in the final answers. The exact magnitude of the error incurred in the case of a set of ill-conditioned equations is still subject to question and requires additional investigation with the aid of high-speed digital equipment.

The usual methods for shifting the decimal point of the given matrix so that the maximum coefficient is near unity are easily effected in this procedure. Furthermore, if it is known (from some physical considerations) that a certain unknown is many times smaller than the rest of the unknowns, its decimal point may be shifted by using -10 or -100 in the corresponding row of the $-I$ matrix.

Obviously the "pivotal condensation" method may be easily incorporated into the described method, since it is a simple matter to select the row with the largest leading coefficient and use it as a pivot in the reduction process.

Computation Time Considerations.—The solution of a single set of equations of the form $(10 \times 10 \times 1)$ requires *four* hours when performed separately. It is quite evident that a multiple set of equations would require less time.

A set of 10 equations with 10 unknowns and multiple right hand sides, e.g., a set involving the same A matrix but 14 columns in the B matrix, $(10 \times 10 \times 14)$ requires ten hours, when a single set is computed. Again it is true that concurrent reduction of several sets will reduce the computation time.

The inversion of a single matrix of ten equations with ten unknowns requires eight hours. Again this time can be reduced by concurrent inversion of several matrices.

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[1] J. VON NEUMANN, & H. H. GOLDSTINE, "Numerical inversion of matrices of high order," Amer. Math. Soc., *Bull.*, v. 53, 1947, p. 1021–1099.
[2] A. M. TURING, "Rounding-off errors in matrix processes," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 286–308.
[3] L. FOX, H. D. HUSKEY, & J. H. WILKINSON, "Notes on the solution of algebraic linear equations," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 149–173.
[4] H. F. MITCHELL, "Inversion of a matrix of order 38," *MTAC*, v. 3, p. 161–166, 1948.
[5] Any person interested in obtaining information about the wiring diagrams used in this procedure should write directly to the author.

RECENT MATHEMATICAL TABLES

628[A].—H. E. MERRITT, *Gear Trains including a Brocot Table of Decimal Equivalents and a Table of Factors of all useful Numbers up to 200 000.* London, Pitman, 1947, viii, 178 p. 13.5 × 21.4 cm. Compare *MTAC*, v. 1, p. 21–23, 66–67, 91–92, 100, 132.

T. I. "The factor table," p. 15–54, contains the factors of the 4032 numbers $<200\,000$, having prime factors not less than 7 and not more than 127, the largest convenient tooth number in a change-gear train. **T. II.** "Brocot table," p. 65–95, 6D; the numerator of the fractions is $\leqslant 99$ and the denominator $\leqslant 100$; there is a column of exact remainders for each division.

Extracts from text

629[A, F].—DRAGOSLAV S. MITRINOVICH, "O Stirling-ovim brojevima" [On Stirling numbers], Skoplje, Yugoslavia, Univerzitet, Filozofski Fakultet, *Prirodno-matematichki oddel, Godishen Zbornik* [Year Book], v. 1, 1948, p. 49–95; Russian resumé, p. 90–92; French resumé, p. 93–95. 15.4 × 22 cm.

$x(x-1)(x-2)\cdots(x-n+1) \equiv S_n{}^1x + S_n{}^2x^2 + \cdots + S_n{}^{n-1}x^{n-1} + S_n{}^nx^n$ and $S_n{}^m$ are the integral Stirling numbers of the first kind.

$S_{n+1}^m = S_n{}^{m-1} - nS_n{}^m$.

If $(x-1)(x-2)\cdots(x-n) \equiv x^n - \phi_n{}^1x^{n-1} + \phi_n{}^2x^{n-2} - \phi_n{}^3x^{n-3} + \cdots + (-1)^n\phi_n{}^n$,

$$\phi_n{}^m = (-1)^m S_{n+1}^{n-m+1}, \quad S_n{}^m = (-1)^{n-m}\phi_{n-1}^{n-m}.$$

There are five tables:

I. $S_n{}^{n-1} = -\binom{n}{2}$, $n = 2(1)52$; **II.** $S_n{}^{n-2} = \frac{1}{4}\binom{n}{3}(3n-1)$, $n = 3(1)65$;

III. $S_n{}^{n-3} = -\frac{1}{2}\binom{n}{4}n(n-1)$, $n = 4(1)51$;

IV. $S_n{}^{n-4} = \frac{1}{48}\binom{n}{5}(15n^3 - 30n^2 + 5n + 2)$, $n = 5(1)50$;

V. $S_n{}^{n-5} = -\frac{1}{16}\binom{n}{6}n(n-1)(3n^2 - 7n - 2)$, $n = 6(1)51$.

$S_n{}^m$ for $n = 1(1)9$, $m = 1(1)n-1$ was given by J. STIRLING (1692–1770) in his *Methodus Differentialis*, London, 1730, p. 11; *The Differential Method*, Engl. transl. by F. HOLLIDAY, London, 1749, p. 10. For $n = 1(1)9$, $m = 1(1)n$, also $n = -4(1)-1$, $m = 0(1)8$, see E. P. ADAMS, *Smithsonian Mathematical Formulae*. Washington, 1939, p. 159. For $n = 1(1)22$, $m = 1(1)n$, see J. W. L. GLAISHER, *Q. Jn. Math.*, v. 31, 1900, p. 26. For $n = 1(1)12$, $m = 1(1)n$ see H. T. DAVIS, *Tables of the Higher Math. Functions*, v. 2, Bloomington, Ind., 1935, p. 215; also given, 1933 by CHARLES JORDAN. See also *MTAC*, v. 1, p. 330.

R. C. A.

630[A–E].—H. M. SASSENFELD & H. F. A. TSCHUNKO, *Mathematische Tafeln für Mathematiker, Naturwissenschaftler, Ingenieure. Erster Teil: Elementare Funktionen.* Walldorf bei Heidelberg, Fr. Lamadé, 1949, viii, 36 p. stiff cover. 20.7 × 29.5 cm.

CONTENTS: **T. 1** (p. 1): P.P., $d = 0(.1)199$; also reprinted on a moveable card. **T. 2** (p. 2–3): Prime factors and 6D mantissae of logs of the prime numbers in the range $n = 1(1)2000$. **T. 3** (p. 4–13): n^2, n^3, $n^{\frac{1}{2}}$, and $n^{\frac{1}{3}}$, 4D, $1000n^{-1}$ 5D (6S for $n \lessapprox 100$), πn and $\frac{1}{4}\pi n^2$ 5–7S, $n = 1(1)2000$. **T. 4** (p. 14): For $\phi = 0(1°)180°$, 5D or 5S values of arc-length l; chord s; segment height h; l/h; segment area F. **T. 5** (p. 14): Spherical area, 6S, and volume, 7S, for $d = 1(1)200$. **T. 6** (p. 15): Arc lengths [0(0°.1)180°; 5D]. **T. 7** (p. 16–17): x^n, $\pm n = 2(1)15$, and $n = -1$, $x =$ [1.1(.1)10.9; 5S]. **T. 8** (p. 18): 4D mantissae and antilogs, $x = 10(1)999$. **T. 9** (p. 19): 4D values log sin x, log cos x, log tan x, log cot x, $x = 0(0°.1)90°$. **T. 10** (p. 20–21): tables involving relations between sexagesimal, centesimal and radian systems of measurement. **T. 11** (p. 22–23): sin x, cos x, and tan x, cot x, for $x =$ [0(0°.1)10°, (0°.1)90°; 5D, 4D], [$0(0^g.1)10^g$, $(0^g.1)100^g$; 5D, 4D]. **T. 12** (p. 24): ln x, $x =$ [0(.001)1(.01)10(1)109; 5D]. **T. 13** (p. 25): $e^{\pm x}$, $x =$ [0(.01)10.09; 5S]. **T. 14** (p. 26–28): sin x, cos x, tan x, sinh x, cosh x, tanh x, $x^r =$ [$0(0^r.01)6^r$, mostly 5D]. **T. 15** (p. 29): sin x, cos x, $x^r =$ [$6^r(0^r.01)11^r$; 5D]. **T. 16** (p. 30–31): $\sin^{-1} x$, $\cos^{-1} x$, $x =$ [0(.001).999; 5D], $\tan^{-1} x$, $x =$ [0(.001)1.1(.01)10.09; 5D]. **T. 17** (p. 32): 9D and 10D values of log x, ln x, $e^{\pm n/100}$, sin $(n/10)$, cos $(n/10)$, sin $(n/1000)$, cos $(n/1000)$, sin $(n/100)$, cos $(n/100)$, $n = 0(1)99$. Also sin $n°$, cos $n°$, n = 0(1°)90°. **T. 18** (p. 33): $x^n/n!$ $n = 2(1)12$, $x = 1.1(.1)10.9$. **T. 19** (p. 34): Miscellaneous numerical tables.

631[A–E, I, K].—L. J. COMRIE, *Chambers' Six-Figure Mathematical Tables. Volume I: Logarithmic Values. Volume II: Natural Values.* Edinburgh and London, W. & R. Chambers, 1948, xxii, 576, xxxvi, 576 p. 17 × 25.4 cm. 42 shillings per v. American edition, offset print: New York, D. Van Nostrand Co., 1949, $10.00 per v. or $17.50 for the two v. The binding and type-pages of the British edition are much the more attractive, and the volumes weigh about three quarters of a pound less.

These volumes will be everywhere welcomed as a very valuable compilation of tables. They cover the field of elementary functions, with some minor excursions into no man's land between the frontier of elementary functions and the domain of higher mathematical functions.

In consideration of the fact that even in this day and age there are many who have no access to calculating machines, the editor has considered it worthwhile to compile two volumes, the first, intended for those who have no access to calculating machines, listing the logarithms of functions, and the second, listing natural values of functions, for those to whom calculating machines are available.

The editor has made a convincing case for the desirable tabulation of functions to six decimal places, and has adhered to this policy with but few exceptions (for example: T. II, III, VIII, in v. **I**; T. XIII in v. **II**). The intervals in arguments have been chosen so that most tables are essentially linear, i.e., linear interpolation is adequate for six-place accuracy.

To facilitate interpolation, first differences are given for most (but not all) of the tabulated functions, frequently with appropriate proportional parts; in the case of two tables in **II** second differences are also given. The first differences have been printed in italics, as a warning to the reader, whenever second differences are not negligible.

To obviate interpolation where interpolation is inherently troublesome auxiliary functions have been tabulated. Thus v. **I** contains tables of $S = \log \sin x - \log x$; $T = \log \tan x - \log x$ and of the corresponding Sh and Th functions. V. **II** contains tables of $\sigma = x \csc x$ and $\tau = x \cot x$ and of the corresponding σh and τh functions. Auxiliary functions are also tabulated in the case of inverse circular and hyperbolic functions. Regarding interpolation in the functions $\csc x$ and $\cot x$ it is important to point out the remarkable formulae on p. xiv of **II** which express the values of $\cot x$ and $\csc x$ for small x in terms of the cotangents and cosecants of the angles $10x$ and $100x$. An important feature worth mentioning is the inclusion of a very large number of critical tables, i.e., tables in which the functional values appear at equidistant intervals, the corresponding arguments being unevenly spaced.

The texts preceding the tables consist of introductions followed by detailed descriptions of the various tables, including numerical examples illustrating the use of the tables, with particular emphasis on both direct and inverse interpolation. The introductions contain a fairly extensive discussion of such topics as the motivation for the choice of the range and intervals of the tables, the essential facts underlying the theory of direct and inverse interpolation, numerical differentiation and integration, etc. As was to be expected the introductions include a discussion of the throw-back method of modifying differences, popularized by the editor, his technique of inverse interpolation and his technique of applying the Lagrangean interpolation formula in the case of unequal intervals. The bibliographies at the end of **I–II** contain the most important references to other tables of the functions tabulated in **I–II** with more than 6D.

