phi, k)$ with the Spenceleys means what it ought to mean, namely the second integral $\int_0^\phi (1 - k^2 \sin^2 \phi)^{\frac{1}{2}} d\phi = \int_0^u \mathrm{dn}^2(u, k) du$, not the zeta function. $A(r)$ and $D(r)$ are exactly as in Hippisley. The values were calculated to 15D, checked within a few units of the last decimal, and cut down to 12D for publication. Brown University Library has a photographic reproduction of the original 15-place manuscript. In the page headings for every argument are given the values of $K, K', E, E', D(90)[= 1/\sqrt{k'}], 1/D(90)$ to 15D, and of q, q' to 16S.

The appendix gives a number of formulae for computation, and also two tables: p. 364–365, values to 15D of $r\pi/180$, sinh $(r\pi/180)$, cosh $(r\pi/180)$ for $r = 1(1)90$; p. 366, values to 25D of sin $(r\pi/180)$, cos $(r\pi/180)$ for $r = 1(1)89$.

The reviewer has made no numerical checks beyond comparing about a third of the values of K and E with values to about 13D computed by himself. With the checks described it is difficult to see how there can be much error, even though differencing is not mentioned. The latter would have required a considerable effort, as the tables contain about 720 values for each modular angle, or altogether about as many as in a natural trigonometrical table giving sines and tangents at interval 10″, while the higher differences are much larger in the present tables.

An old-fashioned reviewer, addicted to desk and even to armchair methods, and not disposed to have his standards of measurement completely shot from under him by "what the new machines could do," must consider this a magnificent piece of computing. Professor and Mrs. Spenceley, and the six named National Youth Administration students who shared the work with them in the earlier years, must be heartily congratulated on the final achievement. If a few criticisms and queries follow, one is glad to be able to point out that they relate far more to the Greenhill program than to its splendid implementation.

Let us begin with a point which some may regard as trivial. Greenhill himself made a good start in 1911; his theta functions were to be $A(r^\circ)$, $\cdots$, r a number (integral in the tables). In fact, they could be called $A(x)$, $\cdots$, and x could be measured in other units than degrees. But his notation quickly degenerated into $A(r)$, $\cdots$, and it is evident that Greenhill never made up his mind whether his r was a number or a number of degrees, which for distinction we may call an angle; he used r in both senses cheek by jowl. Matters are not greatly improved by the Spenceleys. It is true that, except in the headings of the small subsidiary tables at the end, their r is incontestably a number; but in the main table their theta functions are $A(r)$, $\cdots$, while in the text of the appendix they are $A(r^\circ)$, $\cdots$. Moreover, their θ is an angle in the main table, both an angle and a number in the appendix. Perhaps this should not be overstressed; positional astronomers, for instance, have often used the convention that x'' can mean not only $x \times 1''$, but also $x/1''$, that is, the number of seconds in an angle x, with only aesthetic loss to their science. Yet I think most mathematicians will agree that $^\circ$ is best regarded as a useful abbreviation for $\pi/180$, so that, for instance, $(1^\circ)^3 = 1''.097$ approximately; and if the columns of a table are not headed r, $\sinh(r\pi/180)$, they will prefer r, $\sinh r^\circ$ to r°, $\sinh r$.

The appendix contains another small infelicity. The reviewer is quite willing to identify $\mathrm{sn}\ \frac{1}{2}K$, say, with $\mathrm{sn}\ (45K/90)$; but to describe $\mathrm{sn}\ \frac{1}{2}K$ as $\mathrm{sn}\ 45^\circ$ is surely an excess of computer's license. Admittedly it is not easy, especially when divisions into 90 parts are used, to find a concise yet respectable notation which will indicate the theta-function argument x in the case of those functions which in analysis customarily have the elliptic-function argument u. There is much less awkwardness when the centesimal system is used (as in small tables by Hoüel[2]), for $x = 50^g$ or $0^q.50$ corresponds to $u = 0.50\ K$.

To come to the most important point: was Greenhill wise to tabulate his theta functions in place of the Jacobian ones? Is there any substantial "advantage" in the fact that the division-values are algebraic functions of the modulus? It is plain that Greenhill revelled in the sort of algebra by which division-values may be obtained, and it is easy to see the attractiveness of an independent check, though the Spenceleys appear to have got on quite well without it. But after all, values for only a few arguments are likely to be checked by this process, and in these cases the values could very easily be multiplied by the necessary factors, ϑ_2 or ϑ_4, if Jacobian functions were being tabulated. The whole subject of elliptic functions has long been notorious for its conflicts of notation; we are now faced by a fresh conflict of notation, between theory and tables. It is safe to say that most mathematicians now expound the theory of theta functions in some kind of ϑ notation; are they to convert the tabulated Greenhill values to Jacobian ones, or to rewrite the theory in terms of Greenhill functions? Greenhill went some way in the latter direction; certainly theory and tables cannot remain for ever divorced, but there appears to the reviewer to be a considerable probability that mathematicians will choose the former course. It has already been mentioned that in the Hippisley tables the necessary factors ϑ_2 and ϑ_4 are given in the headings for each modulus. This is the only respect in which the new tables seem to the reviewer to fall short of Hippisley's. Failing the tabulation of Jacobian theta functions, one could wish that ϑ_2 and ϑ_4 had replaced $D(90)$ and its reciprocal in the headings. After all, $D(90)$ is given in the line $r = 90$ (certainly to "only" 12D, but $D(90) = (\cos\theta)^{-\frac{1}{2}}$, and $\cos\theta$ is given to 25D on the last page). As things stand, ϑ_2 and ϑ_4 are presumably to be computed either from $\sqrt{(2kK/\pi)}$ and $\sqrt{(2k'K/\pi)}$ respectively, or from q-series. The loss of convenience is plain.

A small suggestion may be made, in case these valuable tables are reprinted at any time.

It would be a convenience if the information in the headings, besides being distributed throughout the volume, were also brought together on two pages at one opening, say in the appendix. As far as the reviewer is aware, there are no other tables whatever giving values of K, K', E, E' to 15D, or of q, q' to 16S, at interval 1°.

One is glad to see in the preface that a set of 5-place tables is suggested; the reviewer agrees that they would be very useful. He also sympathizes with the convenient notation snh $u = -i$ sn iu, etc. used in the appendix.

Whatever may be the course of the inevitable rapprochement between the theoretical and computational aspects of elliptic functions, it is plain that the scientific world owes a very great debt to the authors and publishers of the volume under review. There will be few readers of *MTAC* who will not wish to obtain a copy immediately.

ALAN FLETCHER

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[1] EDITORIAL NOTES: Why the Glaisher Tables of Elliptic Functions were set up in type but remained unpublished, has never been revealed, although certain facts are on record, from Glaisher's pen, in BAAS *Report*, 1873, p. 171–172, 1930, p. 250–251 (also with later information); *Mess. Math.*, v. 6, 1876, p. 112; and H. J. S. SMITH, *Coll. Math. Papers*, v. 1, 1894, p. lxiii–lxiv. Among various letters on mathematical tables (now the property of R. C. A.) which formerly belonged to Glaisher, are 17 (Oct. 2, 1873–Sept. 15, 1881) dealing with these tables. One of these, from Glaisher to the printers, (a) Taylor & Francis, (b) Spottiswoode & Co., was as follows:

1 Dartmouth Place, Blackheath, S. E.
1874 September 16

"Dear Sirs:

"I am directed by the Committee of the British Association on mathematical tables to ask you if you would be so good as to state the cost for which you would be willing to print and stereotype for them a large table of which I enclose one page as a specimen.

"The table consists of 356 pp. similar to that which I enclose but the last two figures in all the columns are throughout to be omitted (so that *e.g.* the 1st, 3rd, 5th, & 7th columns would take only 8 figs. after the decimal point).

"Please notice that certain of the first figures on 178 of the pp. (always 1's and 2's) have a rule over them, which might perhaps require special type to be cast. We should like the size of the page to be an octavo of about the same size as that of the Nautical Almanac, the figures in the Diff. columns to be in smaller type than those in the other columns, and a white line to be left between every five lines.

"Of course we should wish to have as many revises as would be necessary to ensure freedom from error, but as the manuscript is a good one & has been read with the originals I do not anticipate that more than one revise will often be required. A proof of the page after it is stereotyped would also be desired.

"The table would be preceded by an introduction of perhaps 100 pp. of mathematics of a character not very different from Booth's *New Geometrical Methods* (but without figures). Could you also let us know (very roughly) about how much per sheet this may be expected to cost. Of course there are details which I have not mentioned, & which could not be explained without you saw the whole ms. of the table, but the specimen page gives a sufficiently good idea of all the others for all purposes short of an actual contract. The number of copies would be 250."

"Believe me, to be Yours Truly
J. W. L. Glaisher"

The last letter is from Taylor & Francis as follows: "I beg to enclose you samples of paper for the Tables. Will you in returning the one you like kindly let me know how many copies you will have printed."

In connection with this project £809 were spent, of which apparently £259 was for computing and £550 for printing. There was a reference to these tables in *Nature*, v. 15, 1877, p. 252. After Glaisher's death in 1928 30 manuscript volumes covering the work of these ten-place tables became the property of BAASMTC.

What was first planned as a 100-page Introduction to the Tables later developed into a planned "Memoir on the theta and omega functions" by H. J. S. SMITH (1826–1883), to follow the Tables; but Smith died before this was quite complete. It is printed in Smith's *Coll. Math. Papers*, v. 2, 1894, p. 415–621. Four of the above-mentioned 17 letters are from H. J. S. Smith (1873–76) and deal in part with the "Introduction."

[2] J. Hoüel, *Recueil de Formules et de Tables Numériques*, third ed., Paris, 1885, p. [57]–[59].

486[D, E, P].—I. V. ANAN'EV, *Spravochnik po Raschetu Sobstvennykh Kolebaniĭ Uprugikh Sistem* [Reference book for computations of inherent vibrations of elastic systems]. Moscow and Leningrad, OGIZ, 1946, 223 p. 12.8 × 19.4 cm. Tables p. 173–220. Bound, 9.50 roubles.

T.I: $\tan\alpha$, $\cot\alpha$, $\alpha = [0(.02)6.3(.1)10; 5D]$.

T.II: $\cos\alpha$, $\sin\alpha$, $\cosh\alpha$, $\sinh\alpha$, $S(\alpha) = \frac{1}{2}(\cosh\alpha + \cos\alpha)$, $T(\alpha) = \frac{1}{2}(\sinh\alpha + \sin\alpha)$, $U(\alpha) = \frac{1}{2}(\cosh\alpha - \cos\alpha)$, $V(\alpha) = \frac{1}{2}(\sinh\alpha - \sin\alpha)$, for $\alpha = [0(.01)4.99; 5D]$.

T.III: $A(\alpha) = \cosh\alpha \cdot \sin\alpha + \sinh\alpha \cdot \cos\alpha$, $B(\alpha) = \cosh\alpha \sin\alpha - \sinh\alpha\cos\alpha$, $C(\alpha) = 2\cosh\alpha \cdot \cos\alpha$, $S_1(\alpha) = 2\sinh\alpha \cdot \sin\alpha$, $D(\alpha) = \cosh\alpha \cdot \cos\alpha - 1$, $E(\alpha) = \cosh\alpha \cdot \cos\alpha + 1$, for $\alpha = [0(.02)10; 5D]$.

T.IV: $F(\alpha) = \alpha\,(\sin\alpha\cosh\alpha - \cos\alpha\sinh\alpha)/(1 - \cos\alpha\cosh\alpha)$, $H(\alpha) = \alpha\,(\sinh\alpha - \sin\alpha)/(1 - \cos\alpha\cosh\alpha)$, $L(\alpha) = \alpha^2 \sin\alpha \cdot \sinh\alpha/(1 - \cos\alpha\cosh\alpha)$, $N(\alpha) = \alpha^2\,(\cosh\alpha - \cos\alpha)/(1 - \cos\alpha\cosh\alpha)$, $R(\alpha) = \alpha^3\,(\sin\alpha\cosh\alpha + \cos\alpha\sinh\alpha)/(1 - \cos\alpha\cosh\alpha)$, $\Pi(\alpha) = \alpha^3\,(\sinh\alpha + \sin\alpha)/(1 - \cos\alpha\sinh\alpha)$, for $\alpha = [0, .5(.01)1(.02)5; 3\text{–}5D]$.

Extracts from text

EDITORIAL NOTE: On turning to Professor W. PRAGER, "Tables of certain functions occurring in dynamics of structures," *MTAC*, v. 1, p. 101–103, it will be noted that tables of $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, $S_1(\alpha)$, $D(\alpha)$, $E(\alpha)$, have been already discussed. There are 5D tables of $\cosh\alpha\cos\alpha$, $\cosh\alpha\sin\alpha$, $\sinh\alpha\cos\alpha$, $\sinh\alpha\sin\alpha$, in M. BOLL, *Tables Numériques Universelles*, Paris, 1947, p. 476–483, possibly for use in the field of the volume under review.

In the book under review the paper is cheap, the ink is pale, the type of figures is not clear-cut and distinct, so that some figures are not seen at all, and sometimes it is difficult if not impossible to distinguish between a 6 or 9 and 0, or between 3 and 8.

On checking the last four columns of pages 178–180 of Anan'ev's work the following errata were found by S. A. J.:

α	Function	For	Read	α	Function	For	Read
0.46	$V(\alpha)$	0.01625	0.01623	1.21	$S(\alpha)$	1.08934	1.08943
0.53	$T(\alpha)$	0.53024	0.53034	1.23	$U(\alpha)$	0.76196	0.76126
0.60	$T(\alpha)$	0.60074	0.60064	1.25	$U(\alpha)$	0.78658	0.78655
0.67	$S(\alpha)$	1.00830	1.00839	1.27	$T(\alpha)$	1.29750	1.29755
0.79	$V(\alpha)$	0.08228	0.08221	1.30	$U(\alpha)$	0.85123	0.85170
0.85	$T(\alpha)$	0.85380	0.85370	1.36	$U(\alpha)$	0.93336	0.93360
0.97	$V(\alpha)$	0.15297	0.15227	1.40	$V(\alpha)$	0.45933	0.45943
1.03	$T(\alpha)$	1.03953	1.03966	1.41	$T(\alpha)$	1.45655	1.45650
1.11	$S(\alpha)$	1.06333	1.06331				

487[D, P].—PIETRO MARCHISIO, *Tavole pel Tracciamento delle Curve. Tavole Trigonometriche, Tavole Tacheometriche centesimali. Tabelle pel Tracciamento delle Curve con Coordinate Polari e con Coordinate Ortogonali. Introduzione in Lingue Italiana e Francese.* Borgo S. Dalmazzo (Cuneo), Istituto Grafico Bertello, [1947], xxxii, 255 p. 15 × 23.6 cm.

