For (15, 2, 1) $(10, 5, 3)_{12}$, $(9, 6, 3)_{11}$, $(8, 7, 3)_2$, $(8, 6, 4)_3$, *read* (15, 2, 1) $(10, 5, 3)_{15}$, $(9, 6, 3)_{13}$, $(8, 7, 3)_3$, $(8, 6, 4)_4$; *for* (14, 3, 1) $(8, 6, 4)_2$, *read* (14, 3, 1) $(8, 6, 4)_3$; *for* (12, 4, 2) $(11, 6, 1)_4$, *read* (12, 4, 2) $(11, 6, 1)_5$. These corrections change the total number of 4 × 4 magic squares from 539136 to 549504.

S. URATA, A. SAKAI & A. HIRAYAMA

Yamagata-Ken
Japan

UNPUBLISHED MATHEMATICAL TABLES

82[F].—L. POLETTI, *Factor Table and List of Primes for the 30000 natural numbers nearest 15,000,000.* Manuscript table deposited in the library of the American Math. Soc. New York.

This table gives new information for the range 14984970–15000000. The second half from 15000000 to 15015000, is also covered by W. P. Durfee's factor table for the 16th million, a table which is in the same library.

The factor table, which the author calls "Neocribrum," is a "type 3 table" arranged in the usual way modulo 30. On p. 1 are given data on the distribution of the primes in this range. Thus there are 1809 primes which are also classified modulo 30. There are 159 prime pairs. There are 113 consecutive composite numbers following 14996687.

Poletti is the author of *Tavole di Numeri Primi entro Limiti Diversi e Tavole Affini*, Milan, 1920.

D. H. L.

83[G, I].—H. E. SALZER, *Coefficients of the first fifteen General Laguerre Polynomials.* Ms. in possession of the author.

The writer announced previously (*MTAC*, v. 2, p. 89) a manuscript giving the coefficients of LAGUERRE polynomials, which are a special case of general Laguerre polynomials $L_n^{(\alpha)}(x)$, namely for $\alpha = 0$. The present manuscript gives the polynomials in α which are the coefficients of x^ν in the general Laguerre polynomial

$$L_n^{(\alpha)}(x) \equiv e^x x^{-\alpha} \frac{1}{n!}\left(\frac{d}{dx}\right)^n (e^{-x}x^{n+\alpha}) \equiv \sum_{\nu=0}^{n} \binom{n+\alpha}{n-\nu} \frac{(-x)^\nu}{\nu!}, \text{ for } \nu = 0(1)n,$$

and for $n = 0(1)15$.

H. E. SALZER

AUTOMATIC COMPUTING MACHINERY

Edited by the Staff of the Machine Development Laboratory of the National Bureau of Standards. Correspondence regarding the Section should be directed to Dr. E. W. CANNON, 418 South Building, National Bureau of Standards, Washington 25, D. C.

TECHNICAL DEVELOPMENTS

Our contribution under this heading, appearing earlier in this issue, is "The California Institute of Technology Electric Analogue Computer" by Prof. G. D. MCCANN.

DISCUSSIONS

Procedure for the Machine or Numerical Solution of Ordinary Linear Differential Equations for Two-Point Linear Boundary Values

Introduction. Increased attention is being focused on machine and numerical solutions of differential equations which cannot be solved by ordinary mathematical methods. There is need for more information on this subject for engineers and others who deal with such equations. This paper describes a procedure applicable to numerical or machine solutions of the general non-homogeneous ordinary linear differential equation with variable coefficients where the form of these coefficients does not easily permit of solution by series. The method is based on the well-known properties of linear differential equations.

Ordinary differential equations of order higher than the first commonly describe problems where the known boundary conditions are expressed at two different values of the independent variable. Such problems are known as two-point boundary-value problems. Although a great many linear equations, such as the BESSEL and LEGENDRE equations, may be rigorously handled by the method of FROBENIUS, there are frequently those where the variable coefficients of the derivatives are so complex that a series solution is not feasible. For such equations, recourse to solution by numerical methods or by some type of computing machine or analyzer may be sought. In this event, a difficulty is at once encountered if divided boundary conditions are present. In the case of an equation of order n, the dependent variable and its $n - 1$ derivatives must possess assigned values at some point within the interval in order that a machine or step-by-step solution may proceed from that point. Consider the case of a second-order equation where the two boundary conditions are divided between both ends of the range of the independent variable. Only one of an unlimited number of possible values of the initially unknown dependent variable or one of its derivatives, as the case may be, at one end of the interval of solution will satisfy the boundary condition at the other end. For a fourth-order equation with equally divided boundary conditions, a double latitude of possible initial choices would exist.

The theory of ordinary linear differential equations appears extensively in the mathematical literature.[1, 2, 3] Application to the two-point boundary problem where numerical or machine solutions are involved does not appear to be generally well known. The purpose of this paper is to show in relatively simple mathematical terms and by graphical illustration how the two-point boundary problem may be handled. The method applies to any ordinary linear differential equation of order n where the boundary conditions are expressible in terms of linear combinations of the dependent variable and its $n - 1$ derivatives and where K boundary conditions are known at one point of the interval of solution and the remaining $n - K$ conditions are known at some other point of the interval. It is shown that for $K \leq n/2$ only $K + 1$ trial solutions with arbitrarily selected values for those derivatives which are initially unknown are required to determine the unique solution satisfying all n boundary conditions. It is also shown that $n + 1$ trial solutions will give data for any solution of the non-homogeneous equation. In addition, a possible graphical procedure is suggested for converging on the solution of non-linear equations.

Theory. Consider an ordinary non-homogeneous linear differential equation of order n with variable coefficients. This is

$$(1) \qquad f_0(d^ny/dx^n) + f_1(d^{n-1}y/dx^{n-1}) + \cdots + f_{n-1}(dy/dx) + f_ny = f.$$

Assume $f_0, f_1, \cdots f_n, f$ are finite, one-valued, and continuous functions of x in the interval $a_0 \leq x \leq b_0$, and that f_0 does not vanish at any point in the interval. Under these conditions, there is known to exist a solution y such that y and its first n derivatives are continuous and have unique values at every point in the interval. The reduced or homogeneous equation corresponding to equation (1) is

$$(2) \qquad f_0(d^ny/dx^n) + f_1(d^{n-1}y/dx^{n-1}) + \cdots + f_{n-1}(dy/dx) + f_ny = 0$$

and is known to have n and only n linearly independent solutions,[4] $y_1, y_2, \cdots y_n$. The known complete solution of equation (1) is then

$$(3) \qquad y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p = y_p + \sum_{i=1}^{i=n} c_iy_i,$$

where y_p is any particular solution of equation (1) and $c_1, c_2, \cdots c_n$ are arbitrary constants to be determined by the n boundary conditions.

