The Accuracy of Linear Interpolation in Tables of the Mathematics of Finance

Many texts in the Mathematics of Finance give empirical statements with respect to the errors due to linear interpolation in the tables contained in these texts. It is the purpose of this paper to derive some formulae which express the amount of these errors. Formulae for the maximum errors will be obtained when solving by linear interpolation (a) for an unknown time and (b) for an unknown rate in tables of finance. The relative size of the errors in different tables will be considered.

(a) *Unknown time.*

Suppose that we are interpolating in the $(1+i)^n$ table or the $s_{\overline{n}|}$ table for an unknown time, n, and that it has been determined that n falls in the interval (n_1, n_2). Then the time, N as given by linear interpolation[1] is

$$N \equiv n_1 + \frac{(1+i)^n - (1+i)^{n_1}}{(1+i)^{n_2} - (1+i)^{n_1}}.$$

Let $n = n_1 + f$ where $(f < 1)$ and since $n_2 - n_1 = 1$, we have

$$N = n_1 + [(1+i)^f - 1]/i = n_1 + s_{\overline{f}|}.$$

The error, E, due to linear interpolation is

$$(1) \qquad E = n - N = n_1 + f - n_1 - s_{\overline{f}|}, \qquad \text{or} \qquad E = f - s_{\overline{f}|}.$$

Taking the first and second derivatives of E with respect to f, we have

$$dE/df = 1 - (1+i)^f \ln(1+i)/i,$$
$$d^2E/df^2 = -(1+i)^f[\ln(1+i)]^2/i.$$

Since d^2E/df^2 is always negative, we have a maximum error given by solving $dE/df = 0$ for f. This gives

$$(2) \qquad f(\max) = \frac{\ln i - \ln\ln(1+i)}{\ln(1+i)} = \frac{1}{2} + \frac{i}{24} + \cdots.$$

Substituting the value of f given by (2) in (1) gives

$$E(\max) = \frac{\ln i - \ln\ln(1+i)}{\ln(1+i)} + \frac{1}{i} - \frac{1}{\ln(1+i)} = \frac{i}{8} - \frac{i^2}{16} + \cdots.$$

The following theorems may now be presented:

THEOREM I: *The error in finding an unknown time by linear interpolation in the $(1+i)^n$ table or the $s_{\overline{n}|}$ table is independent of the interval (n_1, n_2).*

THEOREM II: *The maximum error in finding an unknown time by linear interpolation in the $(1+i)^n$ table or the $s_{\overline{n}|}$ table occurs when f is slightly more than halfway through the interval.*

THEOREM III: *The error in finding an unknown time by linear interpolation in the $(1+i)^n$ table or the $s_{\overline{n}|}$ table is never more than $\frac{1}{8}$ of the interest rate per period.*[2] (N is always less than n.)

The error due to linear interpolation in the $v^n = (1+i)^{-n}$ table or the $a_{\overline{n}|}$ table [3] is given as follows:

$$E = n - n_1 - (v^n - v^{n_1})(v^{n_2} - v^{n_1})^{-1}$$
$$= f - (v^f - 1)(v - 1)^{-1} = s_{\overline{1-f}|} - (1 - f).$$

From this result the following theorem may be stated.

THEOREM IV: *The error due to linear interpolation for a given value f in the* $(1+i)^n$ *table or the* $s_{\overline{n}|}$ *table is the same for* $1-f$ *in the* v^n *table or the* $a_{\overline{n}|}$ *table (opposite in direction). The maximum error in the two tables is the same.*

(b) *Unknown rate.*

Let us now consider the error due to linear interpolation in solving for an unknown rate. If we are interpolating in the $(1+i)^n$ table and it is observed that i falls between the rates i_1 and i_2 which are given in the table (n a given integer), then the interest rate I, given by linear interpolation, is $I = i_1 + (A - A')(i_2 - i_1)(A'' - A')^{-1}$ where $A = (1+i)^n, A' = (1+i_1)^n$, $A'' = (1+i_2)^n$. The error (positive) is

$$E = i - I = i - i_1 - (A - A')(i_2 - i_1)(A'' - A')^{-1}. \tag{3}$$

Taking the first and second derivatives of E with respect to i, we have

$$dE/di = 1 - n(1+i)^{n-1}(i_2 - i_1)(A'' - A')^{-1},$$
$$d^2E/di^2 = -n(n-1)(1+i)^{n-2}(i_2 - i_1)(A'' - A')^{-1}.$$