This work is the second edition of the author's *Tavole Trigonometriche Centesimali. Tables Trigonométriques Centésimales. Applicazione pel Tracciamento delle Curve. Tavole pel Tracciamento delle Curve con Coordinate Polari e con Coordinate Ortogonali. Introduzione in lingue Italiana e Francese*, Milan, Libreria Internazionale Ulrico Hoepli, 1930. xxxii, 255 p. + 2 p. "Errata corrige." The 1947 work is a reprint, with type reset, of a corrected 1930 edition, except for the addition of a brief preface to the second edition, and an extension of the main table.

The work was intended primarily for construction of railways and roads, calling for the exact laying out of circular arcs by the use of centesimal tables. The principal tables of the volume, p. 1–201, are 6D tables, for every centesimal minute, of the six trigonometric functions, versed sin α, versed cos α ($= 1 - \sin \alpha$), arc α, and $\frac{1}{2}\pi - \alpha$; in the present edition are added two more columns, 5D tables of $\cos^2 \alpha$ and $\sin \alpha \cos \alpha$. For each grade of the quadrant the values of these 12 functions are given on four pages.

On p. 219–224 is a table for reading off the polar coordinates of ends of arcs (cords of arcs and centesimal angles) of various lengths of arcs from .1 to 100, on circles of varying radii, r = 15(5)30(10)60, 75, 80, 100(25)200(50)500, 600, 750, 800, 1000. There are similar tables for rectangular coordinates, p. 226–227.

Then on p. 229–232, for circles of radius r = [100(50)300(100)2000(500)4000; 9D] are given (a) circumference; (b) arc subtended by 1^g at center; (c) number of grades at center subtended by unit length of the circumference; (d) arcs corresponding to angles at center of 1°, 1′, 1″; (e) number of sexagesimal seconds at center corresponding to unit lengths of arc; (f) $1000/r$.

P. 233–234. For r = 10(5)80(10)100(25)200 are given angles at center in grades subtending arcs 1(1)10. This is followed (p. 235–239) by lengths of arcs of a unit circle opposite central angles: (a) [$1^g(1^g)100^g(5^g)300^g(10^g)400^g$; 11D], (b) [1°(1°)100°(10°)270°(30°)360°; 11D], (c) [1′(1′)60′; 11D], (d) [1″(1″)60″; 11D]. Conversion of centesimal grades, minutes and seconds, and conversely (p. 241–247). Various trigonometric formulae and constants (p. 250–255).

R. C. A.

488[D, R].—Great Britain, H. M. Nautical Almanac Office, *Five-Figure Tables of Natural Trigonometrical Functions.* London, His Majesty's Stationery Office, 1947, iv, 124 p. 24.8 × 30.5 cm. Bound, 15 shillings. American agent: British Information Services, 30 Rockefeller Plaza, New York 20, N. Y. $4.00.

The present tables were prepared in 1941 by the Nautical Almanac Office, at the request of the War Office, for use in survey calculations; they were not put on general sale to the public either then or when they were reprinted in 1945. It has now been possible to make the tables, which are the only five-figure natural trigonometrical tables at such a small interval, available to the public.

These tables give natural values of the four trigonometrical functions that occur most frequently in surveying and associated problems, and are intended to replace similar tables (*Manual of Artillery Survey*, Part II, 1924) giving logarithmic values of the same four functions. They are provided specially for use with calculating machines and are not of great practical utility unless such a machine is available. The combination of natural trigonometrical functions and a calculating machine provides, however, a more powerful and effective weapon for the practical computation of survey calculations than the superseded combination of logarithmic values and tables; in particular, mechanization allows of a more direct approach, leading to considerable simplification of the formulae employed.

Many sources are available for the trigonometrical functions to five figures, so that it is difficult to quote an authority for the source of the present figures. They have, however, been rigorously compared in proof with (among other tables) Andoyer's 15-figure values, and these can therefore justly be regarded as the relevant source.

There are two separate tables, the "auxiliary" and the "main" tables.

The auxiliary table (p. 2–31) gives, directly, values of the cotangent for every second of arc up to 7°30′ to 5S. Each column gives the 60 values of the function in each minute of arc and there are 15 columns to the page, so that each degree occupies four pages or two openings.

The main table (p. 34–123) contains values of the sine, tangent, cotangent and cosine arranged semi-quadrantally at interval 10″. Five decimals are retained throughout in the sine, tangent and cosine, but the values of the cotangent have been given to 5S only in most of the range; from 27° to 45° 5D have been retained. Each block contains values of all four functions for 10′ and three blocks are given on each page, so that each opening of the book covers a whole degree; thus, the functions for a given sub-division of a degree are always found in the same relative position in the opened book. In each of the blocks the signs of the functions in each of the quadrants are indicated.

The tables have been constructed at such a small interval of angle that, in field survey work, no interpolation between tabular entries is necessary. If for a special purpose more accurate values of the functions are required, the values of sine, tangent and cosine may be interpolated by simple proportion, the maximum difference between consecutive entries being 10; for the cotangent the differences are larger, but simple proportional interpolation is permissible, except in the main table 0°0′ to 0°17′ and in the auxiliary table from 0′0″ to 1′40″.

Extracts from the text

EDITORIAL NOTE: The printing and display of the table are excellent. A list of errata in the 1941 and 1945 editions is given in the preface, signed by the Astronomer Royal, H. SPENCER JONES, April, 1947. The volume was published in July and a brief errata sheet, dated October, has been inserted.

489[F].—F. J. DUARTE, "Sobre la ecuacion $x_1^3 + x_2^3 = y_1^3 + y_2^3$," Acad. Ciencias Fis., Mat. y Nat., Caracas, Venezuela, *Boletín*, no. 23, 1943, 19 p.

This note on the famous Diophantine equation of EULER contains (p. 15–19) a list of 100 solutions in integers of this equation. These solutions are arranged in no definite order and this makes it difficult to compare the list with others. The bibliography of 12 titles should have included the authoritative article by RICHMOND[1] in which is given a list of 63 solutions in integers not exceeding 100 in absolute value. Of these only 25 are given by Duarte who, however, gives the solution 29, 99, 60, 92 overlooked by Richmond. The majority of solutions of Duarte have large and nearly equal values of x_2 and y_2. The largest solution is

309032, 390545, 313532, 387665.

These reflect the author's methods of solution rather than any general property of the class of all solutions.

D. H. L.

[1] H. W. RICHMOND. "On integers which satisfy the equation $t^3 \pm x^3 \pm y^3 \pm z^3 = 0$," Camb. Phil. Soc., *Trans.*, v. 22, no. xix, 1920, p. 389–403.

490[F].—A. FERRIER, *Les Nombres Premiers*, Paris, Vuibert, 1947, vi + 111 p. 14 × 22.5 cm. 280 francs.

This work contains (p. 60–110) a factor table to 100000 of all numbers not divisible by primes less than 17. With each such number which is not a prime is given its least prime factor. Primes are indicated by boldface type. No account of the construction or possible comparison of this table with others is given. In spot checking the table only two errata were discovered by the reviewer: p. 110, the final digits of 99199 should not be boldface whereas the final digits of 99989 should be boldface. This "type 5" table differs from any factor table previously published. The nearest similar table is due to Cahen[1] which omits primes and extends only to 10000.

Two other tables may be mentioned: p. 29, a table of 2^n and the factors of $2^n \pm 1$, for $n \leq 40$; p. 32, a table of the number of primes not exceeding x for $x = k \cdot 10^n$, $k = 1(1)10$, $n = 1(1)6$. The first half of this work consists of an interesting summary of various results and methods having to do with primes. To quote the author, "Mystérieux et inoffensifs, les nombres premiers offrent un refuge attrayant et serein à qui veut s'écarter de la civilisation de l'atome." The material presented is definitely pre-atomic. Twentieth century advances on the problem are, with a few exceptions, entirely omitted.

D. H. L.

[1] E. CAHEN, *Théorie des Nombres*, v. 1, Paris, 1914, p. 378–381.

491[F].—ALBERT GLODEN, (**a**) *Table des Solutions de la Congruence* $x^4 + 1 \equiv 0 \pmod{p}$ *pour* $350\,000 < p < 500\,000$. Luxembourg, author, and Paris, Centre de Documentation Universitaire, 5 place de la Sorbonne, 1946, 41 p. 21.5 × 27.5 cm. Offset print, except title page. (**b**) *Table des Solutions de la Congruence* $x^4 + 1 \equiv 0 \pmod{p}$ *pour* $500\,000 < p < 600\,000$. Luxembourg, author, rue Jean Jaurès 11, December 1947, i, 12 leaves. 20.8 × 29.5 cm. Offset print on one side of each leaf. For $p < 3.5 \cdot 10^5$ see *MTAC*, v. 2, p. 71, 210–211 (reviews by D. H. L.).

The titles of these publications and previous reviews are sufficiently descriptive of the contents of (**a**) and (**b**), forming the bases for factorizations; compare *MTAC*, v. 2, p. 72, 211, 300.

We have also received a carbon of a type-script table (dated Dec. 2, 1947) of single solutions of $x^4 + 1 \equiv 0 \pmod{p}$, $p > 6 \cdot 10^5$, for 243 values of p (from 600 841 to 9 778 057) compiled by means of the author's factorizations of $x^4 + 1$.

In a communication of 29 November, 1947, "Nouveaux compléments aux tables de factorisations de Cunningham," sent for publication in *Mathesis*, Professor Gloden reported having shown, by means of (**b**), that the following 21 numbers are prime:

(i) $X^4 + 1$ for $X = 710, 730, 732, 738, 742, 748, 758, 760, 768, 772$;

(ii) $\frac{1}{2}(X^4 + 1)$ for $X = 843, 845, 855, 857, 879, 883, 891, 895, 913, 917, 919$.

He gives also the new factorizations

$$\tfrac{1}{2}(889^4 + 1) = 505\,777 \cdot 617\,473$$
$$38^8 + 1 = 539\,089 \cdot 8\,065\,073.$$

Compare *MTAC*, v. 2, p. 252.

492[F].—NILS PIPPING, "Tabel der Diagonalkettenbrüche für die Quadratwurzeln aus den natürlichen Zahlen von 1–500." Aabo, Finland, Akademi, *Acta*, *Mathem. et Phys.*, v. 15, no. 10, 1947, 11 p.

MINKOWSKI defined the real semi-regular continued fraction

$$\xi = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots \quad (a_i^2 = 1)$$

to be a *diagonal continued fraction* in case each of its convergents An/Bn differs from ξ by an amount less than $1/(2B_n^2)$ in absolute value. Every real number not equal to half an integer has precisely one such expansion. The present note contains a table of the diagonal continued fraction expansions of all square roots of non-square integers $D \leq 500$. These are periodic and so it suffices to give the period together with the nonperiodic term $b_0 = [D^{\frac{1}{2}}]$. In case $a_k = -1$, b_k is printed with a prime to indicate this fact. For example for $D = 183$ the entry is 183/14 (2′, 8, 2, 28′). This means that

$$183^{\frac{1}{2}} = 14 - \frac{1}{2 +} \frac{1}{8 +} \frac{1}{2 -} \frac{1}{28 -} \frac{1}{2 +} \frac{1}{8 +} \cdots.$$

In 144 of the 478 expansions the diagonal continued fraction is regular, i.e., the a's are all equal to unity.

D. H. L.

493[F].—DOV YARDEN, "Hashlamoth leluah mispare Fibonatsi" [Addenda to the table of Fibonacci numbers], *Riveon Lematematika*, v. 2, Sept. 1947, p. 22. 21.6 × 33.6 cm. See also MTE **127**.

These are the addenda promised in connection with a previous table of the factors of Fibonacci's U_n and $V_n = U_{2n}/U_n$ (see *MTAC*, v. 2, p. 343). New factors are given of U_n

for $n = 67, 115$ and of V_n for $n = 94, 97, 98, 101, 103, 104, 106, 108, 114, 119, 122, 127$. The large factors of uncertain character of the following numbers have no factor under 10^6.

U_n for $n = 79, 91, 93, 111, 115, 125$.
V_n for $n = 73, 74, 79, 83, 86, 91, 98, 103, 106, 108, 110, 117, 119, 126$.

D. H. L.

494[G].—A. Zavrotsky, "Algunas generalizaciones del concepto de campo," Acad. Ciencias Fis. Mat. y Nat., Caracas, Venezuela, *Boletín*, no. 28, año 1946, 1947, 23 p.

This article contains tables of the functions $S_2(x, y; n)$ defined as follows:

$$S_2(x, y; -1) = \log_2 (2^x + 2^y), \qquad S_2(x, y; 0) = x + y,$$
$$S_2(x, y; 1) = xy, \qquad S_2(x, y; 2) = x^{\log_2 y},$$
$$S_2(x, y; 3) = 2^{(\log_2 x)^u}, \qquad u = \log_2 \log_2 y.$$

In general if we denote by $l_n(x)$ the nth iterated logarithm of x to base 2 so that

$$l_{-1}(x) = 2^x, \qquad l_0(x) = x, \qquad l_1(x) = \log_2 x,$$
$$l_2(x) = l[l(x)] = \log_2 \log_2 x, \cdots,$$

then

$$S(x, y; n) = l_{-n}[l_n(x) + l_n(y)].$$

The tables are for $x = 0(1)10$, $y = 0(1)10$, $n = -1, 2, 3$, and are to 5D for $n = -1$, and 3D otherwise.

These same functions may be defined for the non-zero elements in $GF(p)$ and then they become periodic functions of n. This is illustrated by a set of tables for $GF(5)$.

D. H. L.

495[I].—H. E. Salzer, "Tables for facilitating the use of Chebyshev's quadrature formula," *Jn. Math. Physics*, v. 26, 1947, p. 191–194. 17.5 × 25.4 cm.