If the quantities B_k which assume boundary values are linear,[5] we may express them in the following manner:

$$(4) \qquad B_k = \sum_{j=0}^{j=n-1} a_{k,j}(d^jy/dx^j), \qquad k = 1, 2, \cdots n,$$

where each of the coefficients, $a_{k,j}$, represents a constant or some known function of x, and where $|a_{k,j}| \neq 0$ since the boundary conditions are linearly independent. Substituting the general solution, equation (3), into equation (4) and factoring out the c's, we have for the particular point, $x = x_k$, at which B_k has a known value,

$$\left(B_k = \sum_{i=1}^{i=n} c_i \sum_{j=0}^{j=n-1} a_{k,j}(d^jy_i/dx^j) + \sum_{j=0}^{j=n-1} a_{k,j}(d^jy_p/dx^j) \right)_{x=x_k}, \quad k = 1, 2, 3, \cdots n.$$

In this expression for B_k, all terms are of fixed value. Since there are n such linear expressions for the B's, and the product $|a_{k,j}| \cdot |d^jy_i/dx^j|$ does not vanish, we may solve explicitly for the n values of the c's. Carrying this out one obtains linear equations of the form,

$$(5) \qquad c_i = b_i + \sum_{k=1}^{k=n} b_{i,k}B_k, \qquad i = 1, 2, \cdots n.$$

The term b_i is a combination of the constant values of y_p and its derivatives for the particular x involved, and $b_{i,k}$ is a combination of the constants $a_{k,j}$ and the fixed values of y_i and its derivatives. Substituting equation (5) into equation (3) one obtains

$$(6) \qquad y = y_p + \sum_{i=1}^{i=n} b_iy_i + \sum_{k=1}^{k=n} B_k \sum_{i=1}^{i=n} b_{i,k}y_i.$$

This is a formal statement of the fact that the general solution may be expressed directly in terms of linear combinations of the n boundary parameters. It now becomes evident that for a given equation of order n, if n arbitrary but linearly independent solutions y_i of equation (2) and any particular solution y_p of equation (1) are obtained, then the solution to the problem with any desired values of divided boundary parameters, $B_1, B_2, \cdots, B_n$, known at any points within the interval (a_0, b_0), may be obtained by direct substitution into equation (6). It should be emphasized that y_i and y_p are *any* solutions to their respective equations, barring linearly dependent

y_i's, without restriction on initial or final values. It is seen from equation (6) that, if the boundary conditions are all homogeneous, then each B equals zero and the solution is simply

$$y = y_p + \sum_{i=1}^{i=n} b_i y_i.$$

Also, if the equation is homogeneous but not all of the boundary conditions are homogeneous, then

$$y = \sum_{k=1}^{k=n} B_k \sum_{i=1}^{i=n} b_{i,k} y_i.$$

The term in $b_i y_i$ is zero since b_i is a linear combination of the fixed boundary values of y_p and its $n - 1$ derivatives, and these quantities exist only in the non-homogeneous equation. If the equation and boundary conditions are all homogeneous, then it is apparent that $y = 0$, and there is no problem.

The advantage of expressing the solution directly in terms of the boundary parameters will now be illustrated for a fourth-order equation with boundary values B_1 and B_2 existing at $x = a$ and B_3 and B_4 existing at $x = b$. The limits a, b are ordinary points of the interval (a_0, b_0). For $n = 4$, the solution y and its first $n - 1$ derivatives may be written from equation (6) as

$$y = B_1 Y_1 + B_2 Y_2 + B_3 Y_3 + B_4 Y_4 + Y$$
$$dy/dx = B_1(dY_1/dx) + \text{etc.}, \qquad d^2y/dx^2 = B_1(d^2Y_1/dx^2) + \text{etc.},$$
$$d^3y/dx^3 = B_1(d^3Y_1/dx^3) + \text{etc.},$$

where Y_1, Y_2, Y_3, Y_4 are linear combinations of y_1, y_2, y_3, y_4, and Y is a linear combination of the same y's and y_p. Now consider the problem of a machine or step-by-step solution starting at $x = a$ where the two boundary parameters have the desired values $B_1 = \beta_1$ and $B_2 = \beta_2$. From the two boundary relations at $x = a$ given by equation (4) we may write

$$\beta_1 = [a_{1,0}y + a_{1,1}(dy/dx) + a_{1,2}(d^2y/dx^2) + a_{1,3}(d^3y/dx^3)]_{x=a}$$

$$\beta_2 = [a_{2,0}y + a_{2,1}(dy/dx) + a_{2,2}(d^2y/dx^2) + a_{2,3}(d^3y/dx^3)]_{x=a}$$

from which two of the initial values of the derivatives may be solved in terms of the remaining two. Thus, for any arbitrary values assigned to $y_{x=a}$ and $(dy/dx)_{x=a}$, the values of the second and third derivatives at $x = a$ may be calculated.[6] With B_1 and B_2 assigned the values β_1 and β_2, respectively, the solution and its first derivative become

$$y = B_3 Y_3 + B_4 Y_4 + Y_5, \qquad dy/dx = B_3 dY_3/dx + B_4 dY_4/dx + dY_5/dx,$$

where Y_5 is the new function, $Y + \beta_1 Y_1 + \beta_2 Y_2$. At $x = a$, Y_j and dY_j/dx (where $j = 3, 4, 5$) assume fixed values so that

$$\text{(7)} \qquad y_{x=a} = d_3 B_3 + d_4 B_4 + d_5, \qquad (dy/dx)_{x=a} = e_3 B_3 + e_4 B_4 + e_5,$$

where the d's and e's are constants. Equations (7) show that the initial values of y and dy/dx are related linearly to the boundary parameters at $x = b$. This linearity may be represented graphically as shown in Fig. 1. Although the existence of these linear families of boundary parameters is now established, their determination is still unknown for any trial solution.