No one will challenge the editor's statement that "great attention has been paid to typography"; nevertheless the reviewer feels that the legibility of T. VII in **I**, would have been enhanced by lines separating the columns and by providing spaces between groups of values corresponding to five or perhaps ten arguments. Also, the reviewer does not favor the use of the headings sin, tan, or $\sin x$ and $\tan x$ in **I** where $\log \sin x$ and $\log \tan x$ are intended.

While the reviewer agrees with the editor's comments on the shortcomings of interpolation by means of Lagrangean interpolation coefficients, he feels however that interpolation by means of differences is not altogether free from similar shortcomings.

The reviewer has not sampled the volumes for accuracy (freedom from error). There is, however, no doubt in his mind that they come up to the usual high standards which one is accustomed to find in "Comrie" tables.

We shall now indicate some of the essential contents of the volumes; a number of small auxiliary and critical tables have been omitted from this account.

I—T. I (p. 2–181): logarithmic mantissae for 10,000(1)100,009.

T. II (p. 182–191): 8D logarithmic mantissae, and 6D values of Mx and x/M, for $x = 0(.001)1$, together with 6D values of multiples, 1(1)10(10)90, of M and $1/M$, for the purpose of converting from common to natural logarithms and vice-versa.

T. III (p. 192–211): 8D values of logarithms for 1(.00001)1.10009. One use of this table is in compound interest and annuity problems; a second use is in the evaluation of 8D values of logarithms by the factor method in conjunction with T. II.

T. IV (p. 212–231): antilogarithms, giving the values of 10^x for $x = 0(.0001)1$.

The next three tables (p. 232–340) are devoted to the tabulation of the logarithms of trigonometric functions of angles expressed in degrees, minutes and seconds:

T. VA: $\log \sin x$ and $\log \tan x$ for $x = 0(1'')1°20'$.

T. VB: $\log \sin x$, $\log \cos x$, $\log \tan x$, and $\log \cot x$ for $x = 0(10'')10°$.

T. VC: $\log \sin x$, $\log \cos x$, $\log \tan x$, and $\log \cot x$ for $x = 10°(1')45°$.

The next four tables (p. 341–489) give the values of trigonometric functions of angles expressed in degrees and decimal subdivision of the degree and in radians:

T. VIA: critical table for $\log \cos r$ where r varies from 0 to $0^r.025$, and for $\log \sin r$ where r varies from $1^r.546$ to $1^r.571$ ($\frac{1}{2}\pi$).

T. VIB: values of log sin and log tan for angles 0(.001°)5° together with the radian measures of the angles.

T. VIC: rounded off proportional parts for 10(10)170 units of the last decimal of the radian arguments for the differences 2(1)231(3)876.

T. VID: logarithms of sin, cos, tan, and cot for the angles 0(.01°)45° and the corresponding values of the angles in radians.

T. VII (p. 490–499): critical table for the functions $S = \log \sin x - \log x$ and $T = \log \tan x - \log x$ for x expressed in seconds of arc, minutes of arc, degrees and decimal subdivision of the degree, radians, seconds of time and minutes of time. In all cases the values of $\log \sin x$ and $\log \tan x$ corresponding to the unevenly spaced arguments x are also given.

T. VIII (p. 500–539): logarithms of the hyperbolic $\sin x$, $\cos x$, and $\tan x$ for $x = 0(.001)$-3(.01)5 and several auxiliary tables (critical tables, proportional parts, etc.).

T. IX (p. 540–543): logarithms of $\Gamma(x)$ for $x = 1(.001)2$.

T. X (p. 544–549): conversion of degrees, minutes, and seconds into radians, radians into degrees and decimal subdivision of the degree, radians into degrees, minutes and decimals, radians into degrees, minutes and seconds, and other conversion tables.

T. XI (p. 550–569): first nine multiples of the numbers 1(1)999.

II—There is some "overlapping" between **I** and **II**. Specifically the first three tables in **II** (p. 2–215) are the counterparts of T. V, VI, and VII in **I**, and are devoted to the tabulation of the natural values of the trigonometric functions while T. IV is the counterpart of T. VIII in **I**, and contains a tabulation of the hyperbolic functions. T. IVD, IVE, IVF, IVG of **II** include values of $e^{\pm x}$ for 0(.001)3(.01)6(various)∞.

T. V (p. 318–335): natural logarithms for $x = 0(.001)10$.

T. VIA (p. 336–355): $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\sinh^{-1} x$, and $\tanh^{-1} x$ for $x = 0(.001)1$.

T. VIB (p. 356–357): $\sin^{-1} x$ and $\tanh^{-1} x$ and some auxiliary functions for x in the range from .99 to 1.

T. VIC (p. 358–359): $\sec^{-1} x$, $\cosh^{-1} x$, $\coth^{-1} x$, and some related functions in the range from 1 to 1.01.

T. VID (p. 360–401): inverse functions which can have values larger than unity for $x = 1(.001)2(.01)10(.1)35(1)80$. The frequent use of italics indicates that the tables are "non-linear," i.e., linear interpolation will not give six-place accuracy.

T. VII (p. 402–425): gudermannian and its inverse; $\mathrm{gd}x = \int_0^x \operatorname{sech} x\, dx$ is tabulated for $x = 0(.001)4.5(.01)10$; and $\mathrm{gd}^{-1}x$ for $x = 0(.001)1.400(.0001)1.57050(.00001)(1.570780$-$(.000001)1.570796$.

T. VIIIA (p. 426–465): x^2, x^3, x^4, x^5, $x^{\pm\frac{1}{2}}$, x^{-2}, x^{-1}, $x^{1/3}$, $x^{1/4}$, $x^{1/5}$, $x!$, $\log x!$ and the prime factors of $x = 1(1)1000$.

T. VIIIB (p. 466–489): x^2, and $(x/1000)^3$ and the prime factors of $x = 1000(1)3400$.

T. IX (p. 490–491): prime numbers up to 12919.

T. X (p. 492–513): to facilitate the conversion from rectangular to polar coordinates, $1 - k^2$, $(1 - k^2)^{\frac{1}{2}}$, $(1 + k^2)^{\frac{1}{2}}$, $\tan^{-1} k$ (both in degrees and radians) and $\cot^{-1} k$ for $k = 0(.001)1$. If s is the smaller of the numbers x and y and l is the larger, then $(x^2 + y^2)^{\frac{1}{2}} = l[1 + (s/l)^2]^{\frac{1}{2}} = l(1 + k^2)^{\frac{1}{2}}$ and $\tan^{-1}(y/x)$ is either $\tan^{-1} k$ or $\cot^{-1} k$. The values of $1 - k^2$ and $(1 - k^2)^{\frac{1}{2}}$ have been included because of their use in certain statistical problems.

T. XI (p. 514–517): $\Gamma(x)$ for $x = 1(.001)2$.

T. XIIA (p. 518–519): $\operatorname{erf} x$ for $x = 0(.01)4$.

T. XIIB (p. 520–524): $z = (1/(2\pi)^{\frac{1}{2}})e^{-\frac{1}{2}x^2}$, for $x = 0(.01)5$, and $\alpha = (2/\pi)^{\frac{1}{2}} \int_0^x e^{-\frac{1}{2}t^2}dt$, for $x = 5(.01)6$; also the values of certain related functions.

T. XIIC (p. 525–531): probability abscissa x and ordinate z, the argument $\frac{1}{2}(1 + \alpha)$ being the integral between $-\infty$ and x, for respective arguments .5(.001)1 and .99(.0001)1.

T. XIII (p. 532–543): for the coefficients B'' and B''' in Bessel's interpolation formula, where the B's are functions of n, the fraction of the tabular interval for which the fundamental value is desired. Also two short tables of 4-point and 6-point Lagrangean interpolation coefficients.

T. XIV (p. 544–549): formulae for numerical differentiation and integration.

T. XV (p. 550–561): proportional parts for $6''(1'')10''(10'')50''$ for the differences 2(1)1015.

T. XVI (p. 562–567): conversion table identical with T. X in **I**.

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632[A–E, K, M].—Harold D. Larsen, *Rinehart Mathematical Tables*. New York, Rinehart & Co., 232 Madison Ave., 1948, viii, 264, [2] p. 14 × 21 cm. $1.50.

Partial Contents: **T. 1–3**, p. 3–94, 5D tables of logs of numbers, trigonometric functions (sin, tan, cot, cos) at interval 1′, log trig. functions. **T. 4**, p. 95–114: $N = 1(1)1000$, N^2, N^3, $N^{\frac{1}{2}}$, $(10N)^{\frac{1}{2}}$, $N^{\frac{1}{3}}$, $(10N)^{\frac{1}{3}}$, $(100N)^{\frac{1}{3}}$, $1000/N$. **T. 5**, p. 115–116: Four-place log. **T. 6**, p. 117–120. 5D Nat. trig. functs., 0(0°.1)90°. **T. 7**, p. 121–124, 5D log trig. functs. at interval 0°.1. **T. 13**, p. 131: 5D log. factorials. **T. 14**, p. 132–135: Four-place ln. **T. 15**, p. 136–141: e^x, e^{-x}, $\log e^x$, $x = 0(.01)5(.05)10$. **T. 16**, p. 142–145: $\sinh x$, $\cosh x$, $\tanh x$, $x = 0(.01)$-$3(.05)7.5(.25)10$. **T. 17–18**, p. 146–147: Mortality tables. **T. 20–23**, p. 150–153: $(1 + r)^{\pm n}$, $\pm [(1 + r)^{\pm n} - 1]/r$. **T. 24–25**, p. 154–155: Ordinates and areas of the normal probability curve. **T. 26**, p. 156–159: Values of F and t 5% and 1% points. **T. 27**, p. 160: Values of χ^2 corresponding to certain chances of exceeding χ^2. Various formulae and constants, p. 164–199. Curves for reference, p. 200–214. Derivatives, p. 215–217. Indefinite integrals (430), p. 218–250. Definite integrals (431–494), p. 251–255. Series, p. 256–260. Index, p. 261–264. Proportional parts, 2 p.

633[A–F, H, L–N].—M. Boll, *Remarques et Compléments aux Tables Numériques Universelles*. Paris, Dunod, 92 rue Bonaparte (VI), 1949, 32 p. 18.2 × 27 cm. The publisher will supply these pages free to any purchaser of the original work, published in 1947; compare *MTAC*, v. 2, p. 336–338. This pamphlet is dated, inside, 9 Nov., 1948.

It contains a list of corrections and additions to the original work, p. 4–882. Only one of the numerous errors which we listed has been corrected.

As an addition to the original, p. 26–45, a new table of $n^{\frac{1}{2}}$ is given (p. 4–8) for $n = [1(1)1000; 7S]$.

T. 13, p. 62–70: "Nombres premiers et plus petits diviseurs" has been completely discarded in favor of a new table of the same character, p. 10–18.

Various additions including Euler numbers are set forth for p. 232.

A new table of $\tan x - x$, for $x = [1°(1°)60°; 6D]$ is given (p. 21) as a supplement to p. 234.

The old Fresnel integrals tables, p. 362–363, uniformly 4D, are replaced by corrected tables (p. 23–24) with many 5–6D values.

To the regular polyhedra table, p. 374, an addition is given on p. 25.

Many corrections are given (p. 30) for the table of roots of cubic equations p. 710–711.

R. C. A.

634[A, C, D].—HEINZ WITTKE, *Vademekum für Vermessungstechnik.* Stuttgart, Metzlersche Verlagsbuchhandlung, 1948, 334 p. 10.5 × 15.4 cm. See also the author's work on *Rechenmaschine*, *MTAC*, v. 3, p. 390.

This is an elementary miscellany useful for the surveyor. Among the 24 tables, p. 131–334 are the following 5D tables, p. 132–279: $\log x$, $x = 1000(1)10009$, Δ; log sin, log tan, log cot, log cos, for $0(0^g.01)10^g(0^g.1)100^g$, Δ, PP (beginning with 2^g); natural sin, tan, cot, cos, for $0(0^g.1)100^g$, Δ, PP. Then there is also a 4S table of $100 \cos^2 \alpha$ and $50 \sin 2\alpha$ for $\alpha = 0(0^g.1)100^g$, p. 290–299.