Chebyshev suggested in 1874 the quadrature formula[1]

$$\int_{-1}^{1} f(z)dz = 2n^{-1}\sum_{i=1}^{n} f(z_i) + R_n$$

whose coefficients are all equal. The remainder R_n may be made zero for f any polynomial of degree n provided the points z_i are taken as the roots of a polynomial $T_n(x)$ of degree n which is the polynomial part of the function

$$x^n \exp \{-n \sum_{k=1}^{\infty} (2k[2k+1]x^{2k})^{-1}\}.$$

This paper gives the first twelve polynomials. By a curious stroke of misfortune the roots of $T_n(x)$ are all real only for $n = 1(1)7, 9$, so that Chebyshev's idea is practical in these cases only. The problem of checking the smoothness of the computed values $f(z_i)$ is solved by expressing the divided difference of order $n - 1$ of $f(z)$ as a linear combination of these n values. The requisite coefficients are tabulated, mostly to 9S, together with the roots z_i to 10D, for $n = 3(1)7, 9$. For a similar treatment of the Gaussian quadrature formula see *MTAC*, v. 2, p. 256.

D. H. L.

[1] P. L. Chebyshev, (a) "Sur les quadratures," *Jn. de Math.*, s. 2, v. 19, 1874, p. 19–34; (b) Assoc. Française pour l'Avanc. d. Sci., *Compte Rendu*, Lyon, 1873, p. 69–82; (c) *Oeuvres*, ed. by A. Markov & N. Sonin. St. Petersburg, v. 2, 1907, p. 165–180. Chebyshev's formula was suggested by a formula due to B. Bronwin, *Phil. Mag.*, v. 34, 1849, p. 262.

Editorial Notes: The statement of this article which Mr. Salzer modified in referring to Walther's table (1930), MTE **126**, must also be slightly modified in connection with Chebyshev's table (1873) of the roots of $T_n(z) = 0$, $n = [2(1)7; 6D]$, (a) p. 25–26, (b) p. 74–75, (c) p. 170–171. Comparison with the Salzer table shows the following errors in the original table; (a), (b): $n = 2$, *for* .816479, *read* .577350; $n = 3$, *for* .707166, *read* .707107; $n = 4$, *for* .794622 and .187597, *read respectively* .794654 and .187592; $n = 5$, *for* .832437 and

.374542, *read respectively* .832497 and .374541; $n = 6$, *for* .866249, .422540, and .266603, *read respectively* .866247, .422591, and .266635; $n = 7$, *for* .883854, .529706 and .323850, *read respectively* .883862, .529657 and .323912. The two errors on p. 159 of E. T. WHITTAKER & G. ROBINSON, *The Calculus of Observations*, London, 1924, are evidently due to copying Chebyshev's erroneous values for $n = 5$. All of Chebyshev's errors have been corrected in (c).

496[K].—TRUMAN LEE KELLEY, *Fundamentals of Statistics*, Cambridge, Harvard University Press, 1947, xvi, 755 p. 14.5 × 22.1 cm. $10.00. Compare *MTAC*, v. 1, p. 151–152.

The word "statistics" covers many different meanings. Originally the work of the statistician was supposed to consist in collecting, processing, and presenting empirical data. The modern theoretical statistician lives in a different world. The inadequacy of the above description is seen from the fact that in the theory of designs of experiments (as applied in agriculture and industrial experimentation) and in the design of efficient sampling techniques the mathematical statistician deals with data which do not yet exist. He is concerned with abstract models and theorems much in the same way as the theoretical physicist or engineer. Now even the simplest statistical techniques for analyzing empirical data or for judging reliability and significance of apparent differences are based on rather intricate mathematical theories. Clearly the majority of users of such techniques have not the necessary background for a proper understanding of statistical theory. It is therefore necessary to present the result of modern statistical investigation in a descriptive way and give rules rather than proofs. This corresponds to the education of the engineers who use formulae and rules without knowledge of the underlying physical theories.

The present book follows this pattern. It is intended for people without any mathematical background and is to serve as a first introduction into a rational analysis of quantitative data. Accordingly, the first five chapters (198 pages) are devoted to the general cultural background and to a description of various techniques of graphical and tabular presentation. These are a prerequisite in the same sense as arithmetic. The following three chapters (pages 199–310) are devoted to classical descriptive statistics: moments, means, normal distribution, etc. Statistical methods in the proper sense of the word begin only with the ninth chapter. The choice of topics is obviously made with a view to users in the fields of education and psychology. The main subjects discussed are regression and correlation analysis and some chi-square analysis. Various elementary mathematical techniques are described, such as interpolation, curve-fitting, solution of linear algebraic equations, etc. Chapter 13 also describes the fundamental idea underlying the sequential analysis of ABRAHAM WALD.

The following mathematical tables are contained in the book. On pages 628–636 three-point Lagrange interpolation coefficients for [0(.001)1; 5D]; on pages 637–638 four-point interpolation coefficients [0(.01)1; 5D]. On pages 640–652 tables of the error function are arranged as follows: Put $z = (2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$, $I = \int_0^x z dx$. The argument of the tables is $I = 0(.001).5$ (this covers the entire range). The successive columns give x and z to 6D each; then $q = .5 - I$, followed by z/q (to 5D), z/p (to 5D), pq and $p = 1 - q = .5 + I$ (the latter exact). On pages 653–656 we find the square roots of N and $10N$, and the cube roots of N, $10N$, $100N$ for N = [1(.1)10; 5D] together with indications of the maximum error committed when interpolating. In the text there is on p. 587 a table of $\Gamma(x)$ for x = [.99(.01)2.01; 5D] and on p. 596 of $x = 2 \sin^{-1} p^{\frac{1}{2}} - \frac{1}{2}\pi$ for p = [0(.01)1; 4D]. None of the tables offers anything new.

WILL FELLER

Cornell University

497[K, P].—CONNY C. R. A. PALM, *Table of the Erlang Loss Formula.* (*Tables of Telephone Traffic Formulae*, no. 1.) Printed in Göteborg, Erlanders Boktryckeri, 1947; distributed by C. E. Fritzes Hovbokhandel, Stockholm, Sweden, ii, 23 p. 21.3 × 28 cm.

The simplest situation in the theory of automatic telephone exchanges can be described as follows. Incoming calls are "randomly distributed" (that is, at least during the "busy hour"). The probability that exactly k calls will originate during time t is given by the POISSON expression $e^{-at}(at)^k/k!$ where $a > 0$ is a constant. Moreover, it is assumed that the lengths of the individual conversations are statistically independent and that the probability of any conversation lasting for time t or more is $e^{-(t/b)}$. Suppose there are n circuits available so that an incoming call is served whenever a circuit is free. If no circuit is free, the call is "lost" and has no influence on the future traffic. Erlang's formula[1] for the proportion of lost calls is

$$E_{1,n}(A) = A_n(A_0 + A_1 + \cdots + A_n)^{-1}, \ A = ab,$$

where we put for abbreviation $A_k = (ab)^k/k!$ (the notation is not standard and the general conditions of applicability of Erlang's formula are still discussed). The tables are double-entry 6D tables with $n = 1(1)150$, and $A = .05(.05)1(.1)20(.5)30(1)50$, $52(4)100$. Only entries which are significantly different from zero are tabulated, and thus the effective range of A decreases with increasing n. For $n = 100$ entries occur only for $A \geqslant 56$, and for $n = 150$ we have the single entry $E_{1,150}(100) = .000\,001$. The tabular intervals are selected so as to permit linear interpolation in A with an accuracy to four or five decimals. The sixth decimals are said to be accurate throughout the tables. These have been computed from the recurrence formula $E_{1,n}(A)\{n + AE_{1,n-1}(A)\} = AE_{1,n-1}(A)$.

WILL FELLER

Cornell University

[1] AGNER K. ERLANG, (a) "Løsning af nogle Problemer fra Sandsynlighedsregningen af Betydning for de automatiske Telefoncentraler," *Elektroteknikeren*, Copenhagen, v. 13, 1917, p. 5–13; (b) "Lösung einiger Probleme der Wahrscheinlichkeitsrechnung von Bedeutung für die selbsttätigen Fernsprechämter," *Elektrotechn. Z.*, v. 39, 1918, p. 504–508, with tables, translation of (a). Also T. C. FRY, *Probability and its Engineering Uses*, New York, 1928, p. 342–347.

498[L].—JOHN P. BLEWETT, "Magnetic field configurations due to air core coils," *Jn. Appl. Physics*, v. 18, no. 11, Nov. 1947, p. 968–976; tables p. 969–970. 20 × 26.5 cm.

The field configurations around a circular loop of wire bearing current are discussed, and a tabulation is presented for the field component parallel to the axis of the loop. The table gives 4D values of the function $B_z a/uI$, for $z/a = 0(.02).36$, and $\rho/a = 0(.1).5(.02)$-$1.5(.1)2$.

$$B_z a/uI = 2a[(a+\rho)^2 + z^2]^{-\frac{1}{2}}[K + E(a^2 - \rho^2 - z^2)/\{(a-\rho)^2 + z^2\}],$$

K and E being the complete elliptic integrals of the modulus $k^2 = 4a\rho/[(a+\rho)^2 + z^2]$.

499[L].—CHRISTOFFEL JACOB BOUWKAMP, **A.** *Theoretische en Numerieke Behandeling van de Buiging door een Ronde Opening* [Theoretical and numerical treatment of diffraction by a circular aperture]. Diss. Groningen. Groningen, Holland, J. B. Wolters' U. M., 1941, vi, 60, 3 p. 16 × 24.5 cm. **B.** "On spheroidal wave functions of order zero," *Jn. Math. Physics*, v. 26, July 1947, p. 79–92. 17.4 × 25.3 cm.

The differential equation for spheroidal wave functions of order zero is written in the form

$$d[(1-\xi^2)dX/d\xi]/d\xi + (k^2\xi^2 + \Lambda)X = 0, \tag{1.1}$$

"where k is a given parameter and Λ is one of a set of characteristic numbers $\Lambda_0, \Lambda_1, \cdots, \Lambda_m$ such that for $\Lambda = \Lambda_m$, (1.1) has one and only one solution $X_m(\xi)$ which is an integral function of ξ." The function X_m can be expressed by

$$X_m(\xi) = \sum_{n=0}^{\infty} b_{n\,p}^{(m)} P_{2n+p}(\xi),$$

where $p = 0$ if m is even, $p = 1$ if m is odd, and $P_{2n+p}(\xi)$ is the LEGENDRE polynomial of degree $2n + p$. The coefficients $b_{2n+p}^{(m)}$ are so normalized that

$$\int_{-1}^{1} X_m^2(\xi)d\xi = \int_{-1}^{1} P_m^2(\xi)d\xi = 2/(2m+1),$$

and $b_m \to 1$ for $k = 0$.

Some of the tabular material and formulae relating to spheroidal wave functions, given in **A**, are also in **B** in more complete form. We shall indicate by an asterisk all material of **A** reproduced or elaborated upon in **B**.

The following tables are given in **A**:

(1)* $\Lambda_0, \Lambda_1, \cdots, \Lambda_6$ and coefficients b_n for $k^2 = 3(1)10$; 6D.
(2) $\Lambda_0, \Lambda_1, \cdots, \Lambda_6$ and coefficients b_n for $k^2 = 15, 20, 25, 50, 100$; 6D.
$\Lambda_8, \Lambda_{10}, \Lambda_{12}$ and coefficients b_n for $k^2 = 25$; 6D.
$\Lambda_7, \Lambda_8, \Lambda_9, \Lambda_{10}$ and coefficients b_n for $k^2 = 100$; 6D.

(3) $X_{2m}(0), X_{2m}(1), X_{2m+1}(1), X'_{2m+1}(0)$ determined by the coefficients of (1) and (2) above.
(4) Values of σ_n and $2|\sigma_n|^2/(2n+1)$(σ_n complex), for $k^2 = 3(1)10; 15, 25, 50, 100$, and values of n, up to point where $|\sigma_n|^2$ vanishes, to 4D. Let

$$\varphi = (1/r)\exp(-ikr)\sum_{0}^{\infty} \sigma_{2n+p}X_{2n+p}(\cos\vartheta),\ p = 0 \text{ or } 1;$$

then "for the radiation of sound the behavior of the potential at large distances from the aperture" is found to be φ.

The following formulae contained in this paper should be of interest from a computational viewpoint:

(a)* $\rho_0, \rho_1, \rho_2, \rho_3$ of the formula $\Lambda_m = \sum_{\nu=0}^{\infty} \rho_\nu k^{2\nu}$, for general m; also *numerical* values of ρ_2, ρ_3, $m = 0(1)12$; 6D in ρ_2, 9D in ρ_3.

(b) Expressions for $b_{m+2\nu}$, $\nu = 0, 1, 2, 3$ for general m, as a power series in k^2 up to k^6;
(c)* Power series for Λ_0, Λ_2 to terms through k^{10}; for Λ_4, Λ_6, to terms through k^8. Also, of coefficients $b_0, b_2, \cdots, b_8$ associated with Λ_0 to k^8; of $b_0, \cdots, b_8$ associated with Λ_2 to k^6.
(d)* Power series for $X_0(0)$, $X_0(1)$, through k^8; of $X_1'(0)$ through k^6, all coefficients in both fractional and decimal (8D) form.
(e) Asymptotic expansion of Λ_m, general m, in powers of $1/k$, through the power $1/k^2$.
(f) It is shown how to construct a second solution of (1), $Y_m(\xi)$; an error of STRUTT[1] in regard to this solution is noted.
(g)* Method of constructing Λ_m and the coefficients b_n from the continued fractions associated with the characteristic values, and of improving the approximation by Newton's method, applied at a judiciously chosen stage of the computations.
(h) Some graphs and other short tables of a specialized nature, mainly relating to the physical problem.

(A number of changes in the tables and formulae appear to have been made after publication, but it may be that all copies in circulation have been similarly corrected. There is a notation that of the six decimal places given in the tables, roughly four are correct.)