Assume now that the problem at hand requires that $B_3 = \beta_3$ and $B_4 = \beta_4$ at $x = b$. It is readily seen that there exists a point S which defines the proper initial choice of $y_{x=a}$ and $(dy/dx)_{x=a}$. It is also evident from the linearity of equations (7) that, in proceeding on a straight line joining any two points, such as 1 and 2 in Fig. 1, the values of B_3 and B_4 will vary linearly with the distance measured along that line. With these facts in mind, let us make three separate trial solutions with any arbitrary combination of $y_{x=a}$ and $(dy/dx)_{x=a}$ which define three non-collinear points 1, 2, 3. The initial values of the second and third derivatives for each trial are of course determined so as to satisfy $B_1 = \beta_1$ and $B_2 = \beta_2$ at $x = a$. When these trial solutions have reached $x = b$, the values of y and its three derivatives at this point are used to calculate B_3 and B_4. By linear interpolation point P_1 which represents a point where B_3 equals the desired value β_3 may be located on line 1–2. Likewise points P_2 and P_3 which lie on the desired β_3 line are located by linear interpolation and extrapolation along lines 2–3 and 3–1. Similarly points Q_1, Q_2, Q_3 are located on the desired β_4 line. These two parameter lines may now be constructed and the desired solution point S determined by their intersection.

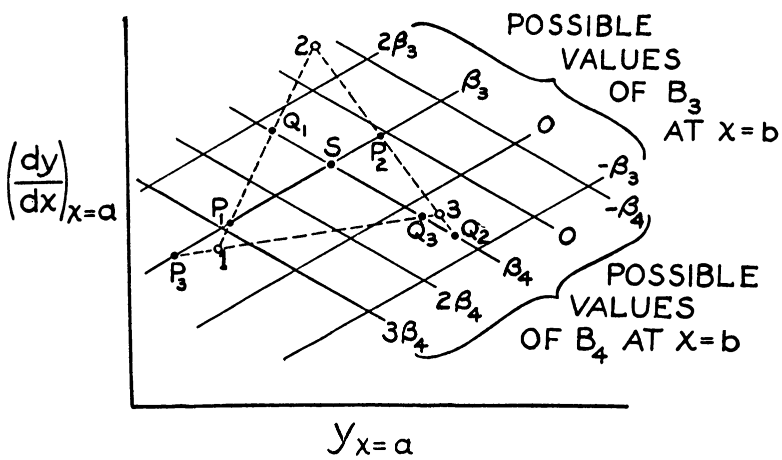

FIG. 1. Control plot for solution of fourth-order linear differential equation with equally divided boundary values.

Although the desired boundary conditions can now be satisfied with a fourth solution which begins with the correct combination of $y_{x=a}$ and $(dy/dx)_{x=a}$, it is not necessary to effect this fourth solution. For any particular value of x in the interval (a, b), the correct values of y and its derivatives may be obtained by linear interpolation and extrapolation to the solution point S from the corresponding values at points 1, 2, 3. This further linearity is at once evident from equations (7) if we replace B_3 and B_4 by any two of the derivatives of y at this particular value of x.

Consider next a sixth-order equation with three boundary conditions known at $x = a$ and the other three at $x = b$. Here, there is a triple latitude of possible choices of the three initially unknown values of y and its five derivatives at $x = a$ and only one unique combination will satisfy the given conditions at $x = b$. To solve this problem, we may visualize, in place of Fig. 1, three families of parallel planes in a three-dimensional space defined by the three initial values of y and its derivatives which are unknown. To determine the solution point S representing the common intersection of the three boundary-value planes will require four different trial solutions, each satisfying the three conditions at $x = a$. These trials will define four non-coplanar points in the three-dimensional space. Linear interpolation and extrapolation along any three non-coplanar lines joining these points will determine three sets of three points, each set uniquely defining one of the desired boundary-value planes. Again the values of y and its derivatives can be obtained at point S for any value of x by a three-dimensional linear interpolation and extrapolation from the four points.

In the case of a second-order equation with its two boundary conditions divided, only two trials, each satisfying the initial condition at $x = a$, are necessary to establish the solution. The relationship is shown graphically in Fig. 2 where the solution point S is determined by the linearity along the line joining the trial points 1 and 2.

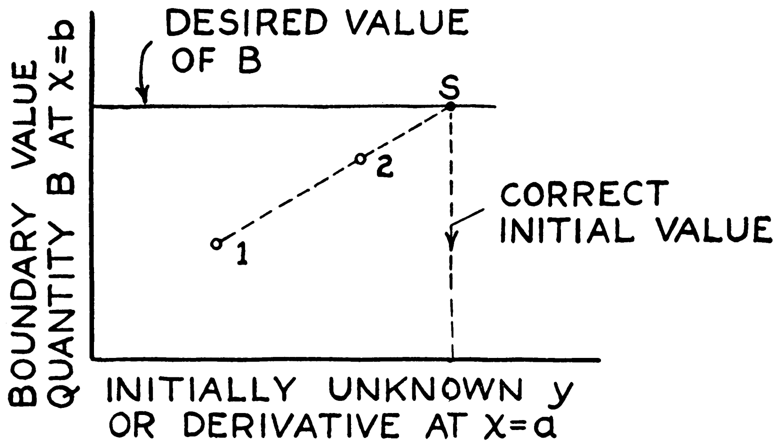

FIG. 2. Control plot for solution of second order linear differential equation with equally divided boundary values.

So far, only cases of equally divided boundary conditions have been discussed. If unequal division occurs, the solution should be started at the point where the greater number of known conditions exists. In the case of a fourth-order equation where three conditions are known for $x = a$, only one derivative of y is unknown, and two trials will be sufficient for solution.

Boundary conditions need not exist at the extremities of the interval of solution. In such cases the solution may begin at a point within the interval where at least half of the conditions are known. It will be necessary to reverse the direction of the independent variable in order to cover the complete range, but otherwise the procedure will be the same.

The following rules may be stated from the foregoing development. They hold for any ordinary linear differential equation of order n defined by equation (1) where K boundary conditions are known at one point within the interval of solution and the remaining $n - K$ conditions are known at some other point within the interval. If $K \leq n/2$, only $K + 1$ trial solutions with $K + 1$ arbitrarily selected initial values of the initially unknown derivatives are required to determine uniquely the desired solution which satisfies all boundary conditions. If data for the solution for any or all possible combinations of the n boundary conditions are required, then any n solutions, barring linearly dependent ones, of the homogeneous equation without reference to initial conditions and any one solution of the complete equation, or a total of $n + 1$ arbitrary trial solutions, must be made.

