where M and e are given and E is required. A small punched-card table gave a rough approximation to E, E_0. This value E_0 was used to enter a standard punched-card table of sin E with tabular interval of 0°.01. From this table were taken sin E_0 and sin $(E_0 + 0°.01)$ by passing the cards through the reproducer in both directions. The relay calculator, with a single run of the cards, evaluated the following formulae:

$$\begin{aligned} \delta_1 &= M - E_0 + 57°.2958e \sin E_0 \\ \delta_2 &= M - E_0 - 0°.01 + 57°.2958e \sin (E_0 + 0°.01) \\ E &= E_0 - 0°.01\delta_1/(\delta_2 - \delta_1). \end{aligned}$$

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W. J. E.

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Inversion of a Matrix of Order 38

The general solution of a system of 38 simultaneous linear equations has recently been obtained by the writer, utilizing the Aiken Relay Calculator, constructed for the Naval Proving Ground, Dahlgren, Virginia, by the staff of the Computation Laboratory of Harvard University. The elements of the reciprocal matrix were computed using an adaptation of the GAUSS method of elimination. This method is particularly suited to machine solution in that it permits the use of a few short computing routines, the number of repetitions of which are governed by simple functions of the system order. The fact that a single system of control tapes may be employed to invert a matrix of any order was considered of paramount importance in the choice of this method.

The remarkable feature of the present computation was the degree of accuracy obtained in the elements of the reciprocal matrix. When tested on known data, the inverse matrix yielded results correct to nine significant digits, notwithstanding the fact that the calculator employed is limited to ten significant digits.

The problem solved originated in the field of mathematical economics. It was posed by Professor WASSILY W. LEONTIEF[1] of the Department of Economics in Harvard University. The original system comprises the output-input relations among the industries of the United States, divided into 38 groups, as compiled by the United States Bureau of Labor Statistics for the year 1939. Thirty-eight equations arise, of the form,

$$(1) \qquad A_i - \sum_{j=1}^{38} B_{ij} = C_i, \qquad j \neq i,\ i = 1(1)38,$$

where

A_i = the output of the ith industry group,
B_{ij} = that portion of the output of group i which is used as input by group j, and
C_i = the "bill of goods," i.e., that portion of the ith group's output consumed, exported or unaccounted for.

For purposes of computation, each element in column j of the coefficients of the left-hand sides of equations (1) is normalized by dividing it by the diagonal element A_j, yielding

$$\begin{aligned} a_{ij} &= B_{ij}/A_j, & i &\neq j, \\ a_{ij} &= A_i/A_j = 1, & i &= j. \end{aligned} \tag{2}$$

Assuming the normalized coefficients a_{ij} to be constant, prediction equations for the industry group outputs X_j,

$$\sum_{j=1}^{38} a_{ij}X_j = Y_i, \qquad i = 1(1)38, \tag{3}$$

may be formed from the a_{ij} and any arbitrary but reasonable "bill of goods," Y_i. Each product $a_{ij}X_j$ represents that portion of the output of group i which is required as input by group j whatever the magnitude of X_j may be. By inverting the square matrix $[a]$ to form $[d] = [a]^{-1}$, the magnitudes, X_j, of the outputs required to produce the assumed "bill of goods," Y_i, may be found from the equations,

$$X_j = \sum_{i=1}^{38} d_{ji}Y_i, \qquad j = 1(1)38. \tag{4}$$

Written in matrix notation, equations (3) and (4) become

$$[a]\{X\} = [I]\{Y\}, \tag{5}$$

$$[I]\{X\} = [d]\{Y\}, \tag{6}$$

where $[I]$ denotes the unit matrix of order 38.

The method of elimination accomplishes the transformation of equation (5) into equation (6) by introducing auxiliary matrices $[b]$ and $[c]$, which satisfy the relation,

$$[b]\{X\} = [c]\{Y\}, \tag{7}$$

where

$$[b] \equiv \begin{bmatrix} 1 & b_{1,2} & b_{1,3} & \cdots & b_{1,38} \\ 0 & 1 & b_{2,3} & \cdots & b_{2,38} \\ 0 & 0 & 1 & \cdots & b_{3,38} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}; \qquad [c] \equiv \begin{bmatrix} c_{1,39} & 0 & 0 & \cdots & 0 \\ c_{2,39} & c_{2,40} & 0 & \cdots & 0 \\ c_{3,39} & c_{3,40} & c_{3,41} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{38,39} & c_{38,40} & c_{38,41} & \cdots & c_{38,76} \end{bmatrix}.$$

Matrices $[a]$ and $[I]$ of equation (5) are transformed into $[b]$ and $[c]$, respectively, of equation (7) by the same operations as are used to reduce determinants, i.e., multiplication by a constant and subtraction from one row of a linear combination of other rows.[2] The same types of operations are employed to convert [b] and [c] into [I] and [d], respectively, of equation (6).

