Complex Zeros of $Y_0(z)$, $Y_1(z)$, and $Y_1'(z)$

One of the authors has obtained conditions for the existence of real and of complex zeros on the various branches of Bessel functions of real order.[1] For the integral order Bessel functions of the second kind, $Y_n(z)$, there are real zeros only on the branch which is real along the positive real axis. (These zeros are positive.) Every other branch has non-real zeros in the right half-plane and all branches have non-real zeros in the left half-plane.

Below are listed the first fifteen (non-real) zeros of $Y_0(z)$, $Y_1(z) = -Y_0'(z)$, and $Y_1'(z)$ in the left half-plane on the branch which has positive real zeros. Approximations to the first few zeros tabulated were obtained from "contour lines" of $Y_0(z)$ and $Y_1(z)$ which are soon to be published.[2] These approximations were refined by Newton's iteration method. The process was not at all simple since it involved the calculation to maximum accuracy of the Y's and their derivatives at points off the rays for which the functions have been tabulated.

Zeros with absolute values greater than 10 (i.e. those outside the range of the above mentioned volume) were computed from the following asymptotic expansions:[3]

$$-z_j \sim \beta - \frac{4n^2 - 1}{8\beta} - \frac{112n^4 - 152n^2 + 31}{384\beta^3} - \cdots,$$

$$-z_j' \sim \beta_1 - \frac{4n^2 + 3}{8\beta_1} - \frac{112n^4 + 328n^2 - 9}{384\beta_1^3} - \cdots$$

where the z_j are zeros of $Y_n(z)$, the z_j' are zeros of $Y_n'(z)$, $\beta = (j + \frac{1}{2}n - \frac{1}{4})\pi - i \operatorname{arctanh} \frac{1}{2}$, and $\beta_1 = (j + \frac{1}{2}n + \frac{1}{4})\pi - i \operatorname{arctanh} \frac{1}{2}$.

Table A below lists to 9D the zeros with absolute values less than 10. With the zeros of Y_0 are the values of Y_1 and Y_1' at the zeros. Similarly for the zeros of Y_1 and Y_1'. **Table B** has fifteen zeros to 5D for each of Y_0, Y_1, and Y_1'.

Table A

Zeros $z_{0,s}$ of $Y_0(z)$ and Values of Y_1 and Y_1' at the Zero

	$z_{0,s}$		Y_1		Y_1'	
s	Real	Imag.	Real	Imag.	Real	Imag.
1	−2.40301 6632	+.53988 2313	+.10074 7689	−.88196 7710	+.11840 7791	−.34042 2712
2	−5.51987 6702	+.54718 0011	−.02924 6418	+.58716 9503	−.01568 8932	+.10481 8434
3	−8.65367 2403	+.54841 2067	+.01490 8063	−.46945 8752	+.00514 0082	−.05392 3912

Zeros $z_{1,s}$ of $Y_1(z)$ and Values of Y_0 and Y_1' at the Zero

	$z_{1,s}$		$Y_0 = Y_1'$	
s	Real	Imag.	Real	Imag.
1	− .50274 3273	+.78624 3714	−.45952 7684	+1.31710 1937
2	−3.83353 5193	+.56235 6538	+.04830 1909	− .69251 2884
3	−7.01590 3683	+.55339 3046	−.02012 6949	+ .51864 2833

Zeros $z_{1,s}'$ of $Y_1'(z)$ and Values of Y_0 and Y_1 at the Zero

	$z_{1,s}'$		Y_0		Y_1	
s	Real	Imag.	Real	Imag.	Real	Imag.
1	+ .57678 5129	+.90398 4792	+.08026 5303	+.89580 3699	−.76349 7088	+.58924 4865
2	−1.94047 7342	+.72118 5919	−.23359 1479	+.40380 0129	+.16206 4006	−.95202 7886
3	−5.33347 8617	+.56721 9637	+.01766 2838	−,11002 8559	−.03179 4008	+.59685 3673
4	−8.53676 8577	+.55606 0704	−.00538 9206	+.05500 9625	+.01541 7716	−.47260 1166

Table B

Complex Zeros of $Y_0(z)$, $Y_1(z)$, and $Y_1'(z)$

	$z_{0,s}$		$z_{1,s}$		$z_{1,s}'$	
s	Real	Imag.	Real	Imag.	Real	Imag.
1	− 2.40302	.53988	− .50274	.78624	+ .57679	.90398
2	− 5.51988	.54718	− 3.83354	.56236	− 1.94048	.72119
3	− 8.65367	.54841	− 7.01590	.55339	− 5.33348	.56722
4	−11.79151	.54882	−10.17358	.55127	− 8.53677	.55606
5	−14.93091	.54900	−13.32374	.55046	−11.70618	.55286
6	−18.07106	.54910	−16.47066	.55006	−14.86367	.55150
7	−21.21163	.54915	−19.61587	.54984	−18.01557	.55080
8	−24.35247	.54919	−22.76009	.54970	−21.16440	.55038
9	−27.49348	.54922	−25.90368	.54961	−24.31135	.55012
10	−30.63461	.54923	−29.04683	.54955	−27.45706	.54995
11	−33.77582	.54925	−32.18968	.54950	−30.60193	.54982
12	−36.91710	.54926	−35.33231	.54947	−33.74619	.54973
13	−40.05843	.54926	−38.47477	.54945	−36.88999	.54966
14	−43.19979	.54927	−41.61710	.54942	−40.03345	.54961
15	−46.34119	.54927	−44.75932	.54941	−43.17663	.54956

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[1] A. Hillman, "On the reality of zeros of Bessel functions," Amer. Math. Soc., *Bull.*, v. 55, 1949.

[2] NBSCL, *Tables of the Bessel Functions $Y_0(z)$ and $Y_1(z)$ for Complex Arguments*, New York, Columbia University Press, publication announced for 1949.

[3] G. N. Watson, *A Treatise on the Theory of Bessel Functions*. Second ed. Cambridge and New York, 1944, p. 505–507.

A Method of Plotting on Standard IBM Equipment

The advent of automatic computing machinery in research in physics and chemistry has eliminated bulky arithmetic procedures, but in many cases this advantage is lost by the bottleneck of plotting the results; as for example in spectrum analysis where many computed curves have to be compared with the data in the form of a graph. If the results are computed on cards, or are in volume enough to be punched on cards, the following procedure simplifies the problem of plotting a large number of points.

The cards are sorted on abscissa and are fed into an IBM tabulator (preferably a 405) fitted with two digit selectors and preferably six class selectors. Each spacing of the platen (corresponding to each line in a typewritten roll of paper) is taken as unit increase in the abscissa. The range of the abscissa is then infinite—with obvious practical limitations. The unit spacing of the ordinate is that between type bars, of which there are 88 across the paper plus a space between the alphabetical and numerical sections. The range of ordinates is therefore limited to 0 to 88. This, however, is a suitable match for the abscissa scale for a reasonable length of paper. If the numbers to be plotted do not lie between 0 and 88 or lie in only a small fraction of this range, say 0 to 20, they can be machine-multiplied beforehand.

The problem then is, for a given value of the ordinate (punched on the card), say 35, to actuate the thirty-fifth type bar, which prints a symbol