Discussion. Although the $n + 1$ trial solutions are sufficient to solve all boundary value problems for a given equation, in most practical cases, where interest is centered on a given set of boundary values, the method described using $K + 1$ trials, where $K \leq n/2$, will be the simpler procedure. Even in this case $K + 1$ families of solutions may be had with the $K + 1$ trials, since any boundary value at $x = b$ for each of the K boundary parameters and any value of b within the interval (a_0, b_0) may be used. All of these solutions must satisfy the same conditions at $x = a$.

Graphical illustration has been used to describe the linearities involved in order to aid in visualizing the problem. The solution point and the values of the various functions at this point may be determined by direct solution of the linear relationships involved. And indeed, if it is necessary to discuss an equation of order higher than the sixth, our spatial visualization which is limited to three dimensions would not aid in this problem. If the method of the $n + 1$ trials is adopted, the use of determinants will facilitate the necessary computation of b_i and $b_{i,k}$ in equation (6).

There are several practical considerations which place some limits on the success of these methods. It is essential that the distances between the trial points be of the same order of magnitude as the distance from any point to the solution point. Although the danger of interpolation and extrapolation on curved lines is absent, still the errors inherent in any machine or numerical solution will limit the extent of accurate extrapolation. It is usually possible in most physical problems by approximation, comparison, and reasonable guessing to predict the general region of solution and thus choose trial points which are not too far removed from the solution point. If a reasonably close estimate cannot be made, any $K + 1$ trials will point to the approximate location of S whereupon $K + 1$ additional trials in the neighborhood of S will yield the solution.

In order to insure sufficient accuracy in the results, a reasonable estimate of the range of magnitudes of y and its $n - 1$ derivatives must be made in the case of a machine solution. It may be necessary to repeat one or more trials with adjusted scale factors if the estimate is far in error.

Although $K + 1$ trial solutions are mathematically sufficient for solution, one or two more may be desirable in the case of a machine solution as a check on the accuracy of the work. In the case of the fourth-order equation with equally divided boundary values, the triangular configuration of trial points shown in Fig. 1 might well be replaced by four points representing the corners of a square. Any three of the six possible line segments joining the four points may be considered as locating these points. These three segments then determine independently three points on the desired parameter line. The remaining three segments may be considered as dependent on the first three and hence will determine three dependent points on the same parameter line.

The method described in this paper has been used successfully by the author in the solution of a problem in the theory of shells involving a fourth-order equation with equally divided boundary conditions. Triangular sets of trial points in the machine solution used were adequate.

In the case of non-linear equations or linear equations where the boundary conditions are non-linear a procedure similar to that described in the foregoing paragraphs is suggested. In this case graphical representation would be indicated, and, in a control plot corresponding to Fig. 1, the families of lines representing boundary parameters would no longer be straight lines or linearly spaced. However, by the principle of uniqueness of solution, it is evident that any one boundary parameter line of one family will not cross any other boundary parameter line of the other family more than once. Also, if a reasonably good initial estimate is made, it should be possible to converge on the solution with a few successive sets of trial points.

J. L. MERIAM

Univ. of California
Berkeley

[1] E. L. INCE, *Ordinary Differential Equations.* New York, 1944.

[2] A. R. FORSYTH, *A Treatise on Differential Equations.* Sixth ed. London, 1933.

[3] I. S. & E. S. SOKOLNIKOFF, *Higher Mathematics for Engineers and Physicists.* New York, 1941.

[4] The criterion for linear independence of these solutions is that the Wronskian of the y_i's (where $i = 1, 2, 3, \cdots n$) and their $n - 1$ derivatives does not vanish.

[5] Boundary values are said to be linear if they can be expressed as linear combinations of the dependent variable and its $n - 1$ derivatives.

[6] If in some actual problem $y_{x=a}$ and $(dy/dx)_{x=a}$, for example, took on assigned boundary values, then these may not be changed. Thus $(d^2y/dx^2)_{x=a}$ and $(d^3y/dx^3)_{x=a}$ would be the unknown initial quantities.

BIBLIOGRAPHY Z–IX

1. ANON., "AMA—an epoch in telephone accounting," *Bell Laboratories' Record*, v. 27, 1949, p. 2–4, illustrs. 25 × 18 cm.

The automatic message accounting equipment is able to "read" perfectly 80 coded digits per second. Paper tape, on which records of thousands of telephone calls are stored, contains information on the calling and called numbers, the month, day, and exact time to tenths of minutes at which a telephone conversation begins and ends. The tapes are run through "reader" machines, and the information is assembled, translated, sorted, summarized, and printed so that telephone bills may be made from it.

2. ANON., "Automatic computer for aircraft-motion mathematics," *Instruments*, v. 22, n. 1, Jan. 1949, p. 69, 70, illustr. 20 × 26.5 cm.

3. ANON., "GE analyzer for Indian Institute," *Electrical Engineering*, v. 68, 1949, p. 84. 29 × 21.25 cm.

An a-c network analyzer, for installation at the Indian Institute of Science, Bangalore.

4. ANON., "Instruments guide industry," *Business Week*, n. 995, Sept. 25, 1948, p. 70–74, illustr. 25.5 × 28.25 cm.

5. David L. Arenberg, "Ultrasonic solid delay lines," Acoustical Soc. Amer., *Jn.*, v. 20, 1948, p. 1–26, bibl., diags., tables. 19 × 27 cm.

Abstract: The development of an ultrasonic delay line for the storage of pulses one microsecond long for periods up to 2000 microseconds is discussed. Theoretical considerations of the piezoelectric transducer involving the quartz crystals and acoustical constants of the medium developed by Emslie, Huntington, and Benfield are applied to determine the band width and power loss of the delay line. Problems, peculiar to the use of solids because of the presence of more than one mode of propagation and the distribution of energy between the different modes on reflection from an air surface, are evaluated for a wide range of Poisson ratios. Measurements of velocity and attenuation in some polycrystalline metals, single crystals, and glasses are given. Fused quartz was found to be the most promising material.

Delay lines constructed using a light source and photoelastic pick-up were not completely developed, but several types using crystal as well as transmitter pick-up were satisfactory. For delay times longer than 300 microseconds, multiple reflection paths in two and three dimensions were used and their preparation from fused quartz discussed.

6. A. R. Boothroyd & E. Colin Cherry, "Analogue calculating machine for functions of a complex variable," *Nature*, v. 163, April 30, 1949, p. 687. 17.4 × 25.4 cm.