Since the same operations are to be performed on the coefficient matrices of both sides of equations (5) and (7), it is convenient to combine the two matrices of each equation into a single augmented matrix. Thus [a] and [I] of equation (5) become a matrix [a, I] of order 38 $\times$ 76, which will be referred to as the given augmented matrix. Similarly, [b] and [c] become auxiliary augmented matrix $[b, c]_{38,76}$, and [I] and [d] of equation (6) become the final augmented matrix $[I, d]_{38,76}$.

The transformation of the given augmented matrix into the auxiliary augmented matrix is accomplished by applying routines (I) and (II), as follows:

(I) the first row of the given augmented matrix is multiplied by the reciprocal of the leading element, yielding the first row of the auxiliary augmented matrix,

$$b_{1,j} = a_{1,1}^{-1}a_{1,j}, \qquad j = 1(1)76.$$

(II) each element of the remaining rows of the given augmented matrix is reduced by the product of the leading element of the row in question by the element of the corresponding column from the first row of the auxiliary augmented matrix, producing $[a^{(1)}]_{(n-1),2n}$

$$a_{i,j}^{(1)} = a_{i,j} - a_{i,1}b_{1,j}, \qquad i = 2(1)38, \qquad j = 1(1)76.$$

The result of routines (I) and (II) is that X_1 is eliminated from all but the first of equations (3).

The matrix $[a^{(1)}]$ is then subjected to routines (I) and (II), yielding the second row of the auxiliary augmented matrix $[a^{(2)}]_{(n-2),2n}$. The sequence of routines (I) and (II) is applied in turn to the residual matrices $[a^{(k)}]_{(n-k),2n}$ until all rows of the auxiliary augmented matrix have been computed.

The last row (38) of the auxiliary augmented matrix now defines explicitly the value of X_{38} (for given Y's), and hence the portion contained in row 38 of [c] is identical with row 38 of [d].

The remaining rows of the final augmented matrix are obtained from the auxiliary augmented matrix by applying routine (III), as follows:

(III) each element of the remaining rows of [c] is reduced by the product of the last element of the corresponding row of [b] by the element of the corresponding column from the last row of [d], to produce $[c^{(1)}]_{(n-1),n}$,

$$c_{i,j}^{(1)} = c_{i,j} - b_{i,38}d_{38,j}, \qquad i = 37(-1)1, \qquad j = 39(1)76.$$

The last row (37) of $[c^{(1)}]$ is identical with row 37 of [d].

Routine (III) is applied to each matrix $[c^{(k)}]_{(n-k),n}$ in turn, the product being formed by the $(n - k)$th element from the corresponding row of [b] and the element of the corresponding column from the $(n - k)$th row of [d]. In each case, the last row of $[c^{(k)}]$ is identical with the corresponding row of [d].

The application of this method of solution to large-scale calculators is greatly facilitated by the fact that the machine operations required to carry out routines (I) and (II) are the same for every row of the series of matrices

$[a]$, $[a^{(1)}]$, $\cdots$ $[a^{(37)}]$. The number of elements in each row is constant if those elements which are either 0 or 1 in the given and auxiliary augmented matrices are provided for by the coding rather than in the data tapes. Two very desirable features derive from this fact: (A) the control tapes may be short, iterative tapes, and (B) the solution can be made independent of the order n of the system. The application of routine (III) is complicated by the presence of the element $b_{i,(n-k)}$ in the routine, but features (A) and (B) can be retained.

The Aiken Relay Calculator is equipped with four typewriters (called printers), four tape punches, four tape readers, and four sequence control mechanisms. These are the limiting elements in the design of the computing routines, and in the present problem a reduction of one-third in the computing time would result from a 50 per cent increase in these components.

It was found necessary to use a pair of identical data tapes for storing the elements of all matrices, simultaneously printing the elements being punched to provide a reference in the event of disagreement between the paired tape values. The printing circuits of the calculator provide a check system to insure that the quantity printed is the same as the quantity registered in the machine. Similar circuits are not provided for in the tape punches since the data being punched can be checked by refeeding.

In carrying out the computation of the auxiliary augmented matrix, one pair of tape readers provided the input matrix by rows, and the associated tape punches produced the output matrix tapes. The row of elements computed by routine (I) was punched into a second pair of data tapes by the third and fourth tape punches; these tapes were then fed into the associated readers and were read and repunched continuously as the remaining rows of the matrix were calculated by routine (II). A similar disposition of the punches and readers was employed in the computation of the final inverse matrix.

Three sets of control tapes were employed in the solution. The first set of two performed the matrix multiplications implied by equations (2) and (4). The second set of four tapes produced the auxiliary augmented matrix. The computation of the final matrix required four more control tapes.