In the paper **B**, the following tables are given:

(5) $\Lambda_0, \Lambda_1, \Lambda_2$ for $k^2 = -10(1)10$; 6D.
(6) $\Lambda_3, \cdots, \Lambda_6$ for $k^2 = 0(1)10$; 6D.
(7) Coefficients b_n associated with $X_0(\xi)$ and $X_1(\xi)$, $k^2 = -10(1)10$; 6D.
(8) Coefficients b_n associated with $X_2(\xi), \cdots, X_6(\xi)$, $k^2 = 0(1)10$; 6D.
Formulae are given in **B** for the following:
(i) Power series for Λ_0, Λ_1 through term involving k^{12}; for Λ_2 through k^{10}; for $\Lambda_3, \cdots, \Lambda_6$ through k^8. All these coefficients of the series are given in both fractional and decimal form, the latter to 12D.

(j) Power series for b_n associated with Λ_0, Λ_1, through k^8; for b_n associated with Λ_2, through k^6, n either odd or even, $\leqslant 9$.
(k) Power series from Λ_m, m general, through term involving k^8.
(l) Details of the method described in (g)* and a list of 31 bibliographic references.

In connection with (i), Bouwkamp mentions a paper by SANDEMAN[2] giving terms for Λ_0 through k^{14}. Although the formulae given in **A** and **B** are not all new, the author has noted some errors made by earlier writers, and the accessibility of **B** makes it a welcome source for the known results.

RELATION TO FUNCTIONS TABULATED ELSEWHERE. The most extensive tables now in existence are probably those contained in J. A. STRATTON, P. M. MORSE, L. J. CHU, and R. A. HUTNER, *Elliptic Cylinder and Spheroidal Wave Functions*, 1941 (see *MTAC*, v. 1, p. 157–160). In this work (later to be designated by S) the notation is as follows:
For Prolate Spheroidal Functions'

$$d[(\eta^2 - 1)dS/d\eta]/d\eta + [A + c^2\eta^2 - m^2/(\eta^2 - 1)]S = 0. \tag{2.1}$$

For Oblate Spheroidal Functions

$$d[(\eta^2 - 1)dS/d\eta]/d\eta + [B - c^2\eta^2 - m^2/(\eta^2 - 1)]S = 0, \qquad 0 \leqslant \eta \leqslant 1. \tag{2.2}$$
$$d[(\xi^2 + 1)dR/d\xi]/d\xi + [B + c^2\xi^2 + m^2/(\xi^2 + 1)]R = 0, \qquad |\xi| > 1. \tag{2.3}$$

(2.2) is obtainable from (2.1) by replacing c by ic.

The *characteristic values* corresponding to a given c and fixed integer m are denoted by $A_{m,l}$ or $B_{m,l}$. The relations between the functions of S and Bouwkamp are therefore as follows:

S Notation	c	η	$B_{0,l}$
Bouwkamp Notation	k	ξ	$-\Lambda_m$

It should be emphasized that "m" of Bouwkamp corresponds to "l" of S, and that Bouwkamp deals *only with functions of order zero.* Where Bouwkamp tabulates results for $-k^2$, they correspond to the *prolate* spheroidal functions and positive k^2; results for positive k^2 corresponding to *oblate* spheroidal functions. Functions tabulated in S: $A_{m,l}$, $B_{m,l}$ and coefficients of the series to about 5 significant figures—to be denoted by (m, l): (0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (3, 0), for $c = 0(.2)5$ and .1; .5(.5)4.5, for both the *oblate* and *prolate* cases. The range of the S functions for $m = 0$ and $l = 0, 1, 2, 3$ is therefore larger than that in Bouwkamp (except for the isolated values corresponding to $k^2 = 50{,}100$). However, since Bouwkamp's parameter is k^2, i.e., c^2 of S—there is little overlapping in results. For $l > 3$, Bouwkamp's results apparently are not duplicated elsewhere except for some unpublished results noted later.

The normalization in S is different from that of Bouwkamp.

Mr. FRED LEITNER of the Applied Mathematics Group, New York University, has the following unpublished results:

$B_{m,l}$ for $c = 1(1)5$; $m = 0, l = 4, 5$; $m = 1, l = 3, 4, 5, 6, 7$;
$m = 2, l = 0, 2, 3, 4, 5, 6, 7$; $m = 3, l = 1(1)7$; $m = 4, l = 0(1)8$;

with coefficients of another type of series (used by LEIGH PAGE) and values of functions. The characteristic values for $m = 0$, $l = 4, 5$ therefore duplicate those of Bouwkamp, but the rest are mostly new.

An interesting formula for obtaining $B_{m,l}$ has recently been communicated to the NBSCL by Dr. WM. F. EBERLEIN of the Institute for Advanced Study. This formula is in reciprocal powers of c (through c^{-3}) for *general values* of m and l. It is therefore of wider application than the one given in Bouwkamp. This formula will probably be published by its author at some future date.

GERTRUDE BLANCH

NBSCL

[1] M. J. O. STRUTT, *Phys. Z.*, v. 69, 1931, p. 597, and *Lamésche, Mathieusche und verwandte Funktionen in Physik u. Math., Erg. d. Math.*, v. 1, no. 3, 1932.
[2] IAN SANDEMAN, R. Soc. Edinburgh, *Proc.*, v. 55, 1935, p. 77.

500[L].—J. G. CHARNEY, "The dynamics of long waves in a baroclinic westerly current," *Jn. of Meteorology*, v. 4, Oct. 1947, p. 135–162; tables p. 160–162. 21.5 × 27.8 cm.

T.1 gives 4S (a few 5S) values of the function $\psi_1(\xi, r)$ satisfying the confluent hypergeometric equation

$$(1) \qquad \xi\psi'' - \xi\psi' + r\psi = 0, \text{ for } \xi = 0(.1).2(.2)1(1)10,$$

$r = -1.5, -0.5, .1(.1).9, 1.5(1)5.5$ **T.2** gives 3D values of the function $\psi_1' = d\psi_1/d\xi$ for the same values of ξ and r; for $r > .8$ the values are given to 4 or 5S. $\psi_1(\xi, r) = \frac{\sin \pi a}{\pi}$
$\times \left\{ a\xi M(a+1, 2, \xi)\left[\ln \xi + \frac{\Gamma'(a)}{\Gamma(a)} - 2\frac{\Gamma'(1)}{\Gamma(1)} \right] + 1 + \sum_{n=1}^{\infty} B_n \frac{a(a+1)\cdots(a+n-1)}{(n-1)!n!}\xi^n \right\}$;
$B_n = \sum_{\nu=0}^{n-1}\left(\frac{1}{a+\nu} - \frac{2}{1+\nu}\right) + \frac{1}{n}$; $a = -r$.

T.3 gives values of $\psi_2(\xi, r) = \xi M(a+1, 2, \xi)$, where $a = -r$, and $M(a, b, \xi) = 1 + \frac{a}{1!b}\xi + \frac{a(a+1)}{2!b(b+1)}\xi^2 + \cdots$. The function ψ_2 also satisfies (1). This table, mostly 5D, is for $r = .1(.1).9, 1.5, 2.5$, and $\xi = 0(.1).2(.2)1(1)8$; the values of $\xi = 5(1)8$ being given only for $r > .8$. **T.4** gives 3D values of $\psi_2' = d\psi_2/d\xi$, for $r = .7(.1).9$, $\xi = 0(.1).2(.2)1(1)4$. **T.5** gives 3 or 4S values of $X = \xi[\ln \xi + \sum_{n=1}^{\infty} \xi^n/(n+1)!n] - 1$, and its derivative, for $\xi = 0(.1)$-$.2(.2)1(1)7$.

501[L].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 7, *Tables of Bessel Functions of the First Kind of Orders Ten, Eleven, and Twelve;* v. 8, *Tables of Bessel Functions of the First Kind of Orders Thirteen, Fourteen, and Fifteen.* By the staff of the Laboratory, Professor H. H. Aiken, technical director. Cambridge, Mass., Harvard University Press, 1947, [x, 636] p., [x, 614] p. 19.5 × 26.7 cm. $10.00 + $10.00. Offset print. Compare *MTAC*, v. 2, p. 176f, 185f, 261f, 344.

These two new volumes of the magnificent series eventually to tabulate $J_n(x)$, $n = 0(1)100$, are 10D tables for $x = 0(.001)25(.01)99.99$, and $n = 10(1)15$. The first significant values, .00000 00001, are for $J_{10}(.847)$, $J_{11}(1.140)$, $J_{12}(1.471)$, $J_{13}(1.837)$, $J_{14}(2.236)$, $J_{15}(2.663)$.

An incidental use for such tables is to determine the 3D or 2D values (<100) of Bessel function zeros; for example, the first 25 zeros of $J_{13}(x)$ are (unrounded) as follows: 17.801, 21.956, 25.70, 29.27, 32.73, 36.12, 39.46, 42.78, 46.06, 49.33, 52.57, 55.81, 59.04, 62.25, 65.46, 68.66, 71.86, 75.05, 78.24, 81.42, 84.60, 87.78, 90.96, 94.13, 97.30.

Since $J_{14}(18.899) = 0.00015\ 08840$ and $J_{14}(18.900) = -0.00000\ 03095$, the first zero of $J_{14}(x)$ is, to 3D (unrounded), 18.899. The correctly rounded value 18.9000 is given by D. B. SMITH, L. M. RODGERS & E. H. TRAUB (1944, see *MTAC*, v. 2, p. 48–49).

The results in these volumes are almost entirely new. Among published tables to at least 10D, for $n = 10(1)15$, are only those of MEISSEL (1895) for $x = [0(1)24; 18D]$; and of HAYASHI (1930) for $x = 1, 2, 10(10)50$, to at least 15D. For $n = 10(1)13$, AIREY (1915) gave a table for $x = [6.5(.5)16; 10D]$. For various differences between Airey and Harvard see MTE **124**.

R. C. A.

502[L].—H. KOBER, *Dictionary of Conformal Representations.* Admiralty, Department of Physical Research, Mathematical and Statistical Section. Part III. *Exponential Functions and some related Functions,* [$w = e^z$, $w = \ln z$, and simple combinations of these functions with functions discussed in earlier parts], iv, 52 leaves. Part IV, *Schwarz-Christoffel transformations representable in terms of elementary transformations,* iii, ix, 17 leaves. Numbers SRE/ACS 109 and 110 = ACIL/ADM 47/562 and 47/818 [ACIL = Admiralty Center for Scientific Information and Liaison]. London, 1947. This publication is not available for general distribution. 20.2 × 33.1 cm. The author is HERMANN KOBER (1888–), Ph.D., Univ. Breslau, 1910.

We have already reported on the publication of the first two parts in *MTAC,* v. 2, p. 296–297. Part V, *Higher Transcendental Functions,* is yet to appear. We are told in Part IV that "The reference to Admiralty Computing Service, Department of Scientific Research and Experiment, on the cover of this report has been used for the sake of conforming with earlier parts of the Dictionary. Correspondence relating to this and other Reports 'SRE/ACS' should be addressed to: Mathematical and Statistical Section, Department of Physical Research, Admiralty, Fanum House, Leicester Square, London W.C.2, which is a part of the Royal Naval Scientific Service and has superseded the earlier Organization."

503[L].—WILHELM MAGNUS & FRITZ OBERHETTINGER, *Formeln und Sätze für die speziellen Funktionen der mathematischen Physik* (*Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete,* v. 52). Berlin, Springer, 1943, viii, 172 p. At the time of publication 13.20 marks, bound. 15.7 × 22.7 cm.

To give an adequate idea of the contents of this valuable non-numerical tabular work it seems desirable to transcribe the following contents of the nine chapters:

I(1–7): *Die Gammafunktion.*

II(7–16): *Die hypergeometrische Funktion*: 1. Die hypergeometrische Reihe; 2. Die Riemannsche Differentialgleichung.

III(16–48): *Die Zylinderfunktionen*: 1. Definitionen, Differentialgleichung, Rekursionsformeln, Reihenentwicklungen, Mehrdeutigkeit, unbestimmte Integrale; 2. Additionstheoreme, Multiplikationstheorem; 3. Asymptotische Entwicklungen, Multiplikationstheorem; 4. Nullstellen, Produktzerlegung für $J_\nu(z)$, Eine Partialbruchzerlegung; 5. Integraldarstellungen; 6. Integralbeziehungen zwischen Zylinderfunktionen; 7. Bestimmte Integrale mit Zylinderfunktionen, insbesondere diskontinuierliche Faktoren und Integraldarstellungen elementarer Funktionen; 8. Den BESSELschen Funktionen zugeordnete Polynome; 9. Die Funktionen von STRUVE, ANGER, und WEBER; 10. Die Funktionen von LOMMEL; 11. Beispiele KAPTEYNscher Reihen; 12. SCHLÖMILCH-Reihen; 13. MATHIEUsche Funktionen.

IV(49–78): *Kugelfunktionen*: 1. Differentialgleichung, Definitionen und Bezeichnungen; 2. Die LEGENDREschen Polynome; 3. Die zugeordnete LEGENDREsche Differentialgleichung erster Art; 4. Die Lösungen der LEGENDREschen Differentialgleichung; 5. Allgemeine Kugelfunktionen [(a). Darstellung durch hypergeometrische Funktionen; (b). Rekursionsformeln und Beziehungen zwischen verschiedenen Kugelfunktionen; (c). Formeln für spezielle Werte von x, μ, ν; (d). Analytische Fortsetzung und Verhalten für $|z| \geqslant 1$; (e). Integraldarstellungen; (f). Einige Integrale mit Kugelfunktionen; (g). Das Additionstheorem; (h). Sätze über Nullstellen; (i). Asymptotisches Verhalten für grosse Werte von $|\nu|$; (k). Ergänzungen]; 6. Kegelfunktionen; 7. Ring- oder Torusfunktionen; [8.] Die Funktionen von GEGENBAUER.

V(78–86): *Orthogonale Polynome*: 1. TSCHEBYSCHEFFsche Polynome; 2. HERMITEsche Polynome; 3. JACOBIsche Polynome; 4. LAGUERREsche Polynome.

VI(86–98): *Die konfluente hypergeometrische Funktion und ihre Spezialfälle*: 1. Die Funktionen von KUMMER; 2. Die Funktionen von WHITTAKER; 3. Die Funktionen des parabolischen Zylinders; 4. Übersicht über die Spezialfälle der konfluenten hypergeometrischen Funktion [(a). LAGUERREsche Funktionen, (b). Die Funktionen des parabolischen Zylinders, (c). Die Zylinderfunktionen, (d). Die unvollständige Gammafunktion, (e). Das Fehlerintegral und die FRESNELschen Integrale, (f). Integrallogarithmus, Exponentialintegral, Integralsinus, Integralcosinus].