R. C. A.

635[C, D].—VEGA, *Seven Place Logarithmic Tables of Numbers and Trigonometrical Functions.* New York, Hafner Publishing Co., 1948, XVI, 575 p. 15 × 22.9 cm. $3.00.

CARL BREMIKER (1804–1877) edited the fortieth edition of the *Logarithmisch-Trigonometrisches Handbuch* of GEORG, FREIHERR VON VEGA (1756–1802), published in Berlin by Weidmann, in 1856, XXXII, 575 p. The interesting preface occupied p. I–XVI, and the Introduction p. XVII–XXXII. In the following year the same publisher issued an English translation, prepared by W. L. F. FISCHER (1814–1890), F.R.S. 1855, with the title: *Logarithmic Tables of Numbers and Trigonometrical Functions by Baron von Vega. Translated from the fortieth or Dr. Bremiker's thoroughly revised and enlarged edition.* Fischer was a fellow of Clare College Cambridge and professor, at the University of St. Andrews, of natural and experimental philosophy (1849–59), and of mathematics (1859–1879). *Alumni Cantabrigienses* tells us that his Christian names were "Frederick William Lewis (or Wilhelm Ferdinand Ludwig)," that he was born in Magdeburg, Prussia and naturalized in 1848, and that he was fourth wrangler in 1845. Of this Fischer edition there were numerous reprints such as the 83rd edition of Vega in 1912. The first edition of Vega's *Handbuch* was in 1793, and the 19th edition, 1839, was edited by J. A. HÜLSSE. The Italian translation of the Vega-Bremiker *Handbuch* by LUIGI CREMONA (1830–1903), and a French translation were also published by Weidmann in 1857, as well as a Russian translation in 1858.

The volume under review, denuded of every name except that of Vega, is simply Fischer's English translation of Bremiker's edition of Vega's tables with the 16 pages of Bremiker's historical preface eliminated. In 1941 G. E. Stechert & Co. (the forerunner of the Hafner Publishing Co.) published a similar English edition, but with the original German preface and introduction.

The first long table (p. 1–185) is $\log N$, $N = 1(1)10009$, P.P., with S and T tables at bottoms of pages.

The second principal table (p. 187–287) is of $\sin x$ and $\tan x$, $x = 0(1'')5°$.

The third main table (p. 289–559) is of $\sin x$, $\tan x$, $\cot x$, $\cos x$, $x = 0(10'')45°$, Δ.

An Appendix (p. 561–575) contains: Tables for the conversion of (i) sidereal time into mean time; (ii) mean time into sidereal time. Tables of refraction. Constants.

See J. HENDERSON, *Bibliotheca Tabularum Mathematicarum*, Cambridge, 1926, p. 126–127.

R. C. A.

636[F].—G. N. WATSON, "A table of Ramanujan's function $\tau(n)$," London Mathematical Soc., *Proc.*, s. 2, v. 51, 1949, p. 1–13; in the press since 1942. Compare *MTAC*, v. 2, p. 26–27; v. 3, p. 23, 177, 298.

This table gives the coefficients $\tau(n)$ of x^{n-1} in the power series expansion of the 24th power of EULER's product $(1 - x)(1 - x^2)(1 - x^3) \cdots$ for $n = 1(1)1000$. As noted above it has already been discussed in this journal. Its appearance will be very welcome to investigators of this mysterious function since it more than doubles the range of previous tables, described in the above reference. Besides the function $\tau(n)$ the table gives $\tau(n)n^{-\frac{11}{2}}$ to 5D. According to a conjecture of RAMANUJAN this function is less than 2 in absolute value when n is a prime. The table shows that this function with n a prime lies between -1.91881 and $+1.90410$ and attains these values for $n = 103$ and 479 respectively. For the composite number 799 the function is equal to -2.01623.

The method used to construct the table is the same as that described and used by the reviewer.[1] When n is composite short methods of calculation yield $\tau(n)$. When n is a prime a comparatively long recurrence formula, based on Euler's pentagonal number theorem, is used. In this latter case the author used the long calculation for both n and $n + 1$ to ensure accuracy.

D. H. L.

[1] D. H. LEHMER, "Ramanujan's function $\tau(n)$," *Duke Math. Jn.*, v. 10, 1943, p. 483–492. See *MTAC*, v. 1, p. 183–184.

637[G, I, K].—WILLIAM EDMUND MILNE, *Numerical Calculus. Approximations, Interpolation, Finite Differences, Numerical Integration, and Curve Fitting.* Princeton, N. J., Princeton Univ. Press, 1949, x, 393 p. Offset print. 15 × 22.8 cm. $3.75.

This introductory treatise on numerical analysis contains one of the finest and most thorough presentations of the subject by a well-known specialist in that field. In the space of a modestly-sized volume the author succeeds in explaining the essentials of finite differences, interpolation, numerical integration and differentiation, important techniques in smoothing and approximations by least squares, and the solution of simultaneous linear equations, algebraic equations, and difference equations. The text is supplemented by an adequate bibliography and useful tables. Also the presentation of the theory is reinforced with clear illustrative examples and sets of exercises at the ends of the chapters. Besides serving well as a standard text for an undergraduate course in finite differences and numerical methods, this book provides more than an adequate foundation in numerical analysis for those who seek to specialize in applied mathematics or branches of science involving calculation, such as statisticians, computers, actuaries, engineers, physicists and biologists. Furthermore, all mathematicians, even those of the purest type, should be cognizant today of the subject matter in this book because of the enormous development of numerical analysis by present-day electronic computational devices, and its probable influence upon mathematics as a whole. The primary purpose of this book is to enable one with a previous background of only high school mathematics, analytic geometry, and elementary calculus, to bridge the gap between classroom mathematics and the important practical applications where concrete numerical results are required, or to paraphrase the author in his preface, "the gap between knowing of a solution and actually obtaining it." The entire work is written simply, clearly and directly; the treatment is elementary. But besides covering most of the fundamentals of numerical calculation, a number of special topics (e.g., treatment of the remainder in quadrature formulae, alternative treatment of smoothing formulae, exceptional cases in reciprocal differences) are treated with an appealing thoroughness which is also capable of stimulating the reader to further deeper investigations. Even the experienced worker in numerical mathematics, who has in his possession the other standard books of MILNE-THOMSON, SCARBOROUGH, STEFFENSEN, JORDAN, WHITTAKER & ROBINSON, etc., could still find it advantageous to add this treatise to his collection, for its value in ready reference, as a quick refresher, and for its few features that are not in those other texts, but which are otherwise available only in scattered sources.

Chapter **I** treats determinants, systems of linear equations, matrices, and homogeneous equations. Its most noteworthy feature is its exposition of one of the most convenient methods for solving a system of linear equations, the concise scheme of elimination described by P. D. CROUT, which is suited to the evaluation of determinants and calculating the inverse of a matrix, and is thus adapted to solving a number of different sets of equations having the same left members but different right members. There follows a discussion of the magnitude of the inherent error, the uncertainty in the result due to the initial uncertainty in the values of the coefficients of the unknowns, which cannot be reduced by any improvement in the technique of solution.

Chapter **II** deals with the solution of non-linear equations in one or more variables by various methods of successive approximations. For one equation in one unknown, there is described Newton's method and some simpler variations (as a fixed slope m in place of $f'(x_n)$ which is employed at the nth step of the iteration to obtain x_{n+1}). Also the exceptional case, where $f'(x)$ vanishes or is small near the root, is explained. For solving two equations in two unknowns there is given both the extension of Newton's method and successive substitutions, together with a detailed study of the exceptional cases where there may be multiple solutions, distinct solutions close together, or no solution. A procedure is given for finding the complex roots of algebraic equations with real coefficients, by synthetic division by quadratic factors, the end result being the real quadratic factor that yields a pair of complex roots. A very simple iterative scheme is described for the solution of symmetrical λ-determinants whose elements a_{ij} correspond to a positive definite quadratic form. The method enables one to find simultaneously with the λ's, the sets of x_j's corresponding to the different roots λ, which satisfy the system of equations

$$\sum_{j=1}^{n} a_{ij}x_j = \lambda x_i, \quad i = 1, 2, \cdots, n.$$

Chapter **III** introduces the notion of interpolating function of polynomial, rational, or trigonometric type, and then is devoted entirely to polynomial interpolation. AITKEN'S process for reducing any degree polynomial interpolation to a succession of linear interpolations is explained together with NEVILLE'S variation of Aitken's principle. Then, Aitken's method is applied to inverse interpolation. The usual expression for the error in nth degree polynomial approximation in terms of $f^{(n+1)}(x)$ is derived from ROLLE'S theorem. The rest of the chapter contains a detailed account of LAGRANGE'S interpolation formula and its usefulness for functions that are tabulated at equally spaced intervals when tables of Lagrangian interpolation coefficients can be employed.

Chapter **IV**, on numerical differentiation and integration, deals mainly with the latter topic, which is the more important. For differentiation (first derivative only) based upon the approximation by Lagrange's polynomial, an estimate for the error is obtained for the derivative at one of the points of interpolation, and differentiation formulae are given up to the seven-point case. Numerical integration is based also upon Lagrange's polynomial and the method of "undetermined coefficients" is described, which merely means the finding of coefficients by solving a system of linear equations arising from the integration (or differentiation) of a very simple set of polynomials like x^n, $n = 0(1)n - 1$. The investigation of the error is interesting and somewhat advanced, but well worth mastering since the final result has wide application. The author considers the remainder of a quadrature formula $R(f)$, as special cases of operators of degree n, when $R(x^m) = 0$ for $m \leqslant n$, but $R(x^{n+1}) \neq 0$. First $R(f)$ is obtained in the form $\int_{-\infty}^{\infty} f^{(n+1)}(s)G(s)ds$, where $G(s) = (n!)^{-1}R_x[\overline{(x - s)^n}]$, with $\overline{(x - s)^n} = (x - s)^n$ if $x > s$, and $\overline{(x - s)^n} = 0$ if $x < s$, so that $G(s)$ can be evaluated. Under a few mild restrictions $|R(f)| < \max|f^{(n+1)}(s)| \cdot \int_{-\infty}^{\infty}|G(s)|ds$. When $G(s)$ does not change sign, $R(f) = f^{(n+1)}(z)R[x^{n+1}/(n + 1)!]$, least $x < z <$ greatest x. The advantage in this latter form is that $R[x^{n+1}/(n + 1)!]$ is easier to evaluate than $\int_{-\infty}^{\infty} G(s)ds$. This theory is then applied to the trapezoidal rule, to several "trapezoidal rules with corrections," to

Simpson's rule, and finally to the series of Newton-Cotes quadrature formulae. Both closed and open types of Newton-Cotes formulae are tabulated as far as the nine-point case and seven-point case respectively, with a discussion of their merits and the remainder terms.

Chapter **V** is on the numerical solution of differential equations by stepwise methods. A very simple integration formula is used to introduce the subject. A second method employs two formulae, one as a "predictor," and the other as a "corrector." Special formulae are given for second-order differential equations. Simultaneous equations are also touched upon. The short chapter concludes with some useful five-term integration formulae.