7. J. G. Brainerd & T. K. Sharpless, "The ENIAC," *Electrical Engineering*, v. 67, n. 2, Feb. 1948, p. 168–172, diags. 21.5 × 29 cm.

8. J. A. Bronzo & H. G. Cohen, "A note on analog computer design," *Rev. Sci. Instrs.*, v. 20, n. 2, Feb. 1949, p. 101–102. 19 × 26.5 cm.

A method is proposed for the simplification of computers designed to solve certain types of partial differential equations.

9. C. Chu, "The recent development in the electronic digital variable computer," Chinese Institute of Engineers, *Jn.*, v. 6, no. 1, May 1948, p. 58–65. 14.5 × 23 cm.

10. Harry M. Davis, "Mathematical machines," *Scientific American*, v. 180, April 1949, p. 28–39, illustrs. 21.25 × 29 cm.

This article is principally concerned with the mathematical machines which count, i.e., the digital machines, and a detailed description of the digital idea is presented. It emphasizes the fact that, although these machines are powerless to do anything creative, they are extremely useful in that they are not subject to human intervention and are capable of enormous speed. Current large-scale computer developments are mentioned including: (1) the IBM Selective Sequence Calculator and the electronic Type 604 Calculator, which is ten times faster than the previous IBM commercial calculating punches; (2) the Moore School EDVAC. having a far more capacious memory and greater versatility than its forerunner, the ENIAC; (3) the twin computers known as BINAC and the UNIVAC, developed by John Mauchly and J. P. Eckert, Jr.; (4) the colossal computing device which is being designed at the Institute for Advanced Study for purposes of weather prediction and possibly for work on questions of atomic physics and economic statistics; (5) the Raytheon Manufacturing Company computer; (6) the Harvard Mark III computer; (7) the MIT "Project Whirlwind" for the construction of a computer for military purposes; and (8) advanced computers now under construction by four British Laboratories. Every current computer will soon become outdated in this new age of electronic thinking; however, the older machines will not become obsolete, because problems exist suitable for every type to handle. Also the relay, as a computing device, and the much speedier electron tube is discussed with a detailed explanation of the operation of the 20 ENIAC accumulators and of the binary representation of numbers. The application of the binary system to logic as well as numbers is stressed. In particular the author mentions the various memory devices now being developed and outlines future applications to the 1950 census, adjustment of maps, engineering computations, hydrodynamics, large-scale planning in the Air Force, etc.

11. HERMAN H. GOLDSTINE & JOHN VON NEUMANN, *Planning and Coding of Problems for an Electronic Computing Instrument.* Part II, v. 3. Princeton, N. J. Institute for Advanced Study, 1948, iii + 23 p. 21.6 × 27.9 cm.

This volume deals with the manner of incorporating subroutines into a complete problem. After considerable explanation, the authors conclude that this is best accomplished for small subroutines by recording them into the main routine as needed. For larger subroutines a preparatory routine automatically modifies the addresses of the subroutine relating to positions within itself by adding a quantity corresponding to the difference between the actual location of the subroutine in the memory and the assumed location at the time the subroutine was initially coded. There is a detailed discussion of suitable preparatory routines for a single subroutine and for a group of subroutines all of which are considered as coding problems. The volume closes with a description of proposed details for handling main routines and subroutines and the preparatory routine for machine input.

While the subject matter is of appreciable importance and the manner of handling it shows ingenuity, there is some question as to whether its treatment in this report is optimum. The following are some of the points one might question:

1. In order to use the method described, the coding of subroutines must be handicapped somewhat by certain limitations. For example, it is not permissible to use one half of a word for numerical storage and the other half as an order. It is also not permissible to build up addresses by the addition of numerical quantities, some of which occupy address positions, unless these quantities satisfy certain requirements. While such limitations are simple in character, their existence lends some importance to use of other methods which impose fewer restrictions to achieve the same result.

In addition to other methods which utilize auxiliary input equipment (for example, that suggested by SAMUEL LUBKIN in "Proposed programing for the EDVAC," Aberdeen Proving Ground, Jan. 1947), we may mention one suggested by ALAN L. LEINER and RALPH J. SLUTZ of the Electronic Computer Section of NBS. It is based upon the provision of extra binary digits in the coding as an indication of whether the corresponding address is to be taken as an absolute location in the memory or as "relative" to the memory location in which it occurs. Also, there is a method based upon the provision of one or more "base numbers" for automatically modifying appropriate classes of addresses (suggested by Samuel Lubkin primarily for utilization in connection with iterations but incidentally of value in connection with the present problem). In addition, it is possible to use a different criterion than that chosen by the authors for determining which addresses are to be modified. For example, NBSMDL has recently coded an analogous preparatory routine for the EDVAC, using the first binary digit in each address to indicate whether or not the address requires modification.

2. Time requirements for machine modification of subroutines are certainly negligible compared to those for manual operations relating to insertion of subroutines from a "library" into the memory. This does not, however, necessarily rule out special routine-preparing equipment, since the latter may facilitate the manual operations themselves and may make coding more convenient.

3. There seems to be little reason for not including the constants of the preparatory routine in the main routine instead of inserting each constant manually as indicated by the authors. Also, the space taken up by the preparatory routine might be used by the main

routine at some time after the address modification is completed. For this purpose, it might be more convenient to locate the preparatory routine at the beginning of the memory rather than at the end. After reading in the subroutines and the preparatory routine (with constants), the latter could be carried out and followed by instructions to read in the balance of the main routine, re-using memory locations originally containing the preparatory routine.

4. The authors mention the possible convention of having all subroutines coded as if they begin at 0, but they do not require this in their preparatory routine. Such a convention seems simpler to use than the requirement that the actual position occupied by a subroutine be further along in the memory than the position for which it is coded. Use of the convention would cut the length of the preparatory routine. It might also be noted that one of the subroutines (by choice, the longest) can always be inserted in the memory in the same position for which it was originally coded and, in such a case, requires no modification.

5. The coding given by the authors for the preparatory routine can be materially shortened. A new code of the routine has been prepared by OTTO T. STEINER & Samuel Lubkin requiring only $36 + 2I$ "words" (where I represents the number of subroutines) as compared to the $58 + 4I$ "words" required in the original routine. It was prepared without the use of flow diagrams or sub-sectioning. Indeed, most MDL coders find such aids, in the form used by the authors, confusing and even detrimental.