Since d^2E/di^2 is always negative ($n > 1$), we have a maximum error given by solving $dE/di = 0$ for i. This gives

$$i(\max) = \left[\frac{A''-A'}{n(i_2-i_1)}\right]^{1/(n-1)} - 1 \tag{4}$$
$$= \tfrac{1}{2}(i_1+i_2) + \tfrac{1}{24}(n-2)(i_2-i_1)^2 - \tfrac{1}{24}(n-2)\tfrac{1}{2}(i_1+i_2)(i_2-i_1)^2 + \cdots.$$

Substituting the value of i given by (4) in (3) gives

$$E(\max) = \left[\frac{A''-A'}{n(i_2-i_1)}\right]^{1/(n-1)}\left(1-\frac{1}{n}\right) - (1+i_1) + \frac{A'(i_2-i_1)}{A''-A'},$$

or

$$E(\max) \doteq [1 + \tfrac{1}{2}(i_1+i_2) + \tfrac{1}{24}(n-2)(i_2-i_1)^2$$
$$- \tfrac{1}{24}(n-2)\tfrac{1}{2}(i_1+i_2)(i_2-i_1)^2]\left(1-\frac{1}{n}\right)$$
$$-(1+i_1) + \frac{1}{n}[1 + \tfrac{1}{2}(n+1)i_1 - \tfrac{1}{2}(n-1)i_2 + \tfrac{1}{12}(n^2-1)(i_2-i_1)^2$$
$$- \tfrac{1}{8}(n^2-1)(i_2-i_1)^2(i_1+i_2)]$$
$$\doteq \tfrac{1}{8}(n-1)(i_2-i_1)^2 - (n-1)(7n+4)(48n)^{-1}(i_1+i_2)(i_2-i_1)^2,$$

or

$$E(\max) < \tfrac{1}{8}(n-1)(i_2-i_1)^2. \tag{5}$$

The following theorems may be presented:

THEOREM V: *The maximum error in solving for an unknown rate in the* $(1+i)^n$ *table occurs when i is slightly more than the mean of the table rates,* i_1, i_2.

THEOREM VI: *The error* [4] *due to linear interpolation in solving for an unknown rate in the* $(1+i)^n$ *table is never more than* $\frac{1}{8}(n-1)(i_2-i_1)^2$.

We may now state and prove a theorem in regard to the size of the error obtained in the $(1+i)^n$ table and the $s_{\overline{n}|}$ table.

THEOREM VII: *The error obtained in solving for an unknown rate by linear interpolation is always less in the* $s_{\overline{n}|}$ *table than the corresponding error in the* $(1+i)^n$ *table.*

This theorem states that

$$i-i_1-(s-s')(s''-s')^{-1}(i_2-i_1)<i-i_1-(A-A')(A''-A')^{-1}(i_2-i_1)$$

where the errors are positive, and

$$s'=[(1+i_1)^n-1]/i_1,\quad s=[(1+i)^n-1]/i,\quad s''=[(1+i_2)^n-1]/i_2,\quad i_1<i<i_2.$$

This inequality may be reduced to

$$\frac{s-s'}{s''-s'}>\frac{A-A'}{A''-A'},\tag{6}$$

or

$$\frac{C_2\alpha_1+C_3\alpha_2+C_4\alpha_3\cdots C_n\alpha_{n-1}}{C_2\beta_1+C_3\beta_2+C_4\beta_3\cdots C_n\beta_{n-1}}>\frac{C_1\alpha_1+C_2\alpha_2+C_3\alpha_3\cdots C_n\alpha_n}{C_1\beta_1+C_2\beta_2+C_3\beta_3\cdots C_n\beta_n}\tag{7}$$

where

$$\alpha_K=i^K-i_1{}^K,\qquad K=1(1)n,\qquad \beta_r=i_2{}^r-i_1{}^r,\qquad r=1(1)n,$$

and the C's are the coefficients in the binomial formula. By multiplying the means and extremes of inequality (7), we have

$$(\alpha_1\beta_2-\alpha_2\beta_1)(C_2^2-C_1C_3)+(\alpha_1\beta_3-\alpha_3\beta_1)(C_2C_3-C_1C_4)$$
$$+(\alpha_1\beta_4-\alpha_4\beta_1)(C_2C_4-C_1C_5)+(\alpha_2\beta_3-\alpha_3\beta_2)(C_3^2-C_2C_4)+\cdots>0.$$