Chapter **VI** introduces the subject of finite differences. The fundamental properties of factorial polynomials and binomial coefficient functions precede the definition and illustration of differences, with an indication of the important role of differences in the detection of errors in tabulated functions. Then NEWTON'S interpolation formulae, employing forward and backward differences, are readily derived from the properties of Δ^k and factorial polynomials, which are also employed to give SHEPPARD'S rules. GAUSS'S forward and backward interpolation formulae are obtained here from Sheppard's rules. Then the definitions of central and mean-central differences are introduced, followed by STIRLING'S central difference interpolation formula. From Gauss's forward formula, in a few neat steps, the highly useful and elegant EVERETT central difference interpolation formula is derived. Finally, from Gauss's forward and backward formulae BESSEL'S interpolation formula is derived. The tabulation of polynomials is discussed from the standpoint of building them up from differences. The important role played by differences in subtabulation of a function from interval H to the smaller h is shown from the symbolic formula $\Delta_h^m = [(1 + \Delta_H)^{h/H} - 1]^m$. Formulae for derivatives in terms of differences are found by differentiation of Stirling's and Newton's interpolation formulae. By integrating Newton's interpolation formula, the author derives both LAPLACE'S formula for numerical integration over a single interval, and several more general formulae for integration over a number of intervals, the most noteworthy being GREGORY'S formula where the differences involve only ordinates in the range of integration. Integration of Stirling's formula is employed to yield a number of elegant formulae for $\int_{x_{-n}}^{x_n} f(x)dx$ (symmetric integrals), in terms of f_0 and $\delta^{2m}f_0$ as far as $\delta^{10}f_0$, for $n = 1(1)5$. Finally, Everett's formula is integrated to give the rapidly convergent GAUSS-ENCKE formula, whose differences use ordinates outside the range of integration.

Chapter **VII** is devoted to divided differences, their definition, proof of symmetry in their arguments, and the derivation of the fundamental Newton divided difference formula for polynomial interpolation. Then the flexibility of divided differences in giving a number of different versions of the same interpolating polynomial is shown by means of Sheppard's rule.

Chapter **VIII** treats reciprocal differences and the approximation of functions by rational fractions. The general idea of the reciprocal difference is arrived at from the determinantal form of the interpolating rational fraction. THIELE'S definition of reciprocal differences and his interpolating continued fraction are discussed with great thoroughness. The author shows both symmetry, and the expression of reciprocal differences as a quotient of two determinants. Then, after defining the "order" of a rational fraction, a uniqueness theorem is proven for interpolation by a rational function of given order k for $k + 1$ points. Also there is proved the constancy of the kth order reciprocal differences for irreducible rational fractions of order k, and the converse. The chapter ends with a detailed study of exceptional cases and a sufficient condition for the existence of a unique irreducible fraction of order k, which is determined by $k + 1$ points.

Chapter **IX** deals with the important subject of polynomial approximations by least squares. The normal equations are derived for the unknown coefficients as a necessary criterion for minimizing the sum of the squares of the errors, and with the aid of determinants whose elements are $s_k \equiv \Sigma x_i^k$, these equations are shown always to possess a unique solution. Furthermore, a sufficient condition for this solution to give a minimum, is shown to hold always. When the integral of the squares of the errors, instead of the sum, is to be minimized, the approximating polynomial is given as a sum of LEGENDRE polynomials. The explicit expression for the mth Legendre polynomial $P_m(x)$ is derived from the orthogonality condition $\int_0^1 x^s P_m(x)dx$, $s = 0(1)m - 1$. $P_m(x)$ is given explicitly as far as $m = 8$; it is shown that $\int_0^1 P_m^2(x)dx = (2m+1)^{-1}$; and the roots of $P_m(x)$ are shown to be real, distinct, and between 0 and 1. Since the method of least squares via the solution of the normal equations becomes too laborious beyond the third or fourth degree approximation, a more convenient alternative method utilizes sets of polynomials which possess orthogonality properties relative to sums, entirely analogous to the Legendre polynomials for integrals (first considered by CHEBYSHEV in 1858), and the author develops the subject in a manner similar to that for the Legendre polynomials. The subject of graduation or smoothing of data is treated first from the least squares viewpoint, employing these orthogonal polynomials under the assumptions that the observations of the unknown function $f(x)$ are equally spaced in x, and that $f(x)$ can be represented with sufficient accuracy by a polynomial of degree m over several consecutive values. Explicit formulae are given for $m = 3$ and 5, for 7(2)21 points of observation. An alternative treatment of smoothing formulae, under four reasonable assumptions involving the random errors, leads to those same formulae, but in terms of central differences, together with an estimate of the improvement in the smoothing formula over the use of unsmoothed entries. Gauss's quadrature formula using the zeros of the Legendre polynomials is derived, and the zeros with corresponding weight factors are tabulated for $P_m(x)$, $m = 2(1)9$, to 7D.

Chapter **X** considers other approximations by least squares, not necessarily using polynomials, for both the continuous and discrete cases. After introducing the general problem of least squares approximation, and the role of orthogonal functions, a fundamental theorem is proved which shows that when the "true function" u is assumed to be a sum of orthogonal functions, a function y, obtained as an approximation to an observed function z, is closer to u than z itself. Trigonometric interpolation is developed first from the standpoint of the FOURIER series, whose coefficients minimize the integral of the square of the difference, and then from the standpoint of harmonic analysis whose coefficients (resembling Fourier coefficients, but in terms of sums instead of integrals) minimize the sum of the squares of the difference. A method for computing the coefficients in harmonic analysis is described. The GRAM-CHARLIER approximation for functions approaching zero very rapidly as $|x|$ becomes infinite leads, in the continuous case, to the HERMITE polynomials $H_m(x)$ and the calculation of the coefficients c_i of $f(x)$ as a sum $(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2} \sum_{i=0}^{n} c_i H_i(x)$, from the moments of $f(x)$. This method is applied to obtain the Gram-Charlier approximation to the function $G(s)$ of Chapter **IV**. For the discrete case, with points equally spaced, the author employs $H_t^{(n)}(s) = \Delta_s^t \binom{n-t}{s-t}$, n, s, and t integral, whose orthogonality property $\sum_{k=0}^{n} H_k^{(n)}(s) H_t^{(n)}(k) = 0$ if $s \neq t$, 2^n if $s = t$, makes them suited to least squares approximation.

Chapter **XI** contains introductory material on simple difference equations. First the author distinguishes between particular discrete solutions, particular continuous solutions, and general solutions. Then several pages are devoted to a list of differences of functions, to be used in solving the difference equation $\Delta u_s = f(s)$, where $f(s)$ might be a polynomial, rational fraction, exponential, trigonometric, or logarithmic function. It is shown how certain difference equations are converted into exact equations by multiplication by a suitable factor. For homogeneous linear difference equations of order higher than the first, the solution by the substitution $u_s = a^s$, depends upon the solution of an algebraic equation in a. Also, it is shown how to solve a few cases of nonhomogeneous linear difference equations with constant coefficients, by means of undetermined coefficients. The author concludes by touching on the solution of linear equations, with variable coefficients, by means of factorial series, and the vanishing of all coefficients of factorials of like degree by virtue of the given difference equation.

Appendix **A** clarifies notation and symbols. Appendix **B** lists 21 reference texts, 13 tables available in book form, 9 tables in journals, and 4 bibliographies. Appendix **C** furnishes a classified guide to formulae and methods in this book. The section on **Tables** contains: T. I. Binomial coefficients $\binom{n}{k}$, $k = 0(1)10$, $n = 1(1)20$. T. II. Interpolation coefficients $\binom{s}{k}$ for Newton's binomial interpolation formula, $k = 2(1)5$, $s = [0(.01)1; 5D]$. T. III. Everett's interpolation coefficients $\binom{s+1}{3}$, $\binom{s+2}{5}$, $s = 0(.01)1$; 5D. T. IV. Lagrange's coefficients for five equally spaced points, $L_i(s)$, $i = -2(1)2$, $s = [0(.01).5; 6D]$. T. V. Legendre's polynomials $P_m(x)$ normalized to the interval $0 \leqslant x \leqslant 1$, $m = 1(1)5$, $x = 0(.01)1$; $m = 1, 2$, exactly; $m = 3, 4, 5$, 5D. T. VI. Orthogonal polynomials $P_j(s)$, (for sums, properties in Chapter **IX**), for $n + 1$ equally spaced points, given exactly for $j = 1(1)5$, $n = 5(1)20$, $s = 0(1)n$. T. VII. Integrals of binomial coefficients $\int_0^s \binom{t}{k}dt$, $k = 0(1)9$, $s = -1, 1(1)8$; exactly. T. VIII. Gamma function $\Gamma(x + 1)$ and digamma function $d \ln \Gamma(x + 1)/dx$ (also known as the "psi function"), for $x = [0(.02)1; 5D]$.

HERBERT E. SALZER

NBSCL

638[H, L].—R. WESTBERG, "On the harmonic and biharmonic problems of a region bounded by a circle and two parallel planes," *Acta Polytechnica*, Stockholm, no. 18, 1948, 66 p. 17.5 × 24.6 cm. Also Physics and Appl. Math. series v. 1, no. 3. Also as Ingeniörsvetenskapsakademiens *Handlingar*, no. 197.

On p. 61 are tables of $I_m = \int_0^\infty t^m dt/\Sigma$, 5S; $J_m = \int_0^\infty t^m e^{-2t}dt/\Sigma$, 1–5S; $K_m = I_m + 2I_{m+1} - J_m$, 5–6S; $m = 1(2)15$, $\Sigma = \sinh 2t + 2t$. This table is an abridgement of one given by HOWLAND [1] for $m = 1(1)20$.

On p. 64 are given 9D values of the first 11 zeros[2] z_n, $n = 0(1)10$, of $\sinh z + z$, each to 9D. For example:

$$z_0 = 2.250728611 + i\,4.212392231$$
$$z_{10} = 4.907438417 + i\,67.471628635$$

Extracts from text

[1] R. C. J. HOWLAND, "On the stresses in the neighborhood of a circular hole in a strip under tension," R. Soc. London, *Trans.*, v. 229A, 1930, p. 67; correction v. 232A, 1933, p. 169.

[2] The first four zeros were given, 4–5S, by F. SEEWALD, "Die Spannung und Formänderungen von Balken mit rechtwinkligem Querschnitt," Aerodynam. Inst. Aachen, *Abhand.*, Heft 7, 1927, p. 16. These values differ in some cases in the third figure from those obtained by the present author.

639[I].—HERBERT E. SALZER, "Coefficients for facilitating trigonometric interpolation," *Jn. Math. Phys.*, v. 27, p. 274–278, 1949. 17.4 × 25.3 cm.

The problem of expressing the trigonometric sum,

$$f(x) = C_0 + (C_1 \cos x + S_1 \sin x) + \cdots + (C_n \cos nx + S_n \sin nx),$$

so that $f(x)$ assumes given values, $f_0, f_1, \cdots, f_{2n}$, when x assumes the values, $x_0, x_1, \cdots, x_{2n}$, leads to what is commonly called Gauss's formula for trigonometric interpolation, namely,

$$f(x) = \sum_{i=0}^{2n} \Pi'^{2n}_{j=0} \sin \tfrac{1}{2}(x - x_j) f_i / \Pi'^{2n}_{j=0} \sin \tfrac{1}{2}(x_i - x_j).$$

The symbol Π' has its customary significance that the factor corresponding to $i = j$ is omitted from the product.

The present paper is concerned with the tabular values of the coefficients,

$$A_i^{(2n+1)} = 1/\Pi'^{2n}_{j=0} \sin \tfrac{1}{2}(x_i - x_j).$$

If the values of x_i are equally spaced, then the coefficients satisfy the relationship,

$$A_i^{(2n+1)} = A_{2n-i}^{(2n+1)}.$$

The author describes his tables as follows: "Coefficients $A_i^{(2n+1)}$ are given for the 3-, 5-, 7-, 9-, and 11-point cases, all at intervals in x equal to 1, .5, .2, .1, .05, .02, and .01; also

$A_i^{(2n+1)}$ are given for functions tabulated at $2n + 1$ equally spaced points over a range of π and $\frac{1}{2}\pi$, for $2n + 1 = 3(2)11$, since those ranges, i.e., 180° and 90°, are important for many periodic functions. All the quantities $A_i^{(2n+1)}$ are given to eight significant figures."