An appreciation of the magnitude of the computation can best be had from a few statistics. Although the application of routines (I), (II) and (III) required but three seconds per element, the total uninterrupted machine time needed to produce the solution was $59\frac{1}{2}$ hours. The corresponding time for a matrix of order n would be approximately $0.001\ n^3$ hours. These estimates should be doubled to include the time required for set-up, tape changing, check re-runs and trouble shooting. Nearly 300000 ten-digit numbers were punched into and read from tapes, and 1000000 digits were printed. Approximately 100000 multiplications and 600000 additions were performed.

The present problem was the first serious computation to be attempted on the calculator, and its solution was beset by all the difficulties inevitably associated with the prototype of a new design. Overcoming these difficulties resulted in (a) improvements in the calculator amounting to at least 25 per cent greater operating efficiency; (b) uncovering a number of mistakes in wiring and other errors of construction prior to shipment to the Naval Proving Ground; (c) development of operating and trouble-shooting procedures; and (d) familiarization of all personnel—mathematicians, operators, and tech-

TABLE: *Comparison of Original Data with Those Computed from the Inverse Matrix*

i	Original Data A_i	Computed Values X_i	Absolute Error $\times 10^{-5}$	Relative Error $\times 10^{-9}$
1	10121	10120.999 99	1	1
2	13282	13281.999 91	9	7
3	2622	2622.000 001	0.1	0.4
4	493	492.999 999 7	.03	.6
5	438	437.999 999 8	.02	.4
6	443	443.000 000 3	.03	.6
7	134	133.999 999 1	.09	7
8	2581	2580.999 999	.1	0.4
9	269	269.000 000 1	.01	.4
10	266	266.000 000 5	.05	2
11	2223	2222.999 993	.7	3
12	441	440.999 999 3	.07	2
13	331	331.000 000 3	.03	1
14	1853	1852.999 997	.3	2
15	2287	2287.000 005	.5	2
16	1616	1615.999 999	.1	0.6
17	2082	2081.999 993	.7	3
18	4871	4871.000 017	1.7	4
19	1737	1737.000 001	0.1	0.6
20	2864	2864.000 007	.7	3
21	1516	1515.999 999	.1	0.7
22	3434	3434.000 003	.3	1
23	1245	1245.000 001	.1	0.8
24	1188	1187.999 999	.1	.8
25	1712	1711.999 997	.3	2
26	2265	2264.999 999	.1	0.4
27	3187	3187.000 003	.3	1
28	3454	3454.000 005	.5	2
29	990	989.999 998 9	.11	1
30	892	891.999 999 3	.07	0.8
31	1666	1665.999 999	.1	0.6
32	10089	10089.000 01	1	1
33	3167	3167.000 007	0.7	2
34	4310	4310.000 001	.1	0.2
35	16571	16571.000 01	1	.6
36	18525	18524.999 99	1	.5
37	4152	4152.000 005	0.5	1
38	22192	22191.999 97	3	1

$$X_i = \sum_{k=1}^{38} d_{ik} C_k, \qquad i = 1(1)38,$$

where X_i = computed total output of industry group i;
d_{ik} = computed coefficients of the inverse matrix $[d]$;
C_k = "bill of goods" actually consumed or exported during the year 1939 (not previously utilized in the computation);
and A_i = actual output of industry group i for the year 1939.

nicians—with the capabilities, limitations, and operating characteristics of the calculator and its associated equipment.

Three levels of checks were employed in the computation. The machine operations required for each element were performed in duplicate, using distinct machine components, and the results compared before they were printed or punched. A check number for each row of the given augmented matrix was formed during the normalization routine by summing the elements in the row, the check number sum becoming the 77th element. The completed rows of the auxiliary and final matrices were then summed, and that sum compared with the modified check number. A complete system of checks was found necessary for all control numbers used to regulate the repetitions of the control tapes. Finally, the original right-hand-side elements C of equations (1) were utilized as the Y's of equations (4). The X's obtained from the solution of equations (4) were then compared with the original A's of equations (1). The comparisons are given in the table.

The remarkably small differences resulting from these comparisons are attributed to two factors. First, the adders and multipliers of the calculator are equipped with round-off compensation circuits; and, second, each row of the given matrix possessed several dominant coefficients, the remaining ones being quite small in comparison. Round-off errors in the latter were largely absorbed, in the summing process, by the larger terms from the dominant coefficients.

Recently von NEUMANN & GOLDSTINE[3] have shown that the loss of significant digits to be expected, under certain rather general assumptions, when numerically inverting an arbitrary large matrix, is very great. However, as the results of the present computation show, the presence of a relatively small number of dominant coefficients in the given matrix may materially reduce this loss. Under this special condition, it may be expected that the inverse of a high-order matrix derived from a stable physical system will be as accurate as the input data. The rapid development of large-scale digital calculating machines should soon provide more examples of this operation, supplying further information concerning the relation between the number of dominant coefficients and the loss of significant digits.

The writer wishes to express his appreciation to Professor LEONTIEF and to Mr. FREDERICK MILLER and the operating staff of the calculator, without whose conscientious efforts the solution of the problem could not have been obtained.

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