All the terms of this expansion may be written in the form,

$$(C_{K+1}C_r-C_KC_{r+1})(\alpha_K\beta_r-\alpha_r\beta_K)\tag{8}$$

where $K<r$ except the last term which is $C_n^2(\alpha_{n-1}\beta_n-\alpha_n\beta_{n-1})$.
Now

$$C_{K+1}C_r-C_KC_{r+1}=C_KC_r\frac{(n+1)(r-K)}{(K+1)(r+1)}$$

which is positive. Also, we must consider the sign of

$$\alpha_K\beta_r-\alpha_r\beta_K=(i^K-i_1{}^K)(i_2{}^r-i_1{}^r)-(i^r-i_1{}^r)(i_2{}^K-i_1{}^K),$$

which is positive if

$$\frac{i_2{}^r-i_1{}^r}{i_2{}^K-i_1{}^K}>\frac{i^r-i_1{}^r}{i^K-i_1{}^K}\qquad\text{where}\qquad r>K,\qquad i_2>i>i_1$$

or

$$\frac{(1+\alpha)^r-1}{(1+\alpha)^K-1}>\frac{(1+\beta)^r-1}{(1+\beta)^K-1}\quad\text{where}\quad \alpha=\frac{i_2}{i_1}-1,\quad \beta=\frac{i}{i_1}-1.\tag{9}$$

Expanding inequality (9) by multiplying the means together and the extremes together, we have terms of the form $C_\phi C_{K+\theta}(\alpha^{K+\theta}\beta^\phi - \alpha^\phi\beta^{K+\theta})$ where $\phi = 1, 2, \cdots K, \theta = 1, 2, \cdots (r - K)$, or $C_\phi C_{K+\theta}\alpha^\phi\beta^\phi(\alpha^{K+\theta-\phi} - \beta^{K+\theta-\phi})$ which is positive since $\alpha > \beta$. Hence all the terms (8), of the expansion of (7) are positive so that (6) is true and Theorem VII is proved.

By Theorem VII, *the maximum for the error obtained in* (5) *also applies to the* $s_{\overline{n}|}$ *table.*

If we interpolate for an unknown rate in the v^n table, the error is given by

$$E = i_1 + (V - V')(V'' - V')^{-1}(i_2 - i_1) - i \tag{10}$$

where

$$V = (1 + i)^{-n}, \qquad V' = (1 + i_1)^{-n}, \qquad V'' = (1 + i_2)^{-n}.$$

A maximum error is obtained by solving

$$\frac{dE}{di} = -1 - \frac{n(1+i)^{n-1}}{V'' - V'}(i_2 - i_1) = 0 \text{ for } i$$

since d^2E/di^2 is always negative. Hence

$$i(\max) = \left[\frac{V' - V''}{n(i_2 - i_1)}\right]^{-1/(n+1)} - 1 = \frac{i_1 + i_2}{2} - \frac{n+2}{24}(i_2 - i_1)^2 + \cdots. \tag{11}$$

Substituting this value of i given by (11) in (10) gives

$$E(\max) = -\left[\frac{V' - V''}{n(i_2 - i_1)}\right]^{-1/(n+1)}\left(1 + \frac{1}{n}\right) + (1 + i_1) - \frac{V'(i_2 - i_1)}{V'' - V'} = \frac{n+1}{8}(i_2 - i_1)^2 - \cdots. \tag{12}$$

The following theorems may be presented:

THEOREM VIII: *The maximum error in solving for an unknown rate in the* v^n *table occurs when* i *is slightly less than the mean of the table rates* i_1, i_2.

THEOREM IX: *The error due to linear interpolation in solving for an unknown rate in the* v^n *table is never more than* $\frac{1}{8}(n + 1)/(i_2 - i_1)^2$.

We may now state and prove a theorem in regard to the size of the error obtained in the v^n table and the $a_{\overline{n}|}$ table.