H. T. D.

640[J].—E. H. Copsey, H. Frazer, & W. W. Sawyer, (a) "Empirical data on Hilbert's inequality," *Nature*, v. 161, 6 March 1948, p. 361. (b) "A research project," *Math. Gazette*, v. 32, May 1948, p. iii–iv.

The problem of Hilbert's inequality is discussed in detail in *MTAC*, v. 3, p. 399–400, where the results of (a) and (b) are set forth. The tables of (a) and (b) are of the largest latent root of the matrix of the nth order for which

$$a_{ij} = (i + j - 1)^{-1}$$

(a) gives this root λ_n to 9D for $n = 1(1)5, 10, 20$ while in (b) will be found λ_n to 5D for $n = 1(1)20$.

D. H. L.

641[K, L].—Zdeněk Kopal, "A table of the coefficients of the Hermite quadrature formula," *Jn. Math. Phys.*, v. 27, p. 259–261, Jan. 1949. 17.5 × 25.3 cm. Compare *MTAC*, v. 1, p. 152–153, v. 3, p. 26.

A Gauss-type quadrature formula for an approximate evaluation of definite integrals with doubly infinite limits appears to have first been established by Gourier,[1] who proved that if $f(x)$ is a function of degree not in excess of $2n - 1$,

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx = 2^{n+1}n!\pi^{\frac{1}{2}} \sum_{i=1}^{n} f(x_i)/[H_n'(x_i)]^2, \tag{1}$$

where $H_n(x)$ denotes the polynomial defined by

$$H_n(x) = e^{x^2}d^n(e^{-x^2})/dx^n, \tag{2}$$

and the x_i's in (1) are roots of the Hermite polynomials of nth order (2). Numerical values of the Christoffel numbers

$$p_i = 2^{n+1}n!/[H_n'(x_i)]^2 \tag{3}$$

for $n = 2(1)4$ were given by Berger,[2] and a more complete set of 7D values corresponding to $n = 2(1)9$ was later completed by Reiz;[3] the latter's paper was the first one in which the respective Christoffel numbers were given in decimal form.

In certain computations performed recently at the MIT, a need arose for the Christoffel numbers of the quadrature formula (1) corresponding to $n = 10(1)20$. Since their values do not appear to have been evaluated before and are apt to be frequently needed in the future, a 6D table for x_i and a 6–13D table for p_i in this range are given.

In computing the Christoffel numbers by equation (3), use has been made of values of the roots of the Hermite polynomials published previously by Smith;[4] their accuracy of 6D imposed the limit to the accuracy with which the corresponding Christoffel numbers could be evaluated. The error of no published value of p is expected to exceed one unit of the last place.

Extracts from the text

[1] G. Gourier, Acad. d. Sciences, Paris, *Comptes Rendus*, v. 97, 1883, p. 79–82.
[2] A. Berger, K. Vetenskaps Societen i Upsala, *Nova Acta*, s. 3, v. 16, no. 4, 1893, p. 3.
[3] A. Reiz, *Arkiv f. Math. Astr. och Fysik*, v. 29A, no. 29, 1943, p. 6.
[4] E. R. Smith, *Amer. Math. Mo.*, v. 43, 1936, p. 354.

642[K, L, M].—J. BARKLEY ROSSER, *Theory and Application of* $\int_0^z e^{-x^2}dx$ *and* $\int_0^z e^{-p^2y^2}dy \int_0^y e^{-x^2}dx$. *Part I. Methods of Computation.* Brooklyn 4, N. Y., Mapleton House, 5415 Seventeenth Ave., 1948, iv, 192 p. 13.8 × 21.6 cm. $8.00. Offset print, bound in cloth.

This is a second edition of the quarto-format report, published in November 1945, which has been already reviewed in *MTAC*, v. 2, p. 213f. In smaller format the reprint is practically identical with the original, and the smaller number of pages was made possible by five times putting the material of two pages of the first edition on one in the second.

The volume contains a survey of various methods of calculating the integrals mentioned in the title. Known methods are discussed and new ones developed. The accuracy of various methods is subjected to a painstaking analysis. The author's attention is focussed on complex values of the variables, and a great number of new asymptotic developments are derived.

The book contains two tables, each covering four functions. Put

$$Rr(u) = \tfrac{1}{2}\cos\tfrac{1}{2}\pi u^2 - \tfrac{1}{2}\sin\tfrac{1}{2}\pi u^2 + \int_0^u \sin\tfrac{1}{2}\pi(u^2 - x^2)dx,$$

$$Ri(u) = \tfrac{1}{2}\cos\tfrac{1}{2}\pi u^2 + \tfrac{1}{2}\sin\tfrac{1}{2}\pi u^2 + \int_0^u \cos\tfrac{1}{2}\pi(u^2 - x^2)dx.$$

Table 1 gives values, to 12D, for $Rr(u)$, $Ri(u)$, $Rr^2(u) + Ri^2(u)$, and $\int_0^u Rr(u)dx$. The range covered is $-.06(.02) + 3(.05)5.15$, except for the integral, for which the last three entries are missing.

Table 2 covers the following four functions to 10D:

$$e^{-w^2}\int_0^w e^{y^2}dy, \quad e^{w^2}\int_w^\infty e^{-y^2}dy, \quad \int_0^w e^{-y^2}dy\int_0^y e^{x^2}dx, \quad \int_0^w e^{y^2}dy\int_y^\infty e^{-x^2}dx.$$

The first is for $w = -.2(.05)4(.1)6.5(.5)12.5$. The second is for $w = -.2(.05)3.8(.1)6.3$; the third and fourth are for $w = -.2(.05) + 3.5(.1)6$.

The book reviewed is "Part I." Portions of Part II, which are unrestricted, have been expanded and completely rewritten and now appear as chapters III and IV of *Mathematical Theory of Rocket Flight* by J. B. ROSSER, ROBERT R. NEWTON, & GEORGE L. GROSS, New York, McGraw-Hill, 1947, viii, 276 p.

WILL FELLER

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643[L].—J. DESCHODT, *Arcs-Aires-Volumes, Centres de Gravité, Moments d'Inertie.* Paris, Office National d'Études et de Recherches Aéronautiques (ONERA), 3, rue Léon-Bonnat, 1948, Div. no. 3, vi, 73 p. 21.2 × 27 cm. Offset print from ms.

A useful summary of formulae indicated by the title. Each formula is accompanied by a figure clearly indicating the meaning of every formula element. There are more than seventy formulae for moments of inertia of plane surfaces and of solids.

644[L].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 11: *Tables of the Bessel Functions of the First Kind of Orders Forty through Fifty-One.* Cambridge, Mass., Harvard Univ. Press, 1948, x, 620 p. The volume is dated 1948, although not published until March 1949. 19.5 × 26.7 cm. $10.00. Compare *MTAC*, v. 2, p. 176f, 261f, 344; v. 3, p. 102, 117–118, 185–186, 367.

This is the ninth of the thirteen planned volumes of the monumental edition of tables of Bessel functions, up to and including $J_{100}(x)$, prepared at the Harvard Computation Laboratory. Practically all of the results set forth in this volume are entirely new. We are given $J_{40}(x) - J_{51}(x)$ for $x = [18.38(.01)99.99;\ 10D]$; the first significant values, .00000 00001, for $J_{51}(x)$ are when $x = 26.75(.01)27.40$ inclusive.

In K. HAYASHI, *Tafeln der Besselschen, Theta-, Kugel-, und anderer Funktionen*. Berlin, 1930, are the following values of $J_n(x)$: $n = 40$–51, $x = 20$, and at least 23D; $x = 30$, and at least 27D; $x = 40$, and at least 32D; $x = 50$, and at least 29D.

The values of Hayashi, rounded to 10D, in every case agree with those in the Harvard volume.

JOHN A. HARR designed the sequence control tapes, supervised the computation, and prepared the manuscript for publication.

R. C. A.

645[L].—G. G. MACFARLANE, "The application of Mellin transforms to the summation of slowly convergent series," *Phil. Mag.*, s. 7, v. 40, Feb. 1949, p. 188–197. 17.2 × 25.2 cm.

Many problems in applied mathematics are reduced to the numerical calculation of a series. To those who have spent any time with such problems, it is common knowledge that the series oftentimes converge slowly in some sense or other. That is, the series might be of the nature

$$(1) \qquad \sum_{n=1}^{\infty} 1/n^{\alpha}, \quad 1 < \alpha < 2$$

and the time necessary to sum such a series to a given number of significant figures can be prohibitive. On the other hand for such a simple series as (1) (which is, incidently $\zeta(\alpha)$, where ζ is the Riemann zeta function), the method of Euler serves quite adequately. The author discusses here a method which is particularly useful for the summing of series of the form

$$(2) \qquad \sum_{n=0}^{\infty} f[(n + a)^{\nu}]$$

presuming, of course, that $f(x)$ is sufficiently well behaved to accept the Mellin transform and that the interchange of summation and integration at certain stages of his procedure is permissible.

Subject to appropriate conditions, the Mellin transform pair is

$$(1) \qquad F(s) = \int_0^{\infty} f(x) \cdot x^{s-1} dx$$

and

$$(2) \qquad f(x) = (2\pi i)^{-1} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) x^{-s} ds, \quad \sigma_1 < \sigma < \sigma_2$$

where $\sigma = Re(S)$ and the σ_1 and σ_2 define the abscissae of convergence of the integral (1). [See E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, Oxford, 1937]. From (2) we note that for n integer

$$f(n + a) = (2\pi i)^{-1} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s)(n + a)^{-s} ds$$

and hence

$$(3) \qquad \sum_{n=0}^{\infty} f(n + a) = (2\pi i)^{-1} \int_{\sigma - i\infty}^{\sigma + i\infty} F(s) \zeta(s, a) ds, \quad \sigma_1 < \sigma < \sigma_2,$$

where

$$\zeta(s, a) = \sum_{n=0}^{\infty} (a + n)^{-s}, \quad Re(s) > 1$$

is the generalized Riemann zeta function. It is, of course, assumed that interchange of sum and integral is permissible. By various devices known to those versed in the methods of contour integration and by use of the properties of $\zeta(s, a)$ it is possible to do something about the evaluation of the integral in equation (3) for a known function $f(x)$.

The author considers three examples of this procedure. First, the well-known sum

$$\sum_{n=1}^{\infty} (\cos ny)/n^2$$

is treated. Here $f(x) = \cos x$ and hence

$$F(s) = 2^{s-1}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}s)/\Gamma(\tfrac{1}{2} - \tfrac{1}{2}s). \quad 0 < \sigma < 1$$

Therefore $\cos x/x^2$ has the transform

$$F(s) = 2^{s-3}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}s - 1)/\Gamma(\tfrac{3}{2} - \tfrac{1}{2}s). \quad 2 < \sigma < 3$$

Hence

$$\sum_{n=1}^{\infty} (\cos ny)/(ny)^2 = (2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty} 2^{s-3}\pi^{\frac{1}{2}}\Gamma(\tfrac{1}{2}s - 1)y^{-s}\zeta(s)ds/\Gamma(\tfrac{3}{2} - \tfrac{1}{2}s), \quad 2 < \sigma < 3.$$

Since the integrand has simple poles at $s = 0$, 1 and 2 and behaves properly at infinity, we get

$$\sum_{n=1}^{\infty} (\cos ny)/(ny)^2 = \pi^2/(6y^2) - \pi/(2y) + \tfrac{1}{4}.$$

A second series is

$$\sum_{n=0}^{\infty} (-1)^n(2n + 1)^{-1} \quad J_1[(2n + 1)y], \tag{4}$$

where $J_1(y)$ is the finite Bessel function of the first order. Here we get for $y < 1$, the rapidly convergent series

$$-\sum_{n=0}^{\infty} \frac{(-)^n}{2} \frac{B_{2n+1}(\frac{1}{4})(2y)^{2n+1}}{(2n + 1)\Gamma(n + 1)\Gamma(n + 2)}.$$

The $B_n(a)$ are the Bernoulli polynomials of order n.