VII(98–114): *Thetafunktionen, elliptische Funktionen und Integrale*: 1. Thetafunktionen; 2. Die WEIERSTRASSche $\wp$-Funktion; 3. Die elliptischen Funktionen von JACOBI; 4. Elliptische Integrale.

VIII(114–143): *Integraltransformationen und Integralumkehrungen*: 1. Die FOURIER-Transformation; 2. Die LAPLACE-Transformation; 3. Die HANKEL-Transformation; 4. Beispiele zur MELLIN-Transformation; 5. Über die GAUSS-Transformation; 6. Verschiedene Beispiele von Integralgleichungen erster Art [(i). Die Reziprozitätsformel von HILBERT für den Cotangens-Kern, (ii). Modifikationen der Formel von HILBERT, (iii). Die ABELschen Integralgleichungen, (iv). Integralumkehrungen vom Typ der MELLIN-Transformation, (v). Weitere Beispiele].

IX(144–161): *Koordinaten-Transformationen*. 1. Differentialoperationen in orthogonalen Koordinaten; 2. Beispiele zur Trennung der Veränderlichen; 3. Lineare Differentialgleichungen 2. Ordnung.

Then there are also: Collection of abbreviations used (p. 162–163), List of function symbols (p. 164–166), Literature list (p. 167–170), Subject and name list (p. 171–172). LAMÉ functions are not considered and MATHIEU functions little more than mentioned.

The copy of this work which we were permitted to see contained the following corrections by the authors: P. 1, l. 21, *for* $(\Gamma(\frac{1}{4})^4)$, *read* $(\Gamma(\frac{1}{4}))^4$; p. 3, l. 4, *for* $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right.$, *read* $\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\right.$; p. 4, l. 19, *for* $\binom{n+m-1}{n-1} = \binom{n+m-1}{m-1}$, *read* $m\binom{n+m-1}{n-1} = n\binom{n+m-1}{m-1}$; p. 5, l. 5, *for* $1 - y$, *read* $y - 1$; p. 9, l. 3, *for* $a - b$, *read* $b - a$, and *for* $a - c$, *read* $c - a$; l. 4, *for* $b - a$, *read* $a - b$, and *for* $b - c$, *read* $c - b$; p. 17, l. -5, *for* $e^{-i\pi\nu}$, *read* $e^{i\pi\nu}$; p. 18, l. -6, *for* $(-t)^n$, *read* $(-t)^{-n}$; p. 20, l. 7, *for* $\nu > 0$, *read* $r > 0$; p. 21, l. 4, *for* $J_n(kr)\cos\phi$, *read* $J_n(kr)\cos n\phi$; p. 22, l. -9, *for* $(\nu, 2m)$, *read* $(\nu, 2m + 1)$; p. 23, l. 10, *for* $\cos\alpha$, *read* $\sin\alpha$; l. -9 and -5, *for* $H_\nu^{(2)}(z)$, *read* $H_\nu^{(1)}(z)$; l. -5, *for* $3^{\frac{1}{2}}$, *read* $3^{-\frac{1}{2}}$ and *for* $\Gamma(\frac{3}{2})$, *read* $\Gamma(\frac{2}{3})$; p. 24, l. 10, *for* Es sei x reell, *read* Es seien x und ν reell; l. 11, *delete*; l. 14 and 15, *for* $e^{-i\nu}$, *read* $e^{i\nu}$, and *for* $\frac{1}{3}[-2\tau]^{\frac{3}{2}}$, *read* $\frac{\nu}{3}w^3$; l. -11, *delete* $-\frac{\pi}{2} \leqslant \arg(-2\tau)^{\frac{1}{2}} < \frac{\pi}{2}$; l. -9, *delete* für reelle Werte von τ; p. 26, l. 3, *for* $4z$, *read* 4, and *delete* $j_{\nu,n}$; p. 33, l. -6, *for* $\ln\frac{a + \sqrt{a^2 + b^2}}{b}$ *read* $\ln\frac{a + \sqrt{a^2 + b^2}}{b}\cdot\left(\frac{-2}{\pi}\right)$; p. 41, l. 10, *for* 3, *read* 3^2; p. 44, l. 7, between terms of summation *for* $-$, *read* $+$; p. 46, l. 13, for $(\lambda - 2h^2\cos 2x) = 0$, *read* $(\lambda - 2h^2\cos 2x)y = 0$; p. 49, l. 13, *for* anderrseits, *read* andrerseits; l. -12, *for* nubeschränkt, *read* unbeschränkt; p. 51, l. 8, *for* $+5$, *read* $+3$; p. 54, l. -5, *for* $P_n^{m'}(x)$, *read* $P_{n'}^{m}(x)$, and *for* $m \neq m'$, *read* $n \neq n'$; p. 61, l. -7, *for* $(2\nu + 1)$, *read* $-(2\nu + 1)$; p. 66, l. -9, *after* $=$ *insert* $\pi^{\frac{1}{2}}$; p. 68, l. 6, *delete* $\sqrt{\pi}$; p. 71, l. 10, *for* $\frac{2}{\pi}$, *read* $\frac{\pi}{2}$, p. 73, l. 3, 5, *for* $(\mathfrak{Sin}\,\alpha)^{\mu/2}$, *read* $(\mathfrak{Sin}\,\alpha)^{\mu}$; l. 9, *for* $-\nu$, *read* ν; p. 83, *after* l. 4, *insert* $-\gamma \neq 0, 1, \cdots, n - 1$; *after* l. 11, *insert* α, x reell, $\gamma > 0$, $\alpha > \gamma + 1$; p. 84, l. 9, *for* $\frac{5}{2}$, *read* $\frac{5}{3}$; p. 87, l. 3, *for* $c \neq 1, 2, 3, \cdots$, *read* $c \neq 0, \pm 1, \pm 2, \cdots$; p. 88, l. -1, *for* Zahlen ± 1, *read* Zahlen $0, \pm 1$; p. 89, l. -5, -7, *for* $W_{\mu,K}$, *read* $W_{K,\mu}$; p. 91, *for* $(z + t)^{-\frac{1}{2}-K-\mu}$, *read* $(z + t)^{-\frac{1}{2}+K-\mu}$; p. 92, l. -2, *for* e^{-2t^2-2itz}, *read* e^{-2t^2+2itz}; p. 95, l. -7, *for* $\frac{1}{2}$, *read* $\frac{1}{2}i$; p. 100, l. -1, *for* reell ist, *read* reell und irrational ist; p. 103, l. 6, *for* $e^{-\pi K/4K'}$, *read* $e^{-\pi K'/4K}$; p. 108, l. 3, *for* x (three times), *read* t; p. 114, l. 5, *for* $\mathbf{K}/\sqrt{2}$, *read* $\mathbf{K}\sqrt{2}$; l. 6, *for* $\mathbf{K}/\sqrt{3}$, *read* $\mathbf{K}\sqrt{3}$; l. 7, *for* tang, *read* tang^2; p. 119, in heading *for* §2, *read* §1, and l. 9, *for* b^2, *read* x^2; p. 125,

l. -5, *for* π, *read* $\sqrt{\pi}$; p. 131, l. -2, *for* $a^{\nu+1}$, *read* a^{ν}; l. -1, Unterfunktion, *read* $\Gamma(\mu+\nu+1) \times (\mu^2+a^2)^{-(\nu+1)/2}P_\nu^{-\mu}[\mu/(p^2+a^2)^{\frac{1}{2}}]Re(\mu+\nu) > -1$; p. 132, l. 6, *for* p^ν, *read* $p^{\nu+1}$; l. -4, *for* $J^2_{\nu+\frac{1}{2}}$, *read* $J_{\nu+\frac{1}{2}}$; l. -3, *for* K, *read* K_0; p. 139, l. -3, *for* $y-x$, *read* $x-y$; p. 156, l. 6, *for* $1/\sqrt{r}$, *read* $(2n+1)/\sqrt{r}$; l. 7, *for* $1/\sqrt{r_0}$, *read* $(2n+1)/\sqrt{r_0}$; p. 157, l. -2, *for* $(\xi^2-\eta^2)=0$, *read* $(\xi^2-\eta^2)F=0$; p. 158, l. -2, *insert* F before $=$; p. 161, *for* unabhängigen Lösungen, *read* unabhängigen reellen Lösungen.

R. C. A.

504[L].—Samuel P. Morgan, Jr., *Tables of Bessel Functions of Imaginary Order and Imaginary Argument.* Pasadena, California Institute of Technology Book Store, 1947, vi, 61 leaves. 21.5 × 28 cm. Paper cover $2.75. Photographic reproduction, in an edition of 175 copies, of the original copy printed by IBM machines on one side of each leaf.

Various problems from different branches of mathematical physics give rise to the differential equation

$$v^2d^2w/dv^2 + vdw/dv - (v^2 - \nu^2)w = 0, \tag{1}$$

in which v and ν are real quantities. Equation (1) is a special case of Bessel's equation,

$$z^2d^2w/dz^2 + zdw/dz + (z^2 - \rho^2)w = 0, \tag{2}$$

in which $z = iv$, $\rho = i\nu$; and its solutions are therefore Bessel functions whose order and argument are both purely imaginary.

A fundamental real pair of solutions of (1) may be defined as follows:

$$F_\nu(v) = (\pi/\sinh \nu\pi)ReI_{i\nu}(v),$$
$$G_\nu(v) = -(\pi/\sinh \nu\pi)ImI_{i\nu}(v) \equiv K_{i\nu}(v) \equiv \tfrac{1}{2}\pi i e^{-\frac{1}{2}\nu\pi}H^{(1)}_{i\nu}(iv).$$

For brevity the functions $F_\nu(v)$ and $G_\nu(v)$ may be called "wedge functions" of the first and second kind respectively, since in potential theory they show a certain analogy to the solutions of Legendre's equation called "cone functions."

Representations of $F_\nu(v)$ and $G_\nu(v)$ in terms of series of modified Bessel functions of positive integral order are given by

$$F_\nu(v) = (\nu\pi/\sinh \nu\pi)^{\frac{1}{2}}[A(\nu,v)\cos\theta(\nu,v) + B(\nu,v)\sin\theta(\nu,v)],$$
$$G_\nu(v) = (\nu\pi/\sinh \nu\pi)^{\frac{1}{2}}[B(\nu,v)\cos\theta(\nu,v) - A(\nu,v)\sin\theta(\nu,v)],$$

where

$$\theta(\nu,v) = \nu\ln\tfrac{1}{2}v - \arg\Gamma(i\nu),$$

$$A(\nu,v) = \sum_{m=1}^{\infty} m(-1)^m(\tfrac{1}{2}v)^mI_m(v)/[m!(m^2+\nu^2)],$$

$$B(\nu,v) = \sum_{m=0}^{\infty} \nu(-1)^m(\tfrac{1}{2}v)^mI_m(v)/[m!(m^2+\nu^2)],$$

or

$$A(\nu,v) = -\sum_{n=1}^{\infty}\sum_{k=0}^{[\frac{1}{2}n-\frac{1}{2}]}\frac{(-1)^k(n)_{n-1-2k}\nu^{2k}v^{2n}}{4^nn!(1^2+\nu^2)\cdots(n^2+\nu^2)},$$

$$B(\nu,v) = \nu^{-1} + \nu^{-1}\sum_{n=1}^{\infty}\sum_{k=0}^{[\frac{1}{2}n]}\frac{(-1)^k(n)_{n-2k}\nu^{2k}v^{2n}}{4^nn!(1^2+\nu^2)\cdots(n^2+\nu^2)},$$

where $[s]$ represents the greatest integer contained in s and the symbol $(p)_q$, where p and q are any positive integers such that $q \leqslant p$, denotes the sum of all the different products which can be formed by multiplying together q of the p factors $1, 2, \cdots, p$, $(p)_0$ being equal to 1 by definition. A short table of values of $(p)_q$ was given by M. Bôcher, "On some applications of Bessel's functions with pure imaginary index," *Annals Math.*, v. 6, 1892, p. 144. [This table, $p = 1(1)8$, $q = 0(1)8$, was taken from O. X. Schlömilch, *Kompendium der* *höheren Analysis*, v. 2, fourth ed., Brunswick, 1895, p. 31.] Definite integral representations of these functions are

$$F_\nu(v) = (\sinh \nu\pi)^{-1} \int_0^\pi e^{v \cos\theta} \cosh \nu\theta d\theta - \int_0^\infty e^{-v \cosh t} \sin \nu t dt,$$

$$(3) \qquad G_\nu(v) = \int_0^\infty e^{-v \cosh t} \cos \nu t dt.$$

Since the wedge functions have an oscillatory singularity at $v = 0$, it is more convenient to tabulate the related quantities $F_\nu(e^x)$ and $G_\nu(e^x)$ as functions of x. These latter functions satisfy the differential equation

$$(4) \qquad d^2w/dx^2 + (\nu^2 - e^{2x})w = 0,$$

obtained from (1) by the transformation of variable $v = e^x$, and $x = \ln v$, which takes the triad of points $(0, 1, \infty)$ of the v-axis into the triad $(-\infty, 0, \infty)$ of the x-axis. The functions $F_\nu(e^x)$ and $G_\nu(e^x)$ have no singularities on the finite part of the x-axis, and they approach sinusoids in νx as $x \to -\infty$ $(v \to +0)$.

In the tables $\nu = .2(.2)10$, $x = -.49(.01)2.5$, to 5S mostly. $F_\nu(x)$ is tabulated over the complete ranges. In the region where $F_\nu(e^x)$ is oscillatory the error in the last figure given should not exceed 5 units. In the region where $F_\nu(e^x)$ is non-oscillatory the error in the tabulated values should not exceed 5 parts in 10 000.