THEOREM X: *The error obtained in solving for an unknown rate by linear interpolation is always less in the* $a_{\overline{n}|}$ *table than in the* v^n *table.*

This theorem states that

$$i_1 + \frac{a - a'}{a'' - a'}(i_2 - i_1) - i < i_1 + \frac{V - V'}{V'' - V'}(i_2 - i_1) - i$$

where the errors are both positive, and

$$a' = \frac{1 - (1 + i_1)^{-n}}{i_1}, \qquad a = \frac{1 - (1 + i)^{-n}}{i}, \qquad a'' = \frac{1 - (1 + i_2)^{-n}}{i_2},$$

$$i_1 < i < i_2.$$

The inequality (8) may be reduced to

$$\frac{a - a'}{a'' - a'} < \frac{V - V'}{V'' - V'}$$

or,

$$a(V'' - V') + a'(V - V'') + a''(V' - V) < 0,$$

or

$$(13) \quad \frac{s}{A}\left(\frac{1}{A''} - \frac{1}{A'}\right) + \frac{s'}{A'}\left(\frac{1}{A} - \frac{1}{A''}\right) + \frac{s''}{A''}\left(\frac{1}{A'} - \frac{1}{A}\right) < 0.$$

Multiplying both members of (13) by $A'AA''$ gives

$$s(A' - A'') + s'(A'' - A) + s''(A - A') < 0$$

which is an expanded form of inequality (6) which was proved to be true.

By Theorem X *the maximum error given by (12) also applies to the* $a_{\overline{n}|}$ *table.*

Since the yield on a bond may be found approximately by interpolating in the $a_{\overline{n}|}$ *table, the maximum error is given by (12).*

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[1] It can be shown that the value of N obtained by interpolation is the exact value of n if simple interest is used for the fractional interest period involved.

[2] In W. L. HART, *Mathematics of Investment*, second ed. Boston, 1929, p. 244, a proof is given that the error is at most $\frac{1}{2}$ of the interest rate per period.

[3] The value of $n = n_1 + f$ $(f < 1)$ obtained in the $a_{\overline{n}|}$ table has the following useful interpretation: f is the final payment due at the end of $n + 1$ interest periods.

[4] See THEODORE E. RAIFORD, *Mathematics of Finance.* Boston, 1945, p. 25, note, and W. L. Hart, *Mathematics of Investment*, third ed., Boston, 1946, p. 75 and p. 138. In these texts the following statement is made. Experience shows that it is safe to assume that a value of I found by interpolation is in error by not more than $\frac{1}{20}$ of the difference of the table rates used in the interpolation.

RECENT MATHEMATICAL TABLES

608[A, D, S].—G. H. GOLDSCHMIDT & G. J. PITT, "The correction of X-ray intensities for Lorentz-polarization and rotation factors," *Jn. Sci. Instrs.*, v. 25, Nov. 1948, p. 397–398. 20.2 × 27.2 cm.

There are two tables. **T. 1**, Inverse Lorentz-polarization factor as a function of $\rho = 2\sin\theta$; $(\mathrm{LP})^{-1} = \sin 2\theta/(1 + \cos^2 2\theta) = \frac{1}{2}\rho(4 - \rho^2)^{\frac{1}{2}}/(2 - \rho^2 + \frac{1}{4}\rho^4)$, for $\rho = [0(.01)2; 4D]$. **T. 2**, Rotation factor D as a function of ξ and ζ for equi-inclination conditions, $\xi = \zeta D/(1 - D^2)^{\frac{1}{2}}$; $D/(1 - D^2)^{\frac{1}{2}}$ is given, 2–3 decimal places, for $D = -1, .2(.1).9, .95, .975$.

Extracts from text

609[B].—LUDWIG ZIMMERMANN, *Vollständige Tafeln der Quadrate aller Zahlen bis 100 009 berechnet und herausgegeben.* Fourth edition, Berlin Grunewald, 1941, xix, 187 p. (*Sammlung Wichmann Fachbücherei für Vermessungswesen und Bodenwirtschaft*, v. 8.) 19.4 × 24.9 cm.

In the publisher's preface we are told that Zimmermann died 15 Aug. 1938. Compare *MTAC*, v. 2, p. 206–207; the errors of the third edition (1938), in T. III, there noted, here persist. The second edition was published at Liebenwerda in 1925; and the first in 1898.

Zimmermann was also the author of: (a) *Rechentafeln, grosse Ausgabe.* Liebenwerda, 1896, xvi, 205 p.; second ed., 1901; third ed., 1906; fourth ed., 1923, xxxix, 225 p. (b) *Rechentafeln, kleine Ausgabe.* Liebenwerda, 1895; fourth ed., 1926, xxv, 38 p. (c) *Tafeln für die*