A final example is

$$\sum_{m=1}^{M} (1 - xm^{\frac{2}{3}})^{\frac{1}{2}}/m^{\frac{1}{3}},$$

where M is the largest integer such that $xM^{\frac{2}{3}} < 1$. In this case we get an asymptotic series for small x, that is M large. The answer in this case is expressed in terms of $\zeta(s, 1)$ and powers of x. The technique employed in the second and third examples is similar to the one employed in the first one. Observe that the summing of the series (4) involves terms which are

$$O[(2m + 1)^{-\frac{3}{2}}(-)^m \cos \{(2m + 1)y - 3\pi/4\}]$$

for m sufficiently large and hence great difficulties would be encountered in getting the sum of the series directly from (4).

The note ends with a table of 72 Mellin transforms which, of course, is equivalent to a table of bilateral Laplace transforms under proper substitution.

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Editorial Note: The origin of the transforms of Robert Hjalmar Mellin (1854–1933) is indicated on p. 7 of the work of Titchmarsh referred to above. The idea of the reciprocity exhibited in (1)–(2) above, occurs in the famous memoir on prime numbers by G. F. Bernhard Riemann (1826–1866), "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse," Preuss. Akad. d. Wissen. zu Berlin, *Monatsberichte*, 1859, p. 671–680; Riemann, *Werke*, Leipzig, 1876, p. 136–144. It was formulated explicitly by Eugène Cahen (1865–) in his doctoral diss., "Sur la fonction $\zeta(s)$ de Riemann et sur des fonctions analogues," *Ann. de l'École Norm. Sup.*, s. 3, v. 11, 1894, p. 75–164. But the first accurate discussion was given by Mellin: (i) "Ueber die fundamentale Wichtigkeit des Satzes von Cauchy für die Theorien der Gamma- und der hypergeometrischen Funktionen," Finska

Vetenskaps Societeten, *Acta*, v. 21, no. 1, 1896, p. 1–115; (ii) "Über den Zusammenhang zwischen den linearen Differential- und Differenzengleichungen," *Acta Math.*, v. 25, 1902, p. 139–164.

646[L].—K. MITCHELL, "Tables of the function $\int_0^x - \log|1 - y|dy/y$, with an account of some properties of this and related functions," *Phil. Mag.*, s. 7, v. 40, Mar. 1949, p. 351–368. 17.2 × 25.6 cm. Compare *MTAC*, v. 1, p. 189, 457–459; v. 2, p. 180, 278.

Let $f(x) = \int_0^x - \ln|1 - y|dy/y = - Rl(1 - x) = \zeta(1, 2|x), Rl(x) = \int_1^x \ln tdt/(t - 1) = - f(1 - x)$. Some properties of an integral of this type have been given by POWELL [1] with a table of the function for $x = [0(.01)2(.02)6; 7D]$. An earlier table, NEWMAN,[2] gives the function $Rl(1 \pm x)$, for $x = [0(.01).5; 12D]$, and a comparison of these two tables over their common range (FLETCHER [3]) reveals discrepancies which are traced to errors in Newman's table. Powell's table, over the common range, contains only rounding errors in the last decimal place. Another table (SPENCE,[4] 1809) gives $Rl(x)$, $x = [1(1)100; 9D]$; and yet another (KUMMER,[5] 1840), gives a function akin to $f(x)$ for $x = [- 11(1) + 10; 11D]$. These tables offer only isolated points of comparison.

The author's attention was drawn to the function by a physical problem in 1940, and the tables of $f(x)$, for $x = [- 1(.01) + 1; 9D]$, $x = [0(.001).5; 9D]$ were calculated and are given in this paper, p. 357–363. Comparison of the author's table with that of Powell reveals one error in the latter; at $x = .01$ (Powell's variable), *for* 1.5886 274, *read* 1.5886 254. There are several other rounding errors in the last decimal place, which have not been separately listed. Errors in Newman's table are listed by Fletcher.

Extracts from text

[1] E. O. POWELL, *Phil. Mag.*, s. 7, v. 34, 1943, p. 600–607.
[2] F. W. NEWMAN, *The Higher Trigonometry. Superrationals of Second Order.* Cambridge, 1892.
[3] A. FLETCHER, *Phil. Mag.*, s. 7, v. 35, 1944, p. 16–17.
[4] W. SPENCE, *An essay on the Theory of the Various Orders of Logarithmic Transcendents*, London, 1809, p. 24.
[5] E. E. KUMMER, *Jn. f. d. r. u. angew. Math.*, v. 21, 1840, p. 74–90, 193–225, 328–371.
EDITORIAL NOTE. The table in question is on p. 88 and the function is $\Lambda(x) = \int_0^x \ln(1+t)dt/(1 + t)$. $\Lambda(x) = 0$, for $x = 4.50374185563$.

647[L].—NBSCL, *Tables of Bessel Functions of Fractional Order. Volume II.* New York, Columbia University Press, 1949, xviii, 365 p. Offset print. 20 × 26.6 cm. $10.00. The foreword by Prof. R. E. LANGER occupies p. vii–x. Introduction, p. xiii–xvii, by MILTON ABRAMOWITZ. Compare *MTAC*, v. 1, p. 93, 300; v. 3, p. 187, 339.

The present volume, devoted to the tabulation of $I_\nu(x)$ for $\pm \nu = \frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$, is a sequel to the volume of 1948 containing $J_n(x)$ for the same orders. The functional values in both volumes are given either to 10D or 10S. The interval in the argument has been so chosen that interpolation with the aid of the tabulated second central differences will yield the maximum attainable accuracy over most of the range covered. In some regions, where this desideratum is not met, fourth central differences are also given; in such regions it is always possible to obtain an accuracy of at least 7S by Everett's interpolation formula involving only second differences. Interpolation is not feasible close to the origin (for $x < .05$), and therefore in this region the auxiliary function $x^{-\nu}I_\nu(x)$ has been tabulated, together with its second central differences.

The tables of $I_\nu(x)$ cover a range of x from 0 to 25. The function $e^{-x}I_\nu(x)$ is tabulated for $x = 25(.1)50(1)500(10)5000(100)10000(200)30000$. With the aid of these values it is possible to compute $I_\nu(x)$ in this range of x. For $x > 30000$ an accuracy of at least 9S can be obtained with the first two terms of the asymptotic expressions for $I_\nu(x)$.

Values of $I_{-\nu}(x)$ are tabulated up to $x = 13$ only, since for larger values of x they are identical with those of $I_\nu(x)$ to 10S. For $\nu = -\frac{3}{4}, -\frac{2}{3}$, in tables of $I_\nu(x)$, $x = 0(.001)1(.01)13$;

for $\nu = -\frac{1}{3}, -\frac{1}{4}$, $x = 0(.001).8(.01)13$; for $\nu = \frac{1}{4}, \frac{1}{3}$, $x = 0(.001).6(.01)25$; for $\nu = \frac{2}{3}, \frac{3}{4}$, $x = 0(.001).5(.01)25$.

On p. xviii are 15D values of constants including 16 involving the gamma function. Tables of Everett Interpolation Coefficients are given p. 333–343, and of $L_\nu(\mu)$ for interpolation in the ν direction, p. 345–365.

Extracts from text

EDITORIAL NOTE: Except for small unreliable tables by DINNIK these valuable tables are the first published tables of their kind.

648[L].—NBSINA, *Tables of* $I_0(2\sqrt{x})$, $I_1(2\sqrt{x})/\sqrt{x}$, $K_0(2\sqrt{x})$, $K_1(2\sqrt{x})/\sqrt{x}$ *and Related Functions.* $0 \leqslant x \leqslant 410$. Computed under the direction of Dr. GERTRUDE BLANCH, Institute for Numerical Analysis, U.C.L.A., Los Angeles, California, February 1949. v, 26 leaves, with text on only one side. Our leaf numbers do not agree with those of the text because there are no leaves ii or 1 in the original. 20.17 × 35.7 cm. These tables are a sequel to the tables reviewed in RMT **505**, v. 3, p. 107.

T. I is of $I_0(2x^{\frac{1}{2}})$, $I_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}}$, for $x = [0(.02)1.5(.05)6.2$; 8S or 9S], δ^2, second central differences. T. II. $e^{-2x^{\frac{1}{2}}}I_0(2x^{\frac{1}{2}})$, $e^{-2x^{\frac{1}{2}}}I_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}}$, for $x = [6.2(.1)13(.2)36(.5)115(1)160(5)410$; 7S or 8S], δ^2 or modified δ^2. T. III. $K_0(2x^{\frac{1}{2}})$, $K_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}}$, for the same ranges as T. I, and 7S or 8S, δ^2 or modified δ^2, except for a small region near the origin. T. IV. $e^{2x^{\frac{1}{2}}}K_0(2x^{\frac{1}{2}})$, $e^{2x^{\frac{1}{2}}}K_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}}$, for the same ranges as T. II, and 7S or 8S, δ^2 or modified δ^2.

For large values of the argument, $I_0(y)$ and $I_1(y)$ are of the order of magnitude of e^y, while $K_0(y)$ and $K_1(y)$ behave like e^{-y}. For $x \geqslant 6.2$ (T. II and III) the tabulated functions have been multiplied by appropriate exponential factors in order to retain the interpolable character of the table.

Let $u = F_n(x)$ or $G_n(x)$, where $F_n(x) = I_n(2x^{\frac{1}{2}})/x^{\frac{1}{2}n}$; $G_n(x) = K_n(2x^{\frac{1}{2}})/x^{\frac{1}{2}n}$, and $I_n(t)$ and $K_n(t)$ are Bessel Functions of order n. The u satisfies the differential equation

$$x\,d^2u/dx^2 + (n+1)du/dx - u = 0.$$

If, in $F_n(x)$ and $G_n(x)$, the functions $I_n(2x^{\frac{1}{2}})$ and $K_n(2x^{\frac{1}{2}})$ are replaced by $J_n(2x^{\frac{1}{2}})$ and $Y_n(2x^{\frac{1}{2}})$, respectively, we obtain $w_n(x)$ which satisfies the Bessel-Clifford differential equation (see RMT **505**)

$$x\,d^2w/dx^2 + (n+1)dw/dx + w = 0.$$

The functions $K_0(2x^{\frac{1}{2}})$ and $K_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}}$ have singularities at $x = 0$, and interpolation in the region close to the origin, where differences are not given, can be performed more advantageously in at least two other available tables, namely:

(a) *Tables of the Bessel Functions* $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, 1948, for $0 \leq x \leq 1$, 7S in $K_0(x)$, $K_1(x)$; *MTAC*, v. 3, p. 187.

(b) BAASMTC, *Mathematical Tables*, v. VI, Part I, 1937; auxiliary functions E_0, F_0, E_1, F_1, $x = [0(.01).5$; 8D]; *MTAC*, v. 1, p. 361–363.

For $x < 100$ the entries were computed by interpolation from (b). A few entries near $x = 0$ in T. III were computed otherwise.

In the following regions there may be an error of two units in the last place [the pages are those as corrected]:

Table I. $I_0(2x^{\frac{1}{2}})$, entries on p. 1, 3–11; $x^{-\frac{1}{2}}I_1(2x^{\frac{1}{2}})$, p. 11.

Table III. Entries on p. 14, both functions.

Table IV. All entries of $e^{2x^{\frac{1}{2}}}K_0(2x^{\frac{1}{2}})$, and entries of

$$e^{2x^{\frac{1}{2}}}K_1(2x^{\frac{1}{2}})/x^{\frac{1}{2}} \text{ on p. 22–24, } x < 100.$$

All other entries should be correct to within a unit of the last place, and for $x > 100$ to within .6 units.

Extracts from text

649[L].—Lidia Stankiewicz, "Sul calcolo della piastra poggiata su suolo elastico," Accad. Naz. d. Lincei, *Atti, Rendiconti, cl. d. sci. fis. matem. e nat.*, s. 8, v. 5, Dec. 1948, p. 339–344. 18 × 26.6 cm.