SAMUEL LUBKIN

NBSMDL

12. D. R. HARTREE, "A historical survey of digital computing machines"; M. H. A. NEWMAN, "General principles of the design of all-purpose computing machines"; M. V. WILKES, "The design of a practical high-speed computing machine. The EDSAC"; F. C. WILLIAMS, "A cathode-ray tube digit store"; J. H. WILKINSON, "The automatic computing engine at the National Physical Laboratory"; A. D. BOOTH, "Recent computer projects," R. Soc. London, *Proc.*, v. 195A, Dec. 1948, p. 265–287 + 3 plates. Discussion held 4 Mar. 1948. 17.2 × 25.6 cm. Reported in *Nature*, v. 161, May 8, 1948, p. 712–713. 15.5 × 25.5 cm.

The discussion was confined to automatic digital general-purpose computing machines. The EDSAC (Electronic Delay Storage Automatic Calculator) is at the University Mathematical Laboratory, Cambridge, England. Compare *MTAC*, v. 3, p. 214–215.

13. WM. A. MCCOOL, *An Improved Electronic Analog Computing Circuit.* Report no. P-3423. Washington, Naval Res. Lab., Feb. 23, 1949, 15 p. illustr. 26.5 × 20 cm.

Abstract: An improved electronic analog computing circuit, employing only two vacuum tubes and characterized by both simplicity and accuracy of computation, is described and analyzed. The theoretical error is calculated for integration, differentiation, and constant multiplication with step-function and sine-wave forms of excitation. The results of a cursory experiment, made to evaluate the performance of the improved circuit functioning as a differentiator, agree with the theoretical results.

14. T. K. SHARPLESS, "Design of mercury delay lines," *Electronics*, v. 20, Nov. 1947, p. 134–138, illustrs. 28.25 × 20.25 cm.

Mercury is used to obtain millisecond delays for radar, computers, and memory devices because it transmits compression waves relatively slowly, introduces negligible loss, and has an impedance comparable to that of crystal transducers. Recirculation and temperature compensation techniques are presented.

15. NORBERT WIENER, *Cybernetics*. Hermann et Cie, Paris, and John Wiley & Sons, New York, 1948, 194 p. 22.5 × 14.5 cm.

The term, cybernetics, has been chosen by Wiener and his collaborators to designate the entire field of control and communication in the animal and the machine. The justification for a common treatment of apparently diverse phenomena in control and communication, ranging, for example, from the operation of purely mechanical feed-back systems to neurological phenomena like memory and visual perception, would appear to lie in the identity of the mathematical approach. However, as is explained in a thirty-nine page narrative which serves as an introduction to the text of the book, the underlying motif of research in cybernetics is rather the investigation of boundary regions of science, areas formerly "neglected as a no-man's land between the various established fields."

The volume, *Cybernetics*, contains eight formal chapters, beginning with a chapter on Newtonian and Bergsonian time and ending with a chapter on information, language, and society. The remaining chapters are concerned, respectively, with "Groups and statistical mechanics," "Time series, information, and communications," "Feed-back and oscillation," "Computing machines and the nervous system," "Gestalt and universals," and "Cybernetics and psychopathology." In addition there is a note, refreshingly objective by comparison with popular press notices on the subject, in which the capability of a computing machine at chess playing is discussed briefly.

The introduction gives a somewhat detailed account of the origin of Wiener's interest in the investigation of the nature of control and communication in the human being and of the manner in which group effort was developed in the field. It is stated that the book represents the outcome of a program of work undertaken jointly with Dr. ARTURO ROSENBLUETH, formerly of the Harvard Medical School, and now of the Instituto Nacional de Cardiologia of Mexico. The assistance of scientists, distinguished for their outstanding work in such diverse fields as electro-mechanics, anthropology, sociology, and statistical opinion sampling, is acknowledged.

In chapter I, "Newtonian and Bergsonian time," the implications of reversibility of time are discussed. It is pointed out that planned physical experiments introduce asymmetry into reversible time systems. In fact, the very nature of the questions asked, of probability and prediction, must of necessity be asymmetrical, distinguishing between the past held in the present by the fixing of certain quantities, and the future. Newtonian astronomy and meteorology, Newtonian dynamics and Gibbsian statistical mechanics, physics, and biology are discussed from the standpoint of reversibility of time. The difference between the irreversible time of evolution and biology and the reversible time of physics was emphasized by Bergson. In Wiener's opinion, the controversy between vitalism and mechanism had at its kernel the fact that Newtonian physics is not the proper frame for biology.

Chapter II, "Groups and statistical mechanics," is, in a sense, an introduction to the following chapter, "Time series, information, and communications." These two chapters alone, while they are exacting from the standpoint of the mathematical maturity and background required of the reader, would appear to justify the cost of publishing the book. For example, a lucid yet concise account is presented of the foundations of statistical mechanics, including the role of LEBESGUE measure and the metrical invariants of transformation groups in the development of the ergodic theory, and a discussion of the notion of entropy. Also a satisfying formal exposition of a statistical theory of time series is made showing how the theory applies to message-noise problems in communication, to the ensemble of time series associated with Brownian motion, and to prediction problems. Single and multiple time series, both continuous and discrete, are treated from the viewpoint of full knowledge of the parts of the series. The role of the theory of time series in quantum mechanics is touched upon and the line of development of a practicable theory of time series involving conclusions based on a sampling of the past is indicated.

In the remaining chapters of *Cybernetics*, IV, "Feed-back and oscillation," V, "Computing machines and the nervous system," VI, "Gestalt and universals," VII, "Cybernetics and psychopathology," and VIII, "Information, language, and society," the reader of moderate mathematical interest will find the going easier. It is necessary, says the author, to use mathematical symbolism and mathematical techniques in IV, but the mathematical treatment is made more palatable to the average reader by extensive verbal explanation. In IV, feed-backs found in human and animal reflexes and homostasis are discussed. In chapters V, VI, and VII, discussion of both normal and abnormal action of the nervous system centers about the ultra-fast, automatically-sequenced computing machine as a model. The reviewer cannot repress a feeling that the author is unrealistic in his discussion of possible future lines of development of such computing devices, for example, the incorporation in them of the ability to learn; nevertheless, the utility of such devices outside the field of numerical computation is made clear.

The last chapter, "Information, language, and society," ends the *Cybernetics* on a sombre note, seeming to possess a tincture of misanthropy. One can only hope that in its impact upon sociology, the new science of cybernetics will exert a beneficent influence and that the social sciences will not be too greatly retarded in development by the comparatively close coupling of the observer with phenomena mentioned in this chapter.