The function $G_\nu(e^x)$ is tabulated over the following ranges in ν and x:

$.2 \leqslant \nu \leqslant 1, -.49 \leqslant x \leqslant .5$; $\quad 1.2 \leqslant \nu \leqslant 2, -.49 \leqslant x \leqslant 1$;
$2.2 \leqslant \nu \leqslant 4, -.49 \leqslant x \leqslant 1.5$; $\quad 4.2 \leqslant \nu \leqslant 7, \ .49 \leqslant x \leqslant 2$;
$7.2 \leqslant \nu \leqslant 10, \ .49 \leqslant x \leqslant 2.5$.

The error in the last figure of any tabulated value does not exceed 5 units.

As a matter of interest the value of $G_\nu(e^x)$ computed from the definite integral (3) and correct to the last printed figures, 5S, are given (leaf 61) for $x = 1(.5)2.5$, and for those values of ν not included in the main table.

Extracts from the text

Bessel functions of imaginary order and imaginary argument provide solutions of Laplace's equation useful in certain potential and heat flow problems.[1,2] If it is desired to find a potential function depending on the cylindrical coordinates ρ, ϕ, z, which vanishes on the cylinders $\rho = \rho_1$, $\rho = \rho_2$, and on the planes $z = z_1$, $z = z_2$, and takes arbitrary values on the axial planes $\phi = \phi_1$, $\phi = \phi_2$, one assumes an infinite series of terms of the form

$$[AF_\nu(k\rho) + BG_\nu(k\rho)] \begin{matrix}\sinh\\ \cosh\end{matrix} \nu\phi \begin{matrix}\sin\\ \cos\end{matrix} kz,$$

and determines the separation constants ν and k and the coefficients A and B of each term to satisfy the given boundary conditions.

Some hydrodynamical investigations of the stability of flow of superposed streams of fluid in which density and velocity both vary with height have been carried out by TAYLOR[3] and GOLDSTEIN[4] in a form which leads to Bessel functions of imaginary order.

The propagation of transverse seismic waves over the surface of an elastically inhomogeneous medium is of considerable interest in geophysics, and it has been shown[5,6] that the transmission of these so-called LOVE waves is expressible in terms of Bessel functions of (large) imaginary order and imaginary argument if the modulus of rigidity of the elastic medium increases as a quadratic function of depth.

The flow of electric current between coaxial cylindrical electrodes, taking account of both convection and diffusion, has been investigated by BORGNIS[7] using the functions of purely imaginary order; and subsequently EMDE[8] has discussed in some detail asymptotic representations of Bessel functions whose order is large and imaginary.

Finally Bessel functions of imaginary order and imaginary argument occur in solutions of the wave equation, upon proper choice of the separation constants. These solutions arise when one considers the propagation of acoustic or electromagnetic waves through bent pipes,[9] a problem wherein the functions of imaginary order and *real* argument also play an important role.

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[1] M. BÔCHER, *loc. cit.*
[2] J. DOUGALL, Edinb. Math. Soc., *Proc.*, v. 18, 1900, p. 33–83.
[3] G. I. TAYLOR, R. Soc. London, *Proc.*, v. 132A, 1931, p. 499–507.
[4] S. GOLDSTEIN, R. Soc. London, *Proc.*, v. 132A, 1931, p. 524–548.
[5] H. JEFFREYS, RAS, *Mo. Not.*, *Geophys. Suppl.*, v. 2, 1928–31, p. 101–111.
[6] S. SAKURABA, Geophys. Mag., Tokyo, v. 9, 1935, p. 211–214.
[7] F. BORGNIS, *Ann. d. Physik*, s. 5, v. 31, 1938, p. 745–754.
[8] F. EMDE, *Z. f. angew. Math. u. Mech.*, v. 19, 1939, p. 101–118.
[9] P. KRASNUSHKIN, *Jn. Phys.*, S. S. S. R., v. 10, 1946, p. 443.

505[L].—NBSCL, *Table of* $J_0(2\sqrt{u})$, $Y_0(2\sqrt{u})$, $u^{-\frac{1}{2}}J_1(2\sqrt{u})$, $u^{-\frac{1}{2}}Y_1(2\sqrt{u})$. Nov. 1947, ii, 11 leaves. 20.6 × 35.6 cm. Hectographed preliminary typescript on one side of each leaf, not available for general distribution.

These tables of $J_0(2u^{\frac{1}{2}})$, $Y_0(2u^{\frac{1}{2}})$, $u^{-\frac{1}{2}}J_1(2u^{\frac{1}{2}})$, $u^{-\frac{1}{2}}Y_1(2u^{\frac{1}{2}})$ are for u = [0(.05)3(.1)5(.2)-8(.5)50(2)160(5)410; 8–11D]. For most of the values tabulated last-place accuracy is not guaranteed. Second, or second and fourth, differences are given everywhere except for $u > 160$, where the interval in u is too large for interpolation, and near the origin, where $Y_0(2u^{\frac{1}{2}})$ and $u^{-\frac{1}{2}}Y_1(2u^{\frac{1}{2}})$ have singularities. In the regions where differences are not given it is best to use, for interpolation purposes, the tables of $J_0(x)$, $J_1(x)$, $Y_0(x)$, $Y_1(x)$, given in BAASMTC, *Math Tables*, v. 6 or, for $2u^{\frac{1}{2}} < 1$, the NBSCL tables of $Y_0(x)$ and $Y_1(x)$ at interval .0001 (in the press).

Tables of $J_0(2x^{\frac{1}{2}})$ and $x^{-\frac{1}{2}}J_1(2x^{\frac{1}{2}})$, [by J. R. AIREY], were published in BAAS, *Report 1924*, p. 287–295, for x = [0(.02)20; 6D].

Extracts from text

506[L].—U. S. NAVY, OFFICE OF RESEARCH AND INVENTIONS, *Scattering and Radiation from Circular Cylinders and Spheres; Tables of Amplitudes and Phase Angles*, prepared by A. N. LOWAN for NBSCL, and P. M. MORSE, H. FESHBACH, and M. LAX for the M.I.T. Underwater Sound Laboratory. New edition, July, 1946, vi, 124 p. 19.8 × 25.9 cm. See *MTAC*, v. 1, 1945, p. 390.

In its endeavor to disseminate scientific information the Office of Research and Inventions has published this Report, formerly restricted in availability, in a compact new edition (smaller paper and type page, 124 p. instead of 124 leaves), which ought to be of considerable value to physicists and engineers engaged in acoustical and electromagnetic wave research. We refer above to our review of the original edition. The reprint is so exact that the erroneous title-entries for tables 10 and 11 in the Table of Contents, which we earlier noted, still persist.

507[L, M, I].—HERBERT E. SALZER, "Table of coefficients for repeated integration with differences," *Phil. Mag.*, s. 7, v. 38, May 1947 [publ. Oct. 1947], p. 331–338.

The exact values of $G_n^{(2)} \equiv (1/n!)\int_0^1\int_0^t t(t-1)\cdots(t-n+1)(dt)^2$ and $H_n^{(2)} \equiv (1/n!) \times \int_0^1\int_0^t t(t+1)\cdots(t+n-1)(dt)^2$ are given for $n = 1(1)20$. For the same values of $n \leqslant 22 - k$ are also given $G_n^{(k)} \equiv \int_0^1\cdots\int_0^t\int_0^t t(t-1)\cdots(t-n+1)(dt)^k/n!$, for $k = 2$, 12D; 3, 10 or 11D; 4, 9–11D; 5, 8–11D; 6, 7–10D. Also $H_n^{(k)} \equiv (1/n!)\int_0^1\cdots\int_0^t\int_0^t t(t+1)\cdots(t+n-1)(dt)^k$, $k = 2$, 11D; 3, 9–10D; 4, 8–10D; 5, 7–10D; 6, 6–9D.

A table of $G_n^{(1)}$ and $H_n^{(1)}$, $n = 1(1)20$, was given by A. N. LOWAN & H. E. SALZER,[1] *Jn. Math. Phys.*, v. 22, 1943, p. 49–50. $G_n^{(2)}$ and $H_n^{(2)}$ for $n = 1(1)7$ were given by W. E. MILNE, "On the numerical integration of certain differential equations of the second order," *Amer. Math. Mo.*, v. 40, 1933, p. 324.

Extracts from text

[1] Compare *MTAC*, v. 1, p. 157.—EDITOR.

508[M].—A. A. DORODNIT͡SYN, "Asimptoticheskoe reshenie uravneniia͡ Van-der-Polia͡" [Asymptotic solution of Van der Pol's equation]. *Prikladnaia͡ Matem. i Mekhanika. Applied Math. and Mechanics*, v. 11, July 1947, p. 313–328, table p. 327. 16.8 × 25.8 cm.

The table gives 4D values of $Q_0(u)$ and $Q_1(u)$ for $u = -6(1)-4(.2)+4$.
$Q_1(u) = (1/A(u))\int_u^\infty A(u)(uQ_0^{-1} - u^2)du$, $A(u) = \exp(-\int_0^u du/Q_0^2)$,
$Q_0(u) = du/d\tau = u^2 - \tau = u^2 + \tau_1$, where
$u = \sqrt{\tau_1}\{J_{-2/3}(\frac{2}{3}\tau_1^{\frac{3}{2}}) - J_{2/3}(\frac{2}{3}\tau_1^{\frac{3}{2}})\}/\{J_{1/3}(\frac{2}{3}\tau_1^{\frac{3}{2}}) + J_{-1/3}(\frac{2}{3}\tau_1^{\frac{3}{2}})\}$
is a solution of the equation $d^2u/d\tau^2 - 2udu/d\tau + 1 = 0$.

509[M].—G. PLACZEK, "The angular distribution of neutrons emerging from a plane surface," *Phys. Rev.*, s. 2, v. 72, Oct. 1, 1947, p. 556–558. 20 × 26.7 cm.

Of $\phi(u) = \frac{1}{2}(1 + u)^{-\frac{1}{2}}e^w$, $w = \pi^{-1}\int_0^{\frac{1}{2}\pi} x \tan^{-1}(u \tan x)dx/(1 - x \cot x)$, there are two tables, by NBSCL: **T.I**, $u = [0(.1)1; 7D]$, obtained by numerical integration; and **T.II**, obtained from these values by interpolation $u = [0(.01)1; 5D]$, Δ, and Δ^2, 0 to .19. u is the cosine of the angle between the direction of the motion of the neutron and the outward normal.

510[M].—Miss P. M. SKINNER (Mrs. TRUSCOTT), "Numerical tables," in A. HAMMAD, "The primary and secondary scattering of sunlight in a plane-stratified atmosphere of uniform composition.—Part III. Numerical tables and discussion of secondary scattered light," *Phil. Mag.*, s. 7, v. 38, July 1947 (publ. Nov. 1947), p. 515–529. 17 × 25.1 cm.

T.I, p. 518–519, $Ej_n(x)$, for $n = 1(1)5$, $x = [0(.01)1; 6D]$; $Ej_n(x) = \int_1^\infty e^{-tx}t^{-n}dt$; $Ej_1(x) = -Ei(-x)$. The values for $n = 2(1)5$ were obtained from the values for $n = 1$ by applications of the recurrence formula $(n - 1)Ej_n(x) = e^{-x} - xEj_{n-1}(x)$, $n > 1$.

T.II, p. 520–524, $L_n(c, z, \overline{m})$, for $c = .1, .2(.2)1$, $n = 1, 3, 5$,
$\overline{m} = 1 - m = 0(.2)1$, $\sec z = [1(1)4, 1.5, 6; 5D]$;
$L_n(c, z, m) = c^{-1}e^{Z-1}[Ek_n(cm, -\sec z) + Ek_n(c\,\overline{1 - m}, \sec z)]$, where
$Ek_n(\tau, b) = \int_0^\tau e^{bt}Ej_n(t)dt$, $Z = c \sec z$.

T.III, p. 527, $e^\Theta Ld_n(c, z, \overline{m} = 1, \theta)$, $n = 1(2)5$, $\sec z = 1, 2, 4$,
$\sec\theta = [1, 2(2)6, \infty; 5D]$, $c = .2(.2)1$, $\Theta = c \sec\theta$;
$Ld_n(c, z, \overline{m}, \theta) = e^{-\Theta\overline{m}}\int_0^{\overline{m}} e^{-\Theta m'}L_n(c, z, m')d\overline{m}'$.

Extracts from text

EDITORIAL NOTE: A table of $Ej_1(x)$ was given by J. W. L. GLAISHER, R. Soc. London, *Trans.*, v. 160, p. 371, 380–381, 385–386, $x = [0(.01)1; 18D]$, Δ^3, $[1(.1)5; 11D]$, Δ^3, $[6(1)15; 11D]$, $[2; 43D]$, $[20; 12D]$. Tables of $Ej_n(x)$ were given by E. GOLD, R. Soc. London, *Proc.*, v. 82A, 1909, p. 62, for $n = 1(1)3$, $x = [.01(.01).1(.05)1(.1)2(.2)3(.5)5,$-$6; 5D]$. Hammad's description of this range is very incorrect and Glaisher's table, p. 380–381, would never be found from the reference he gives for it. See also FMR, *Index*, p. 207. Miss Skinner's values for $Ej_1(x)$ agree with rounded-off values of Glaisher, except that Glaisher's $Ej_1(.46)$ would produce 0.611387 rather than Miss Skinner's 0.611386. The 9-place table of NBSCL shows Glaisher to be the more accurate.

511[M].—O. A. T͡SUKHANOVA & G. D. SALAMANDRA, "Rasprostranenie tepla ot sfericheskogo istochnika, okhlazhdaemogo v sypuchei̮ srede," [Diffusion of heat from a spherical source, cooled in a granular medium]. Akad. N., SSSR, Leningrad, *Izvestii͡a, Otd. Tekhn. N.*, no. 8, Aug. 1947, p. 977–986, tables, p. 983. 16.5 × 25.8 cm.

There are six tables of $J = \int_c^\infty e^y d\alpha$, where $y = (\beta/\alpha)^2 - \alpha^2$, for (i). $\beta^2 = .001$, c = [.01, .02, .06, .1(.1)1.5; 6D]; (ii). $\beta^2 = .01093$, c = [.056, .066(.02).106(.04).386, .466, .586, .746, .946, 1.186, 1.427; mostly 4D]; (iii). $\beta^2 = .04$, c = [.1(.06).4(.1)2.5; 6D]; (iv). $\beta^2 = .1$, c = [.18(.04).5(.1).8; 6D], [1(.1)2.8; 7D]; (v). $\beta^2 = .5$, c = [.42(.06).6(.1)1.5; 6D], [1.6(.1)2.5; 7D]; (vi). $\beta^2 = 2.4806$, c = [.87(.1)1.57; 4–5D].