$F(x, y) = \int_0^\infty \int_0^\infty \sin(xu) \sin(yv) du\, dv/[uv\{(u^2 + v^2)^2 + 1\}]$ is here tabulated for x and $y = [0(.5)8(1)12, \infty; 3D]$.

650[L, M].—David L. Arenberg & Doris Levin, *Table of Fresnel Integrals and Derived Functions.* Naval Research Laboratory Field Station, 470 Atlantic Avenue, Boston, Mass., [1948], [15] leaves, hektographed. 20.5 × 26.7 cm. Not available for general distribution.

In this publication are given 4D values of

$$C(u) = \int_0^u \cos(\tfrac{1}{2}\pi x^2)dx, \quad S(u) = \int_0^u \sin(\tfrac{1}{2}\pi x^2)dx,$$

for $u = 0(.1)20$, and for $u = 8(.02)16$. The values of $C(u)$ and $S(u)$ for $u \leq 8$ were taken from the tables of Sparrow.[1] For $u > 8$ computations were made by means of approximate formulae from the semi-convergent series given by Watson:[2]

$$C(u) \approx \tfrac{1}{2} + (2\pi z)^{-\frac{1}{2}}[\sin z - (2z)^{-1} \cos z], \tag{1}$$

$$S(u) \approx \tfrac{1}{2} - (2\pi z)^{-\frac{1}{2}}[\cos z + (2z)^{-1} \sin z], \tag{2}$$

where $z = \frac{1}{2}\pi u^2$. For $z > 50$, (1) and (2) are accurate to ± 1 in the fourth decimal place. All other functions of $C(u)$ and $S(u)$ will have corresponding errors.

For $u = 0(.1)20$, there are 4D tables of $[C(u)]^2$, $[S(u)]^2$, $R_{00}(u) = \{[C(u)]^2 + [S(u)]^2\}^{\frac{1}{2}}$, and $R_{\frac{1}{2}\frac{1}{2}}(u) = \{[C(u) - \frac{1}{2}]^2 + [S(u) - \frac{1}{2}]^2\}^{\frac{1}{2}}$; and 5D or 6D tables of $[C(u) - \frac{1}{2}]^2$ and $[S(u) - \frac{1}{2}]^2$. The function $R_{\frac{1}{2}\frac{1}{2}}(u)$ was computed mainly to obtain an independent check of the consistency of the calculations, since this is a monotonic function giving the relative distance between points on the Euler spiral (often called after Cornu) and the limiting value $C(\infty) = S(\infty) = \pm \frac{1}{2}$. By taking differences between the tabulated values of $R_{\frac{1}{2}\frac{1}{2}}(u)$, one easily detected flagrant discrepancies in $S(u)$ and $C(u)$.

These tables were computed to meet the need of more extended tables than those already available.

Extracts from introductory text

[1] C. M. Sparrow, *Table of Fresnel Integrals*, Rouss Physical Laboratory, University of Virginia, 1934.

Editorial Note: This publication, photo-lithoprint reproduction of author's manuscript by Edwards Bros., Ann Arbor, Mich., contains ii, 9 p. (21 × 27.1 cm.), and the tables are of $C(u)$ and $S(u)$, for $u = [0(.005)8; 4D]$. The first paragraph of the text by the author, Carroll Mason Sparrow, 1880–1946, is as follows: "The greater part of the following table was made some years ago in connection with a study of imperfect gratings; existing tables not being adequate by reason of their too large interval. This manuscript table was resurrected and slightly extended to meet a teaching need, and is here reproduced in the hope that it may prove of some use to others." The Arenberg & Levin tables of $C(u)$ and $S(u)$, for $u \leq 8.5$ are identical with those given in Jahnke & Emde, *Tables of Functions*, 1945, p. 34. In no previously published tables has u been greater than 8.5.

[2] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, second ed., Cambridge and New York, 1944, p. 545.

651[L, M].—Harvard University, Computation Laboratory, *Annals*, v. 18, 19: *Tables of Generalized Sine- and Cosine-Integral Functions*, Parts I and II. Cambridge, Mass., Harvard Univ. Press, 1949, xxxviii, 462, viii, 560 p. 20 × 26.7 cm. $10.00 + $10.00.

The Computation Laboratory has here undertaken to present as complete and useful a table of certain integrals as can be included within the scope of 1000 pages. The integrals are

$$(1)\quad \begin{cases} S(a,x) = \int_0^x \sin u\, dx/u, \quad C(a,x) = \int_0^x (1-\cos u)dx/u, \\ \bar{C}(a,x) = \int_0^x \cos u\, dx/u = \sinh^{-1}(x/a) - C(a,x), \\ Ss(a,x) = \int_0^x \sin u \sin x\, dx/u, \quad Sc(a,x) = \int_0^x \sin u \cos x\, dx/u, \\ Cs(a,x) = \int_0^x \cos u \sin x\, dx/u, \quad Cc(a,x) = \int_0^x \cos u(1-\cos x)dx/u, \\ \bar{C}c(a,x) = \int_0^x \cos u \cos x\, dx/u = \sinh^{-1}(x/a) - C(a,x) - Cc(a,x), \end{cases}$$

where $u = (x^2 + a^2)^{\frac{1}{2}}$. Clearly, tables of these functions need include only one of the integrals C, $\bar{C}$, and only one of the integrals Cc, $\bar{C}c$; C and Cc were chosen because they afford much better interpolation near the origin.

The tabulation extends over a set of points (a, x) within the square $0 \leq a \leq 25$, $0 \leq x \leq 25$. The set was so chosen as to provide for interpolation in 6D tables within as large as possible a portion of the square, and yet permit the tables to be encompassed in two volumes.

These volumes are arranged in sections, each section consisting of the complete table, ordered by ascending x, for one value of the parameter a. The sections (about 350) are in turn ordered by ascending a, and those sections corresponding to $0 \leq a < 2$ are contained in Part I, while those corresponding to $2 \leq a \leq 25$ are in Part II.

When $a = 0$, the integrals $S(a, x)$, $C(a, x)$ reduce to the sine-integral and cosine-integral functions

$$Si(x) = \int_0^x \sin x\, dx/x, \quad Ci(x) = \int_0^x \cos x\, dx/x.$$

These functions are classical, thorough studies having been made [1] by 1906. Further they have been exhaustively tabulated.[2] On the other hand, for non-zero a the previously existing tabulations are fragmentary and inadequate.[3–6] The present volumes were prepared with the thought in mind that the class of tabulated functions should be augmented to include the composite functions, that 6D accuracy should be provided, that the domain of tabulation should be large enough to cover the cases that arise in practice, and that the mesh should be fine enough to admit interpolation to the accuracy that the applications demand.

When $a = 0$,

$$\begin{aligned} S(0,x) &= Si(x), \quad C(0,x) = \gamma + \ln x - Ci(x), \\ Sc(0,x) &= \tfrac{1}{2}S(0,2x) = \tfrac{1}{2}Si(2x), \quad Cs(0,x) = \tfrac{1}{2}S(0,2x) = \tfrac{1}{2}Si(2x), \\ Ss(0,x) &= \tfrac{1}{2}[\ln 2 + \gamma + \ln x - Ci(2x)] = \tfrac{1}{2}C(0,2x), \\ Cc(0,x) &= \tfrac{1}{2}[\ln 2 - \gamma - \ln x - Ci(2x)] + Ci(x) = \tfrac{1}{2}C(0,2x) - C(0,x), \end{aligned}$$

where γ is Euler's constant.

When $a \neq 0$, it may be shown that

$$\begin{aligned} Sc(a,x) &= \tfrac{1}{2}[Si(z) - Si(y)], \quad Cs(a,x) = \tfrac{1}{2}[Si(z) + Si(y)] - Si(a), \\ Ss(a,x) &= -\tfrac{1}{2}[Ci(z) + Ci(y)] + Ci(a), \quad \bar{C}c(a,x) = \tfrac{1}{2}[Ci(z) - Ci(y)], \end{aligned}$$

where $z = u + x$, $y = u - x$.

$$\int_0^\infty \sin u\, dx/u = \tfrac{1}{2}\pi J_0(a), \quad \int_0^\infty \cos u\, dx/u = \tfrac{1}{2}\pi Y_0(a).$$

The six functions (1) [omitting $\bar{C}$ and $\bar{C}c$] can be represented as a combination of elementary functions and power series in a and x, convergent for all values of a and x. For these power series in x, 10D tables of the first four or five of the various coefficients $\alpha_j(a)$, $\beta_j(a)$, $\gamma_j(a)$, $\nu_j(a)$, $\lambda_j(a)$, $\mu_j(a)$ are given (p. xxxvi–xxxviii), for $a = 0(.01).99$.

In the half-unit rectangle $0 \leq \alpha \leq 1$, $0 \leq x \leq \frac{1}{2}$, the integrals were computed by means of these series. The error in the coefficients is in all cases less than 6×10^{-9}; the resulting error in the integrals is of lower order than the truncation error. The introduction was written by Mr. SINGER with the help of Mr. GADD. "Computation of the tables," p. xix–xxv.

"Interpolation" by J. ORTEN GADD, JR. & THEODORE SINGER (p. xxvi–xxx). Throughout the tables the remainder after linear interpolation is less than 5 units in the third decimal place, and the remainder after second order interpolation is less than 1.2 units in the third decimal place. This is true regardless of whether the interpolation is accomplished by means of a two-way formula or by several uni-variate interpolations. Actually, the remainders after interpolation are much smaller throughout the greater part of the tables.

Of the eight integrals (1) six are tabulated in these volumes. The others $\bar{C}(a, x)$ and $\bar{C}c(a, x)$ are readily determined from $C(a, x)$, $Cc(a, x)$, and $\sinh^{-1}(x/a)$. It is anticipated that tables of the inverse hyperbolic functions will appear in the *Annals*, v. 20.

"Applications" by RONOLD W. P. KING (p. xxxi–xxxv).

The following list is partly representative of the types of problems that have been or may be investigated by using one-dimensional Helmholtz integrals which lead to the generalized sine- and cosine-integral functions.

(a). Vector and scalar potentials of electric circuits; open-wire transmission lines; linear, loop, and rhombic antennae, and arrays using these as elements. Distributions of current may be uniform, sinusoidal, or exponential.
(b). Self-impedance (including radiation resistance) of a polygonal loop antenna that is not so small that retardation is negligible.[7]
(c). Mutual impedance of rectangular loop antennae separated by an arbitrary distance.[7]
(d). The general analysis of two-wire, four-wire, multi-wire, and polyphase transmission lines, including the determination of radiation resistance.[7,8]
(e). Radiation resistance of a linear radiator with a sinusoidal distribution of current.[9]
(f). Distribution of current and impedance for a cylindrical antenna.[10–17]

Extracts from introductory text

[1] NIELS NIELSEN, *Theorie der Integrallogarithmus.* Leipzig, 1906; the author gives an exhaustive bibliography.

[2] NBSMTC, *Tables of Sine, Cosine, and Exponential Integrals.* v. 1–2, New York, 1940; *Table of Sine and Cosine Integrals for Arguments from 10 to 100.* New York, 1942. The references contained in these volumes include a full bibliography of tables of these functions.

[3] C. J. BOUWKAMP, "Note on an integral occurring in antenna theory," Natuurkundig Laboratorium de N. V. Philips' Gloeilampenfabrieken, Eindhoven, Netherlands, Unpubl. ms.

[4] R. V. D. CAMPBELL, "Evaluation of the function $S(b, h) = \int_0^h \sin (x^2 + b^2)^{\frac{1}{2}}dx/(x^2 + b^2)^{\frac{1}{2}}$" June, 1944; [see *MTAC*, v. 2, p. 218].

[5] H. A. ARNOLD, R. V. D. CAMPBELL, & R. R. SEEBER, JR., "Evaluation of the function $C(b, h) = \int_0^h \cos (x^2 + b^2)^{\frac{1}{2}}dx/(x^2 + b^2)^{\frac{1}{2}}$," Oct., 1944; [see *MTAC*, v. 2, p. 218].