E. W. CANNON

NBS

EDITORIAL NOTE: Professor Wiener's book was printed in France and is filled with typographical errors, sometimes making the mathematics quite unintelligible.

NEWS

Association for Computing Machinery.—CHARLES B. TOMPKINS has resigned from the Council of the Association, and JAN RAJCHMAN has been appointed. At present, the membership totals about 600. Prof. EDY VELANDER of Sweden has been authorized to undertake the formation of a Swedish section, and Mr. ALBERT CAHN of NBSINA is in charge of forming a California section of the Association.

About 200 members of the Association attended the meeting in Oak Ridge, Tennessee, April 18–20, 1949. Those who were present found it to be an informative and satisfying meeting. The Association greatly appreciated the welcome and hospitality of the Oak Ridge National Laboratory, the Oak Ridge Institute of Nuclear Studies, and the other Oak Ridge groups.

The papers listed below were delivered. Plans are being made to furnish summaries (or the full version) of these papers to members. In some cases the full paper will be submitted to the editors of *MTAC*.

1. "Address of welcome," A. M. WEINBERG, Oak Ridge Nat. Lab. 2. "Limitations of electronic digital computing machines," E. W. CANNON, NBS. 3. "Analog and special-purpose computing machines," S. H. CALDWELL, MIT. 4. "Inversion of large matrices," GERTRUDE BLANCH, NBSINA. 5. "Problems in linear programing," EMIL SCHELL, Office of the Air Comptroller, USAF, Washington. 6. "Computing machinery in Australia," T. G. ROOM, Univ. of Sidney, Australia. 7. "The forgotten man of computing," MINA REES, ONR, Washington. 8. "Statistical theory of round-off errors," H. D. HUSKEY, NBSINA. 9. "The relations of symbolic logic and large-scale calculating machines," EDMUND C. BERKELEY, New York. 10. "Calculation of neutron age by Monte Carlo," A. S. HOUSEHOLDER, Oak Ridge Nat. Lab. 11. "Computing statistical atomic fields on the selective sequence electronic calculator," L. H. THOMAS, IBM. 12. "Binary notation in arithmetic algorithms and operation code partitions," B. F. CHEYDLEUR, Naval Ordn. Lab., White Oak, Silver Spring, Md. 13. "602A calculating punch—time saver in computing thermodynamic properties of gases," JACK BELZER, H. L. JOHNSTON, & LYDIA SAVEDOFF, Dept. Chem., Ohio State Univ., Columbus, O. 14. "The register code for the ENIAC," R. F. CLIPPINGER & B. DIMSDALE, BRL, Aberdeen, Md. 15. "Recent progress on the Selectron," JAN RAJCHMAN, Radio Corp. Amer., Princeton, N. J. 16. "Analog computer for FOURIER transforms in crystal analysis," RAYMOND PEPINSKY, Alabama Polytechnic Inst., Auburn, Ala. 17. "The philosophy of computing machine design," GEORGE R. STIBITZ, Burlington,

Vt. 18. "Internal conversion calculation on Mark I," M. E. ROSE, Oak Ridge Nat. Lab., & BERNARD SPINRAD, Argonne Nat. Lab., Chicago. 19. "Experience with the use of the AIKEN Relay Calculator," CLARENCE ROSS, Naval Proving Ground, Dahlgren, Va. 20. "Logical design of direct-coupled vacuum tube control circuits," THEODORE KALIN, Computation Lab., Harvard Univ. 21. "The statistical mechanics of high polymers," GILBERT W. KING, Cambridge, Mass. 22. "The NBS Interim Computer," SAMUEL LUBKIN, NBS. 23. "Uses of pulse transformer in a-c coupling of computers," RALPH SLUTZ, Electronics Div., NBS. 24. "An intercomparison of storage devices," S. N. ALEXANDER, NBS.

NBSINA.—Two symposia were held on June 22 through July 1, 1949, on subjects pertinent to the effective utilization of automatic digital computing machinery. A representative group of the nation's engineers, physicists, and mathematicians participated.

The subject of the first one, June 22–25, was the *Construction and Applications of Conformal Maps*. The applications of conformal maps in such fields as aerodynamics and electronics were emphasized, and special attention was devoted to the actual current needs of research workers in these fields. Construction methods were discussed, along with the problem of programing such methods on existing and proposed automatic digital computing systems. In this connection, particular reference was made to the electronic machine now being designed at NBSINA under the direction of HARRY D. HUSKEY.

The program opened with a one-day course designed to acquaint those attending the Symposium with the preparation of problems for automatic digital computing machines. The following talks were included: "Definition of an automatic digital computing machine" by H. D. Huskey; "Description of a specific automatic computer" by ROSELYN SIEGEL, NBSINA; "Programing the solution of n simultaneous linear equations" by H. D. Huskey; and a laboratory session in which participants were urged to work out detailed routines for causing the computers to perform such operations as division, floating addition, solving a simple differential equation, and possibly solving a set of simultaneous linear equations. On subsequent days the following Sessions were held:

I. June 23, General Session, J. H. CURTISS, NBS, chairman:

"On network methods in conformal mapping computation" by R. VON MISES, Harvard Univ.

"Conformal mapping of domains of higher topological structure illustrated by flow patterns" by R. COURANT, New York Univ.

II. June 23, Session on Physical and Industrial Applications, D. V. WIDDER, Harvard Univ., chairman:

"Applications of conformal mapping to torsional rigidity, principal frequency, and electrostatic capacity" by G. SZEGÖ, Stanford Univ.

"Some industrial applications of conformal mapping" by H. PORITSKY, General Electric Co.

"Conformal maps involving multiple-connected regions and their technical applications" by G. STEIN, Westinghouse Electric Corp.

"On the use of conformal mapping in problems of two-dimensional elasticity" by I. S. SOKOLNIKOFF, UCLA.

III. June 24, Session on Fluid Dynamics, J. L. BARNES, UCLA, chairman:

"On the HELMHOLTZ problem of conformal representation" by A. WEINSTEIN, U. S. Naval Ordn. Lab. and Univ. of Maryland.

"Aspects of conformal mapping in aerodynamics" by I. E. GARRICK, NACA.

"On THEODORSEN's method of conformal mapping" by ALEXANDER OSTROWSKI, Univ. of Basle and NBS.

"On conformal mapping of variable regions" by S. E. WARSCHAWSKI, Univ. of Minnesota and NBS.