512[M, V].—S. A. KHRISTIANOVICH, "Priblizhennoe integrirovanie uravnenii̮ sverkhzvukovogo techenii͡a gaza" [Approximate integration of equations of a supersonic gas flow], *Prikladnai͡a Matem. i Mekhanika. Applied Math. and Mechanics*, v. 11, June 1947, p. 215–222; tables p. 217. 16.8 × 25.8 cm.

There are tables (mostly 4S) of $\sqrt{x} = \sqrt{\{(\lambda^2 - 1)/[1 - \lambda^2(\kappa - 1)/(\kappa + 1)]^\nu\}}$, $\nu = (\kappa + 1)/(\kappa - 1)$, and $\sigma' = \int_1^\lambda \{(t^2 - 1)/[1 - t^2(\kappa - 1)/(\kappa + 1)]\}^{\frac{1}{2}} dt/t$, for λ = 1(.05)-1.5(.1)2.4; κ = 1.405.

513[Q].—DEUTSCHE SEEWARTE, *Ortsbestimmung durch astronomische Beobachtung zweier gegebener Fixsterne mit Hilfe von Höhengleichendiagrammen*, prepared for and issued by the Oberkommando der Kriegsmarine, 1940–1941. Loose leaf, 8 v., 171 + 161 + 165 + 189 + 150 + 146 + 170 + 147 p. 21 × 30 cm.

These volumes were issued at inervals from January 1940 to September 1941; reference is made to volumes for southern latitudes but none is known to have been issued.

The principal use of star altitude curves is to determine position, more particularly latitude and local sidereal time, from the known altitudes of two (or more) stars. Accordingly curves of constant altitude, for two or more stars, are superposed upon a rectangular framework of latitude and local sidereal time, from which both can be read corresponding to any point defined by the altitudes of two or more stars. The classic publication in this field is Weems' *Star Altitude Curves* (see *MTAC*, v. 2, p. 133–134), and comparison is inevitable.

In the volumes under review, the latitude scale (arranged vertically) is linear, 2 cm., to each degree; the horizontal sidereal time scale is 0.5 cm. to a minute of time, half-an-hour to a page. The scale varies slightly from volume to volume and is rather larger in the later (lower latitude) volumes. A rectangular grid of pecked lines (in the early volumes, continuous) is drawn for every 10′ of latitude and $0^m.5$ of sidereal time; the lines for every degree and multiple of 5^m are emphasized slightly. Upon this background are overprinted two sets of star altitude curves, given for every 10′, with the integral and half-integral degree curves emphasized. The border scales are adequately indicated and divided, but the altitude curves are only marked (once) for each degree and this seems insufficient; the pages are well supplied with catch headings. The scale used enables positions to be read to within about 1′ or 2′ in latitude and about $0^m.1$ or 1′.5 in longitude.

In any one volume, any number from six to eighteen different star pairs may be used; an index table (or rather series of diagrams) in the front of each volume indicates the coverage and the pairs that are available at any time. At least two, but usually more, star pairs are always available. The volume is tab-indexed according to star-pair, which are arranged in (overlapping) order of coverage. In some of the later volumes the altitude of *Polaris* is indicated by scales alongside the latitude scales on each side of the page.

The various auxiliary tables and diagrams include a star map, corrections for annual variation in altitude due to precession (the positions of the stars used are those of 1941.0), corrections for refraction at heights above the Earth's surface—the curves as drawn include mean refraction at sea level—tables of sidereal time at 0^h U.T. for various years, various tables for transferring position lines, and finally diagrams for determining approximate azimuths of the stars used.

In Weems' diagrams the latitude scale is not constant, but varies as a Mercator chart. This means that non-simultaneous position lines can be transferred in the normal manner. In these diagrams a constant latitude scale is adopted, and the rules for transferring position lines are necessarily more complicated. Tables 8, 9 and 10 provide the means whereby this is accomplished. For a two star fix, a point on the diagram is obtained as if the two sights were simultaneous. This point is now transferred horizontally along the time scale by an amount equal to the time difference between the sights. From this position a line is drawn whose azimuth depends on the latitude and the course of the aircraft (Table 9). A point on this line is now chosen whose distance from the transferred point depends on time difference and speed of aircraft (Tables 8 and 10). The final fix is obtained by moving this point parallel to the curves corresponding to the first star sighted until the altitude of the second star is consonant with the observed value.

Five pages of explanation and examples are given covering all problems likely to arise in astronomical air navigation.

The early volumes are comparatively crude compared to the later ones and it is clear that many technical advances were made in the course of the production. The later volumes are on a slightly larger scale, printed on better paper and with thinner and clearer lines; although both background and superposed grids are printed in black, there is surprisingly little confusion and there is no doubt that the background grid aids reading of the scales. Weems has no grid and his star curves are beautifully printed in contrasting colours; the *Polaris* curves are liable to mislead in this respect since at first glance they appear to be lines of constant latitude. A combination of the two would seem to provide the optimum method of presentation.

There is a mystery about the production of these volumes which underlines the comparison with Weems' star curves; why should the German OKM produce eight (at least) large unwieldy volumes, clearly designed primarily for air use? There is no indication that they were intended for or used by the Luftwaffe, which was amply provided for by other tables. The difference in bulk, and in elegance, between these volumes and those of Weems is very marked; some of this disparity is accounted for by the fact that, on the average, three star pairs are catered for throughout as against Weems' one pair and *Polaris* (for northern latitudes), and the remainder by the thick paper and the awkward loose leaf binding.

This is a case where graphical or diagrammatical presentation has a distinct advantage over the tabular methods. Direct tabulation of latitude and local sidereal time with the two altitudes as argument is unsatisfactory; it covers two awkward-shaped areas on the Earth's surface and involves double-entry interpolation which is only linear if a small interval of tabulation is used. Tabulation of the two altitudes with arguments latitude and local sidereal time would involve double inverse interpolation, which, even if linear, cannot be attractive.

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514[Q].—Japan, Hydrographic Department, *Altitude and Azimuth Almanac for August–December 1947* (Title also in Japanese). Tokyo, May, 1947, vi, 93 p. + 1 plate. 18.4 × 26 cm.

This almanac is unique among published ephemerides in that it tabulates directly the altitude and azimuth as would be observed from certain specified positions on the surface of the Earth. It appears to be the successor to a war-time publication "Publication No. 9582, Altitude-Azimuth Tables for September 1945 (Sun, Moon, Planets and Fixed Stars) computed for Kisarazu, Naha, Iwo Jima and Kanoya and for Japanese Central Standard Time";[1] however, whereas the earlier publication was designed for air navigation and catered for all celestial bodies, the present almanac is designed for surface navigation and is restricted to the Sun.

Briefly, the altitude, to the nearest minute, and azimuth, to the nearest degree, of the Sun's lower limb are tabulated for every 10 minutes (occasionally 20 minutes) of Japanese Central Standard Time (Time Zone $-$ 9^h) for longitude E. 150° and four latitudes, N. 25°, N. 30°, N. 35° and N.40°; the altitude is corrected by application of semi-diameter, parallax, refraction and dip from a height of 5 meters, so as to be directly comparable with the observed altitude, corrected only for instrumental errors. Tabulations are given whenever the Sun is above the horizon, but the interval of tabulation is increased to 20^m earlier than 6^h and later than 17^h in order to get all the tabulation for one day in one column, containing a maximum of 72 or 73 lines arranged in blocks of 6 (occasionally the last block for the first days in each section contains 7 lines). There are eight such columns on each page, containing the tabulations for eight days for one latitude; each latitude constitutes a separate section.

In addition, there is a one-page table for each latitude giving the altitude of *Polaris* in longitude E. 150°, for every 20^m of standard mean time during darkness and for every sixth day; the altitude is corrected for refraction and dip. There is also a short table of the additional dip correction to be applied if the observation is made from a height differing from 5 meters, and a one-page interpolation table giving ten sub-multiples of the 10^m differences in the range 0°(2′)5°. The range of the latter table is surprising as the maximum difference for a 10^m interval is 2°30′.

The preface, explanation, page headings and in fact everything but the actual figures are in Japanese. The instructions and examples are not difficult to follow, though in one case the reviewer has not been able to reproduce the figures given. The general method of use advocated is to adopt as an assumed position the nearest point with both latitude and longitude multiples of 5°. Interpolation for longitudes other than E. 150° is done by applying an appropriate correction (multiple of 20^m) to the time; some examples seem to advocate correcting the observed altitude graphically for the change in time necessary to allow direct comparison with a tabular entry. In all cases long intercepts may occur.

The Almanac is not very well printed and the figures are rather small for easy reading; the chief fault is, however, a certain unevenness in alignment. Casual examination has failed to find any large errors, though one or two had already been corrected by pen and the end-figure does not always appear too reliable.

The advantages of being able to obtain an intercept by a straight comparison of an observed and a tabulated altitude are so great that it is well worth while considering carefully this form of almanac. As presented here it is intended to cover the relatively small area round Japan, but any universal almanac must cater for the whole of the Earth. Extension to different longitudes is not difficult, since this could be accomplished by using L.M.T. as argument and interpolating for longitude between consecutive days, as is done now for times of rising and setting; this correction cannot strictly be ignored even for the Sun with small longitude differences of the order of 15°, but for the Moon it would be very large and not linear.

Interpolation for latitude, to avoid long intercepts and if necessary to use the D.R. position, can be done fairly quickly and accurately from a knowledge of the azimuth, which is tabulated; but tabulations would have to be given for every 5° of latitude and this would seem to make the method impracticable for a universal almanac. Furthermore an interval of 10^m in time gives rise to quite large second differences in the altitude, so that linear interpolation (as is pointed out in one of the examples) may lead to considerable error; the answer would be a still smaller interval—say of 5^m—near the meridian.

Taking as a basis of comparison an air almanac tabulating G.H.A. and Dec., at intervals of 10^m, for the Sun Moon and planets (the method is not suitable for the stars) it is interesting to find the amount of tabulation required for this method for one latitude. Making reasonable allowances for the known periods of visibility to which tabulations can be restricted, to the small interval required near meridian passage and to the necessity for tabulating the Moon at intermediate longitudes, it works out about one-half the equivalent air almanac. This assumes a tabular accuracy of 1′; an increase to 0′.1 would make interpolation too difficult at the intervals of tabulation suggested. It is thus seen that an almanac of this nature would be an expensive luxury even for a comparatively narrow band of latitude.[2]

D. H. SADLER

[1] See *MTAC*, v. 2, p. 83, RMT **297**, by Professor C. H. SMILEY.
[2] Reprinted, by permission, from Institute of Navigation [British], *Jn.*, v. 1, no. 2, 1948.

515[Q].—ZDENĚK KOPAL, *Theory and Tables of the Associated Alpha-Functions.* (Harvard College Observatory, *Circular* 450), Cambridge, Mass., 1947, 25 p. 24.6 × 29 cm.

The study of variable stars has always been of great interest to the astronomer. A variable star is one whose light intensity or magnitude varies with time. These variations may be self-repeating in a periodic manner as in the case of either the Cepheid-type variables or eclipsing variables, or the light variations may be semi-regular or irregular as in the case of the novae or supernovae. Conclusions of the greatest importance have been drawn from the study of variable stars despite little or no understanding of the details of the mechanism causing the light variation. For example, it was empirically discovered that there was a correlation—the so-called period-luminosity law—between the period of a Cepheid variable and its luminosity, or absolute magnitude. This "law" was used first, to determine the size of our own galaxy or Milky Way system and second, to determine the distances to some of the nearer of the external galaxies; and yet no satisfactory quantitative explanation of the details of the light-curve of a Cepheid has so far been made.

The theory and tables under review are primarily concerned with the theoretical interpretation of the light-curves of *eclipsing* variables. These eclipsing variables, or eclipsing binaries as they are sometimes called, are the only variable stars whose light-curves are subject to precise analysis and interpretation and, strangely enough, eclipsing variables are stars that are not *inherently* variable. Even in our largest telescopes an eclipsing binary looks like a single star; but each such star is actually a pair of stars revolving about a common center of mass, and through the accident of geometry mutually eclipsing one another every half revolution. Stars showing a periodic variation of their radial (line of sight) velocity are known as spectroscopic binaries and every eclipsing binary is also a spectroscopic binary, the light and velocity varying in the same period and in a manner consistent with the hypothesis of a binary system. The interpretation of the combination of photometric and spectroscopic observations of eclipsing binaries has yielded astronomical data of the greatest importance in connection with the determination of the diameters, volumes, masses, densities, temperatures, luminosities, rotations and internal constitutions of the stars.

The original methods of solution of a light-curve of an eclipsing variable were based on the simplest possible assumptions, namely, spherical stars presenting uniformly bright discs and moving in circular orbits. It was discovered that tables easily could be constructed giving α, the fractional loss of light (proportional to the eclipsed area in this simple case) as a function of k, the ratio of the radii of the two stars ($k \leqq 1$), and p, the geometric depth of eclipse, p being equal to $+1$ at first or external contact and equal to -1 at second or internal contact. From these tables it was possible to construct a theoretical light-curve corresponding to any combination of assumed values of r_1 (radius of the star undergoing eclipse), r_2 (radius of the eclipsing star), i, the inclination of the orbit plane, and P the period, r_1 and r_2 being taken in terms of the radius of the orbit as unity. The astronomer's problem was to pick out from this infinity of theoretical light-curves that particular one which best satisfied his observations. Certain auxiliary functions of p and k were tabulated which greatly assisted this process of selection and made the solution of a light-curve a task of but a few hours.

Observations of the sun revealed that the surface brightness at a point near the edge of the disk, or limb, was considerably less than that at the center of the disk, and that the variation of brightness over the disk followed quite accurately the so-called cosine law of limb darkening, the amount of darkening at the limb being a function of the wave-length (color) of observation.