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652[L, M].—S. A. SCHELKUNOFF, *Applied Mathematics for Engineers and Scientists.* (*The Bell Telephone Laboratories Series.*) New York, Van Nostrand, 1948, p. 456.

On this page there is a table of

$$SC(x) = \int_0^x Si\, u\, dCiu = \int_0^x Si\, u \cos u\, du/u,\ x = k\pi,$$

for $k = [.2(.2)2; 5D]$.

653[L, S].—G. H. GODFREY, "Diffraction of light from sources of finite dimensions," *Australian Jn. of Sci. Res.*, s. A. *Phys. Sciences*, Melbourne, v. 1, no. 1, Mar. 1948, p. 1–17.

T. 1, p. 7: Rectangular aperture, $I_x = [Si(2x) - (\sin^2 x)/x]/\pi$, $x = [0(.1)15(.5)34.5; 5D]$. T. 2, p. 8, $I_{x+2.4\pi} - I_x$, $x = [-1.2\pi(.1\pi) + 1.3\pi; 5D]$. T. 3, p. 9, $I_{x+3.4\pi} - I_x$, $x = [-1.7\pi(.1\pi)0; 5D]$. T. 4, p. 13, Circular aperture, $I_t = \int_0^t H_1(2x)dx/(\pi x^2)$, $t = [0(.1)15; 4D]$. T. 5, p. 15, $I_{t+8.4} - I_t$, $t = [-4.2(.1) - 2.8; 4D]$.

654[M].—WOLFGANG GRÖBNER & NIKOLAUS HOFREITER, *Integraltafel. Erster Teil: Unbestimmte Integrale.* Vienna and Innsbruck, Springer-Verlag, 1949, viii, 166 p. 20.7 × 29.8 cm. $4.20. Offset print of a manuscript original. This work was first published at Braunschweig, in 1944. It is stated in the preface (p. iv) that of the first edition a French translation was made by Engineer WEBER (Ministère de l'Armement S.F.I.S., Rapport Nr. 451–01–01/02/03) in which the authors of the original work are not mentioned.

This work by professors at the Universities of Innsbruck and Vienna respectively is divided into three sections, namely: 1. *Rational Integrands* (p. 1–21); 2. *Algebraic Irrational Integrands* (p. 22–106); 3. *Transcendental Integrands* (p. 107–166). Thus elliptic and hyperelliptic integrals are listed in section 2, but the Weierstrass and Jacobi elliptic functions in section 3, along with Bernoulli and Euler numbers, sine integral, cosine integral, exponential integral, etc. There are often 13–17 integrals on a page, all most neatly written, with numerous cross references, some explanations and references to discoverers, and frequent mention of JAHNKE & EMDE'S *Funktionentafeln.* The whole presentation is exceedingly clear and satisfactory. It is remarked in the preface that the second part of the work, on Definite Integrals, is, with the first part, to form a unit, so that gaps now apparent in the first part will later be satisfactorily filled.

The authors state that the following works were at their disposal: H. B. DWIGHT, *Tables of Integrals and other Mathematical Data.* New York, 1934; M. HIRSCH, *Integraltafeln oder Sammlung von Integralformeln.* Berlin, 1810; W. LÁSKA, *Sammlung von Formeln der reinen und angewandten Mathematik.* Braunschweig, 1888–1894; F. MINDING, *Sammlung von Integraltafeln.* Berlin, 1849; C. NASKE, *Integralformeln für Ingenieure und Studierende.* Berlin, 1935; B. O. PEIRCE, *A Short Table of Integrals.* Third ed., Boston, 1929 ("Vorzügliches Buch"); G. PETIT-BOIS, *Tafeln unbestimmter Integrale.* Leipzig, 1906.

R. C. A.

655[M, P].—E. T. GOODWIN & J. STATON, "Table of $\int_0^\infty e^{-u^2}du/(u+x)$," *Quart. Jn. Mech. Appl. Math.*, Oxford, v. 1, Sept. 1948, p. 319–326.

The function $f(x) = \int_0^\infty e^{-u^2}du/(u+x)$ is tabulated for $x = [0(.02)2(.05)3(.1)10; 4D]$. To facilitate interpolation for small x an auxiliary function $g(x) = f(x) + \log x$ is tabulated for $x = [0(.1)1; 4D]$. An asymptotic expression is given for values of x greater than 10. Full details of the method of computation are set forth and an interesting application of the Euler transformation to the summation of asymptotic series is included.

The function $f(x)$ arose in research connected with the determination of the response of a detector to a random noise voltage having a narrow spectrum.

Extracts from text

656[P].—A. C. STEVENSON, "The centre of flexure of a hollow shaft," London Math. Soc., *Proc.*, s. 2, v. 50, p. 536–549, 1949.

The author re-solves the problem of flexure by a transverse force of a bar of circular cross-section with a cylindrical cavity of circular cross-section having any position and size with respect to the outer circle. The load force is assumed to act through the centroid perpendicular to the axis of symmetry of the cross-section. The general solution is obtained in bipolar coordinates in terms of the author's modification of the classical Saint-Venant flexure theory. This involves three plane harmonic functions which are written as appropriate infinite trigonometric series, of which all coefficients are determined so as to satisfy the given boundary conditions. The most complete previous solution was given by SETH;[1] this however did not consider explicitly the centre of flexure or the limiting case when the cavity just reaches the edge of the cylinder. The present paper gives, p. 548–549, 5D tables for determining the torsional moment, the associated twist, the centre of flexure, the "centre of least strain energy," and the centroid, all as functions of two dimensionless parameters which define the ratios between the radii of the inner and outer circles and the distance between their centres, $\lambda = .1(.1).9$, $\mu = .1(.1).9$, $\lambda + \mu \not> 1$. The formulae for the case when the cavity just touches the outer surface are given in terms of trigamma and tetragamma functions.[2]

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[1] B. R. SETH, "On the flexure of a hollow shaft, I–II," Indian Acad. Sci., Proc., v. 4, 1936, p. 531–541; v. 5, 1937, p. 23–31.
[2] H. T. DAVIS, *Tables of the Higher Math. Functions*, v. 2, Bloomington, Ind., 1935; BAASMTC, *Mathematical Tables*, v. 1, second ed. 1946. See *MTAC*, v. 3, p. 424.

657[S].—NBSCL, *Tables of Scattering Functions for Spherical Particles.* (NBS *Applied Mathematics Series*, no. 4, issued 25 Jan. 1949). Washington, D. C., Superintendent of Documents, 1948, xiv, 119 p. 18 × 26 cm. $0.45.

This is a collection of four sets of tables useful in the study of the scattering of electromagnetic radiation by transparent as well as absorbing and dispersing spherical particles over a wide range of ratios of particle radius to radiation wave length.

The scattering of light by particles has been of great scientific interest from the time of the early work of TYNDALL and RAYLEIGH (1870) on the blue color of the sky and on the colors produced in illuminated suspensions. The subject has attracted renewed attention in recent years from the effect of fog and rain on the action of microwave radar, as well as that of colloidal suspensions on visible light.

The tables presented here are based on the fundamental work of GUSTAV MIE,[1] though with some changes in notation. A very helpful feature is the careful definition of all tabulated quantities as well as a description of their physical significance. The principal quantities tabulated are the so-called *intensity functions* i_1 and i_2,

$$i_1 = |i_1^*|^2 = |\sum_{n=1}^{\infty} \{A_n\pi_n + P_n[x\pi_n - (1 - x^2)\pi_n']\}|^2,$$

$$i_2 = |i_2^*|^2 = |\sum_{n=1}^{\infty} \{A_n[x\pi_n - (1 - x^2)\pi_n'] + P_n\pi_n\}|^2,$$

where $\pi_n(x) = \partial P_n(x)/\partial x$, and $\pi_n'(x) = \partial^2 P_n(x)/\partial x^2$, $P_n(x)$ being the Legendre polynomial of degree n. $A_n = a_n/n(n + 1)$, $P_n = p_n/n(n + 1)$, a_n and p_n being complicated expressions involving Bessel functions of half-integral order. i_1 and i_2 give the angular distribution of intensity and the total radiation scattered by a small spherical particle as a function of $\alpha = 2\pi r/\lambda$, where r is the radius of the particle and λ is the wave length of the radiation. Among various tables of **Part I** (p. 1–51) i_1 and i_2 are tabulated for values of the real index of refraction m of the scattering particle equal to 1.33 (that for water), 1.44, 1.55, and 2, and for values of α = .5, .6, 1, 1.2, 1.5, 1.8, 2, 2.4, 2.5, 3, 3.6, 4, 4.8, 5, 6, as the angle of scattering γ (the angle between the direction of propagation of the scattered radiation and the reversed direction of propagation of the incident radiation) ranges 0(10°)180°. The other functions tabulated are real and imaginary parts of (i) A_n, P_n, for various values of n, (ii) i_1, i_2, as well as values of $\frac{1}{2}\alpha^2 K(m; \alpha) = \frac{1}{2}\int_0^\pi (i_1 + i_2) \sin\gamma d\gamma$, where K represents the total scattering coefficient or the total energy scattered per second per unit cross-sectional area of particle, illuminated by radiation of unit intensity.

The scattering of light by colloidal suspensions offers a useful method for the determination of the size of the scattering particles. The present tables afford a means of estimating this from the experimentally observed intensity and scattering coefficient functions.

In **Part II**, p. 53–59, three functions are tabulated, for a transparent particle of refractive index 1.5, for a range of α from .5 to 12, and for pairs of values whose ratio is 1.2, which is the wave length ratio of the two most distinctive colors observed, namely: red light (λ = 6290 A°) and green light (λ = 5240 A°). These functions are $K(m; \alpha)$, $(2n + 1)R(C_n^1)$, $(2n + 1)R(C_n^2)$, where $C_n^1 = (-1)^{n+1} i a_n/(2n + 1)$; $C_n^2 = (-1)^n i p_n/(2n + 1)$.

In the tables of **Part III** (p. 61–81) are presented the values of

$$F(m; \alpha) = (2/\alpha^2) \sum_{n=1}^{\infty} (2n + 1)\{C_n^1(m; \alpha) + C_n^2(m; \alpha)\} = K(m; \alpha) + iL(m; \alpha),$$

for a medium containing *absorbing* particles with extinction coefficient k varying from 0 to .1. This quantity is defined in such a way that $e^{-4\pi k}$ is the fraction of the radiation absorbed in travelling a distance λ through the bulk material. The tables here are restricted to the range m = [1.44(.01)1.55; 4S or 5D] for the real part of the complex index of refraction.

The dispersion and absorption of electromagnetic radiation in liquid water are also of importance in the microwave radar region of wave lengths from 10 centimeters to 3 millimeters. Consequently in **Part IV** (p. 83–119), tables are included giving the real and imaginary parts of C_n^1 and C_n^2, 4D, for significant values of n, and the scattering function $K(m^*; \alpha)$, 3D, for complex indices of refraction m^* corresponding to this wave length interval. The tabulations are for the following special cases of m^*: $4.21 - 2.51i$, α = .1(.05)1(.1)3; $5.55 - 2.85i$, α = .1(.05)1(.1)2; $8.18 - 1.96i$, α = .1(.025)1; $3.41 - 1.94i$, α = .1(.05)-1(.1)5; $7.20 - 2.65i$, α = .1(.025)1(.05)1.3; $8.90 - 0.69i$, α = .1(.01).3(.005).43(.01).6.

This collection of tables will be of considerable value to investigators in the application of visible light scattering to the study of suspensions as well as to those who study microwave scattering. It is to be hoped that similar tables may ultimately be prepared for the scattering of sound by spherical obstacles.

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[1] G. Mie, "Beiträge zur Optik trüber Medien," *Annalen d. Physik*, s. 4, v. 25, 1908, p. 377–445.