"Fluid dynamics, conformal mapping, and numerical methods" by ANDREW VAZSONYI, U. S. Naval Ordn. Test Sta. Pasadena, Cal.

IV. June 24, Session on Theory of Conformal Maps, W. T. Martin, MIT and UCLA, chairman:

"Some remarks on variational methods applicable to multiple-connected domains" by D. C. Spencer, Stanford Univ.

"A variational method for simply-connected domains" by A. C. Schaeffer, Purdue Univ.

"Kernel functions and conformal mapping" by Stefan Bergman, Harvard Univ., & Menahem Schiffer, Stanford Univ.

"The kernel function and canonical conformal maps" by Zeev Nehari, Washington Univ.

"A new proof of the Riemann mapping theorem" by P. R. Garabedian, Stanford Univ.

V. June 25, General Session, J. W. Green, UCLA, chairman:

"Conformal mapping applied to electromagnetic field problems" by Ernst Weber, Polytechnic Inst., Brooklyn.

"An extremal method in conformal mapping" by L. V. Ahlfors, Harvard Univ.

"Some generalizations of conformal mappings occurring in gas dynamics" by Lipman Bers, Syracuse Univ.

"The use of conformal mapping in the study of flow phenomena at the free surface of an infinite sea" by Eugene P. Cooper, U. S. Naval Ordnance Test Sta., Pasadena, Cal.

"Recent contributions of the Hungarian school to conformal mapping" by G. Szegö, Stanford Univ.

VI. June 25, Session on Numerical Methods, H. F. Bohnenblust, Cal. Inst. Techn., chairman:

"Relaxation methods as ancillary techniques" by Sir Richard Southwell, London, England.

"The use of conformal mapping to compute flows with free streamline" by D. M. Young, Harvard Univ.

"An approximation method for conformal maps" by Lee H. Swinford, Univ. of California, Berkeley.

The second Symposium dealt with *Probability Methods in Numerical Analysis*, with special reference to the techniques now known among physicists by the name, "*Monte Carlo.*" It was held on June 29–July 1, under the joint sponsorship of NBSINA and the Rand Corp. with the cooperation of the Oak Ridge Nat. Lab.

A word of explanation concerning the "Monte Carlo" technique is in order here. It is well known in the theory of stochastic processes that the probability distributions associated with certain random walk or random flight problems satisfy certain classical integro-differential equations. This fact provides a novel method for approximate numerical integration of such equations through the medium of building up large samples of trials of corresponding random walks. It also suggests the possibility and desirability of directly constructing appropriate random walks as discrete mathematical models for given physical situations, to be used in place of the more classical continuous models. This method of solution of problems in mathematical physics by sampling techniques based on random walk models constitutes what is known as the "Monte Carlo" method. The method as well as the name for it were apparently first suggested by John von Neumann and S. M. Ulam. "Monte Carlo" techniques are now being used rather extensively in connection with high-speed automatic digital computing machinery.

Many mathematical and physical problems remain to be solved in the theory of "Monte Carlo" techniques. Most of the current "Monte Carlo" applications are classified; however, the proceedings of the Symposium were unclassified. The program was as follows:

I. June 29, Orientation Session, J. H. Curtiss, chairman:

"The Monte Carlo method" by S. Ulam, Los Alamos Sci. Lab.

"Discussion of the Monte Carlo method" by J. von Neumann, Inst. Adv. Study

II. June 29, Session on Physical Applications of Stochastic Methods, J. von Neumann, chairman:

"Introductory remarks" by the chairman

"Showers produced by low energy electrons and photons" by Robert R. Wilson, Cornell Univ.

"Nomograms for Monte Carlo solution of the Milne problem" by B. I. Spinrad, Argonne Nat. Lab. and G. H. Gortzel, New York Univ.

"Neutron age calculations in water, graphite, and tissue" by A. S. Householder, Oak Ridge Nat. Lab.

"Methods of probabilities in chains applied to particle transmission through matter" by Lewis Nelson, Oak Ridge Nat. Lab., & Wendell DeMarcus, Fairchild Engine and Airplane Corp.

"Multiple-scattered gamma rays" by Ugo Fano, NBS

"Stochastic methods in statistical mechanics" by Gilbert W. King, Arthur D. Little Co.

III. June 30, Session on Physical Applications of Stochastic Methods (continued), Frank C. Hoyt, Argonne Nat. Lab., chairman:

"Calculations on a water shield for fast neutrons, I" by Maria Mayer, Argonne Nat. Lab.

"Calculations on a water shield for fast neutrons, II" by Preston Hammer, Los Alamos Sci. Lab.

"A Monte Carlo technique for estimating particle attenuation in bulk matter" by B. A. Shoor, Northrop Aircraft; Lewis Nelson, Oak Ridge Nat. Lab.; Wendell DeMarcus, & Robert L. Echols, Fairchild Engine and Airplane Corp.

"Neutron transmission through thick slabs" by Williston Shor, U. S. Navy.

"Estimation of particle transmission by random sampling" by Herman Kahn, Rand Corp. & T. E. Harris, Rand Corp.

IV. June 30, Session on Random Digits, Jerzy Neyman, Univ. of California, Berkeley, chairman:

"Generation and testing" by George W. Brown, Rand Corp.; Nicholas Metropolis, Los Alamos Sci. Lab.; & George E. Forsythe, NBS.

"Various techniques used in connection with random digits" by J. von Neumann.

General discussion from the floor.

V. July 1, Mathematical Session, John W. Tukey, Princeton Univ., chairman:

"The connection between stochastic processes and partial differential equations" by Will Feller, Cornell Univ. and NBSINA; Mark Kac, Cornell Univ. and NBS; & J. L. Doob, Univ. of Illinois.

Discussion from the floor.

VI. July 1, Round Table Discussion: Critique of Applications and Discussion of Possible New Directions for Research, John W. Tukey, discussion leader:

Panel: J. Neyman; M. A. Girschick, Stanford Univ.; W. A. Wallis, Univ. of Chicago; D. H. Blackwell, Howard Univ.; T. W. Anderson, Columbia Univ.; & L. J. Savage, Univ. of Chicago.

Also on June 27–28, a condensed, relatively elementary course on automatic computation was held at NBSINA. The purpose of this course was to introduce interested persons to the logical theory and performance characteristics of automatic digital calculators being developed at the present time. Preparation of problems for the calculators was emphasized. It was composed of two lecture and two laboratory sessions.