Tables for eclipsing variables, accurate to 3S, were computed on the hypothesis of complete darkening at the limb by RUSSELL & SHAPLEY[1] in 1912 by graphical integration, the analytical solution being one of great complexity involving complete and incomplete elliptic integrals of the first and second kind. It was discovered that either the uniform or darkened hypothesis would satisfy equally well almost all of the observed light-curves then available. This was essentially due to the poor photometric quality of even the best photographic or visual photometric observations, the percentage errors of single observations being about five or ten percent, whereas, in positional work for example, the errors may be perhaps as low as one part in five million.

A number of complications arise whenever the two components in an eclipsing system are very close together. The stars are distorted from spherical figures into approximately tri-axial ellipsoids by the strong tidal and rotational forces involved. The surface brightness of the disk at any point is not only a function of the distance from the center (or limb) but is also a function of local gravity at the point in question, the disk being darker at the equator than at the poles. Furthermore, the hemispheres that face each other will be brighter (by mutual absorption and re-radiation of the other component's radiant energy) than the opposite hemispheres. This last complication, the so-called "reflection" effect, has as yet been analyzed only under the most simplified assumptions and is a problem for the future. Dr. Kopal's paper deals with the effects on the light-curve—and also the radial velocity curve—"of the rotational and tidal harmonic distortions of components in close eclipsing systems—between minima as well as within eclipses—taking account of the appropriate distribution of brightness over the distorted surfaces (due to limb- and gravity-darkening)." This work has been stimulated by recent improvements and developments in the stability and sensitivity of the photoelectric cell and the multiplier phototube that make it possible to observe a large percentage of the two thousand known eclipsing variables with an accuracy of an entirely new order of magnitude.

If we let r_1 and r_2 denote the eclipsed and eclipsing radii of two spherical components of a binary system, and δ the instantaneous apparent separation of their centers projected on the celestial sphere, then the fractional loss of light can be conveniently expressed in terms of a family of associated alpha-functions α_n^m of various integral orders m and indices n which are non-dimensional quantities defined as

$$\pi r_1^{m+n+2}\alpha_n^m = \left\{ \int_s^{r_1} \int_{-[r_1^2-x^2]^{\frac{1}{2}}}^{[r_1^2-x^2]^{\frac{1}{2}}} + \int_{\delta-r_2}^{s} \int_{-[r_2^2-(\delta-x)^2]^{\frac{1}{2}}}^{[r_2^2-(\delta-x)^2]^{\frac{1}{2}}} \right\} x^m z^n dxdy$$

if the eclipse is partial, and

$$\pi r_1^{m+n+2}\alpha_n^m = \int_{\delta-r_2}^{\delta+r_2} \int_{-[r_2^2-(\delta-x)^2]^{\frac{1}{2}}}^{[r_2^2-(\delta-x)^2]^{\frac{1}{2}}} x^m z^n dxdy$$

if it is annular, where

$$z = (r_1^2 - x^2 - y^2)^{\frac{1}{2}} \quad \text{and} \quad s = (r_1^2 - r_2^2 + \delta^2)/2\delta.$$

An inspection of these equations reveals that all associated alpha-functions may be made to depend upon two non-dimensional variables (such as k and p) and that 91 bivariate tables would be necessary for the second, third and fourth harmonic distortions of the photometric and radial velocity curves. The author introduces a new integral which he calls an I-integral defined by

$$\pi r_2{}^q I^m_{\beta,\gamma} = 2^{\frac{1}{2}\gamma} \int_{\delta - r_2}^{c} [r_2{}^2 - (\delta - x)^2]^{\frac{1}{2}\beta}(s - x)^{\frac{1}{2}\gamma}(\delta - x)^m dx$$

where $c = s$ or $\delta + r_2$ for partial or annular eclipse respectively, and $q = 1 + \beta + \frac{1}{2}\gamma + m$. The integral $I^m_{\beta,\gamma}$ is a non-dimensional quantity depending on a *single* variable and all associated alpha-functions of orders $m = 0(1)3$ can be expressed as simple combinations of the I-integrals factored by powers of r_2/r_1 and δ/r_2 or s/r_2.

Dr. Kopal gives the relationships and recurrence formulae necessary to transfer from the alpha-functions to the I-integrals. The properties of I-integrals are then discussed and their evaluation in terms of the complete elliptic integrals of the first and second kind and the variable $\mu = (\delta - s)/r_2$ is next given. It is shown that the use of μ as an argument in the tables would be unwise and a new variable α is introduced, defined by

$$\mu = 1 - 2 \sin^2 \tfrac{1}{2}\alpha \text{ for partial eclipse, and } \mu = 1 - 2 \csc^2 \tfrac{1}{2}\alpha \text{ for annular eclipse.}$$

The introduction of α as argument and the use of modified second differences defined by

$$M'' = \Delta'' - 0.184\Delta^{\text{iv}} + \cdots$$

permits the tabulation of a tremendous amount of information in a relatively small space. The author then gives a numerical example concerning a fictitious eclipsing system. Diagrams are next given, which illustrate the behavior of various I-integrals during partial eclipse. Finally the following I-integrals are tabulated to 5S (in general): $I^m_{-1,0}$ and $I^m_{-1,1}$ for $m = 0(1)5$; $I^0_{-1,\gamma}$ for $\gamma = 2(1)7$; $I^0_{1,\gamma}$ for $\gamma = -1(1) + 6$; and $I^0_{3,\gamma}$ for $\gamma = -1(1) + 3$. These integrals with their modified second differences are tabulated for both partial and annular eclipses (except for $I^m_{-1,0}$ which is tabulated for partial eclipse only) with $\alpha = 0(5°)180°$. Many of these integrals become infinite for $\alpha = 0°$ and interpolation for small α becomes difficult or impractical. The reviewer feels that the tabulation of the reciprocals of the I-integrals, or perhaps the use of a factor such as δ, might be a considerable improvement for these portions of the tables.

This is a very timely and important piece of work. Although the scope and applicability of these new functions are still far from being fully explored, it appears certain that little real progress in the interpretation of a wide range of phenomena observed in close binary systems can be expected until the general properties of the associated alpha-functions are well understood. The author indicates that the functions will be used to predict the light or velocity changes of configurations in free non-radial oscillations, as well as to interpret the forms of line profiles of rotating distorted stars in full light or during partial or annular eclipse. The problems encountered in such close systems are formidable both from the point of view of the astrophysicist and of the mathematician. Dr. Kopal is to be congratulated upon making such a long step forward in the solution of such problems.

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[1] H. N. Russell & H. Shapley, "On darkening at the limb in eclipsing variables," *Astrophysical Jn.*, v. 36, 1912, p. 239–254, 385–408.

516[Q].—O. A. de Azeredo Rodrigues, *Tábuas para Retas de Altura* (Escola Naval no. 23). Rio de Janeiro, Imprensa Naval, 1943, 54 p. 16 × 23.6 cm. Approved for printing, 3 May 1940, by the Minister of Marine.

This volume is quite similar to Fontoura da Costa and Penteado's *Tábuas de Altura e Azímute*[1] (later referred to as F & P) and to Ageton's *Dead Reckoning Altitude and*

Azimuth Table (H.O. 211, RMT **104**), both in contents and in arrangement. The astronomical triangle is divided into two right triangles by a perpendicular from the celestial body upon the meridian. As in F & P, the interval of the argument is one minute of arc, and it is the values of $C = 10^5 \log \csc x$ which are printed in heavy type rather than those of $S = 10^5 \log \sec x$ as in Ageton. This will tend to confuse persons already well acquainted with the latter volume. Values of S and C less than 665 are given to one decimal; in Ageton, only values less than 240 are given to one decimal.

The formulae are essentially the same as in the two earlier volumes; only a slight change in notation is involved. The author calls the length of the perpendicular from the celestial body upon the meridian G (Ageton's R, F & P's $90° - \psi$) rather than $90° - G$; the declination of the foot of the perpendicular, $90° - V$ (Ageton's K, F & P's $90° - \gamma$). As in F & P, the author neglects to warn the user of the inaccuracies which may arise when the foot of the perpendicular lies near the pole.

The arrangement of the tables is almost identical with that of F & P. The only differences noted were that the C column is printed to the left of the S column; the arguments given at the foot of the page are only those from 90° to 180° (the ones from 270° to 360° given in parentheses in F & P are omitted); there is no convenient thumb-tab index.

The usual tables for refraction, dip of the horizon, and parallax are presented early in the volume. Also included are a table of the six natural trigonometric functions [0(1°)90°; 4D], a log sine table [0(10′)90°; 4D], a log tangent table [0(10′)89°50′; 4D] and 4-place tables of logarithms of numbers and anti-logarithms.

One uncommon useful feature of this volume is a nomogram to be used in correcting a circum-meridian altitude. The last sixteen pages of the volume are devoted to explanations of the use of the tables and nomogram.

A comparison of 1200 values of C and S in this book with the corresponding ones in F & P revealed no differences.

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[1] See, in this issue, N 92.

517[U].—THOMAS F. HICKERSON. *Latitude, Longitude and Azimuth by the Sun or Stars*. Chapel Hill, N. C., published by the author, 1947, ii, 101 p. 13.5 × 23.4 cm. $2.00.

This volume is a second edition of *Navigational Handbook with Tables*, published by the author at Chapel Hill in 1944. It differs from the earlier volume in that it contains 50 pages of explanation instead of 29, and includes a table (I) of meridional parts to 0.1 with argument latitude, 0(10′)79°, 79°(2′)79°58′ condensed from table 5 of BOWDITCH, 1938 edition, which is identical with table 3 of Bowditch, editions of 1903 up to that of 1938. There are also tables for Interval to Noon with "Ship moving Westward" (Table III) and "Ship moving Eastward" (Table IV).

The principal table (II) is similar to that given in Ageton's Dead Reckoning Altitude and Azimuth Table, with values of $A = 10^5 \log \csc x$ and $B = 10^5 \log |\sec x|$ generally given to the nearest integer, but with one decimal given for all tabular values less than 106. There are two major exceptions to this similarity: the interval of the argument is 0′.2 instead of 0′.5 and the table does not follow Ageton's pattern in which the letters at the top and bottom of a column are the same and each tabular value appears twice. This table folds back on itself at 45° instead of at 90°. Other tables included are: Conversion of Arc into Time (V), Time into Arc (VI) and Altitude Corrections (A–C). The latter table appears to have been taken directly from the *American Nautical Almanac*.

In the explanation of the use of the tables, the material is presented as a number of "cases"; Case 1, Great Circle Course and Distance; Case 2, Points along a Great Circle Track; Case 3, Line of Position with Appendix giving the "Fix," etc.

The work forms given are essentially those in Ageton, plus about eight new ones. No warning is given of the difficulties which arise when the foot of the perpendicular lies near a pole; the smaller interval of the argument makes this less serious than in Ageton's tables.

The author's Case 7 (Latitude when the Sun or Star is near the Meridian) and Case 9 (Longitude when the Sun or Star is near the Prime Vertical) will probably appeal to many practical navigators. It is unfortunate that the author did not warn the reader of the troubles which arise in these cases if the altitude is near 90°.

The formula used in Case 9 is

$$2B(t/2) = B(s) + B(s - z) - B(d) - B(L)$$

where t and d are the meridian angle and declination respectively of the celestial body, L is the observer's latitude, z is $90° - H$ or the body's zenith distance, $s = (z + d + L)/2$. Table S, based upon this formula, gives t and z for Betelgeuse, each to the nearest minute of arc for L, 0(1°)60° and H, 20°(5°)35°. This table is quite similar to, and possesses many of the advantages of, *Tafeln zur astronomischen Ortsbestimmung*, by ARNOLD KOHLSCHÜTTER, Berlin, 1913. A set of similar star tables prepared today for the northern temperate zone would make a valuable addition to navigational literature. They would possess the great advantage of a direct approach, allowing the navigator to enter a table with the observed altitude as an argument, and with an assumed latitude or longitude, find the corresponding longitude or latitude.

A brief examination of the principal table (II) indicates that the tabular values are much more reliable than those in Ageton.

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518[V].—G. MORETTI, "Scie piane turbolente," *L'Aerotecnica*, v. 27, 15 June 1947, p. 210–221. 20.5 × 29 cm.

The tables, p. 219–220, are (1) of $M(x|\alpha, \xi)$, $N(x|\alpha, \xi)$, $P(x|\alpha, \xi)$, $\alpha = 1.069$, $\xi = .765$, for $x = [0(.1)2, 2.2, 2.5, 3; 3\text{–}4D]$, where

$$M = [\alpha^2 e^{-x^2}/C(\xi)]\begin{vmatrix} A(x)C(x) \\ A(\xi)C(\xi) \end{vmatrix}, \qquad N = [\alpha^2 e^{-x^2}/C(\xi)]\begin{vmatrix} B(x)C(x) \\ B(\xi)C(\xi) \end{vmatrix},$$

$$P = [2\alpha e^{-x^2}/C(\xi)]\begin{vmatrix} x^2-\frac{1}{2} & C(x) \\ \xi^2-\frac{1}{2} & C(\xi) \end{vmatrix}, \qquad A(x) = \tfrac{1}{2}e^{-x^2} + x\int_0^\infty e^{-t^2}dt,$$

$$B(x) = \tfrac{1}{2}e^{-x^2} - x\int_0^\infty e^{-t^2}dt, \qquad C(x) = \tfrac{1}{2}e^{x^2} - x\int_0^\infty e^{t^2}dt;$$

(2) of $y(x|\alpha, \xi, \mu) = M(x|\alpha, \xi) + \mu N(x|\alpha, \xi)$ and of $y(x|\alpha, \xi, \lambda) = M(x|\alpha, \xi) + \lambda P(x|\alpha, \xi)$, where $\mu = 1.5369, 1, 0, -2, -5, -10$, $\lambda = -1.2155, -1(1)2, 5, 10$ and in both cases $\alpha = 1.069$, $\xi = .765$, $x = [0(.1)2, 2.2, 2.5, 3; 3D]$.

Extracts from text

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **477** (Cazzola), **484** (NBSCL), **486** (Anan'ev), **495** (Chebyshev, Whittaker & Robinson), **503** (Magnus & Oberhettinger), **506** (U. S. Navy), **510** (Skinner); N **87** (German Tables), **92** (Fontoura & Penteado).

124. J. R. AIREY, "Tables of the Bessel functions $J_n(x)$," $n = 0(1)13$, $x = [6.5(.5)16; 10D]$, BAAS, *Report*, 1915, p. 30–32.