

used. Even with the faster machines the time required for the solution of a biharmonic equation by the methods considered here is uncomfortably large unless the second order Richardson (or probably also the extrapolated Liebmann) method is used. It is clear that for many problems of interest the simplest iterative procedures will prove impossibly tedious even with the fastest automatic computers.

The apparent likelihood that the extrapolated Liebmann procedure would prove more rapidly convergent and more convenient for electronic computers than the second-order Richardson method would seem to justify an experimental study with such a computer.

The writer is indebted to R. H. MACNEAL and W. E. MILNE for encouragement and helpful discussion.

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² R. V. SOUTHWELL, *Relaxation Methods in Engineering Science*, Oxford, 1940.

³ E. T. WHITTAKER & G. ROBINSON, *The Calculus of Observations*, London and Glasgow, 1932.

⁴ G. SHORTLEY, R. WELLER, & B. FRIED, "Numerical solution of Laplace's and Poisson's equations with applications to photoelasticity and torsion," Ohio State University, *Studies, Engineering Series, Bull.* no. 107, 1942.

⁵ L. F. RICHARDSON, "The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam," R. Soc., London, *Phil. Trans. s. A*, v. 210, 1911, p. 307-357. "How to solve differential equations approximately by arithmetic," *Math. Gazette*, v. 12, p. 415-421, 1925.

⁶ R. COURANT, "Über partielle Differenzengleichungen," Congresso Internazionale dei Matematici, *Atti*, Bologna, v. 3, 1930, p. 83-89.

⁷ This can be shown by the method used by H. LEWY in "On the convergence of solutions of difference equations," *Studies and Essays Presented to R. Courant on his 60th Birthday*, New York, 1948, p. 211-214.

⁸ H. LIEBMANN, "Die angenäherte Ermittlung harmonischer Funktionen und konformer Abbildungen," Bayer. Akad. Wiss., *math.-phys. Klasse, Sitz.*, 1918, p. 385-416. Further references to the Liebmann method are cited in footnote 1 of Shortley, Weller & Fried. See also *MTAC* v. 3, p. 350, footnote 3.

On a Definite Integral

The function

$$f(x) = \int_0^\infty (u + x)^{-1} \exp(-u^2) du$$

is tabulated by E. T. GOODWIN and J. STATON¹ [*MTAC*, v. 3, p. 483] from series expansions and by numerical integration of the differential equation satisfied by the function.

The integral can be evaluated explicitly in terms of two simple tabulated functions. The writer used the LAPLACE transform² to evaluate the integral. This method will now be given.

We may write

$$(1) \quad I(t) = \int_0^\infty (u + t^{\frac{1}{2}})^{-1} \exp(-u^2) du = \int_0^\infty (1 + v)^{-1} \exp(-tv^2) dv$$

and if we let

$$L\{I(t)\} \equiv \int_0^\infty I(t) \exp(-pt) dt$$

denote the Laplace transform of $I(t)$, we have

$$(2) \quad L\{I(t)\} = \int_0^\infty (p+v^2)^{-1}(1+v)^{-1} dv = \frac{1}{2}(1+p)^{-1}[\pi p^{-\frac{1}{2}} + \ln p] \\ = \phi(p)$$

so that by the inverse transformation

$$(3) \quad I(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(p) \exp(pt) dp$$

$$(4) \quad = \frac{e^{-t}}{4\pi i} \int_{c-i\infty}^{c+i\infty} z^{-1} e^{zt} \ln(z-1) dz + \frac{1}{4i} \int_{c-i\infty}^{c+i\infty} z^{-\frac{1}{2}} e^{zt} (1+z)^{-1} dz.$$

The value of these integrals may be found by referring to tables of transforms.^{3,4} The result is

$$(5) \quad I(t) = \frac{1}{2} e^{-t} \int_{-\infty}^t u^{-1} e^u du + \pi^{\frac{1}{2}} e^{-t} \int_0^{t^{\frac{1}{2}}} \exp(u^2) du$$

so that*

$$(6) \quad f(x) = \frac{1}{2} \exp(-x^2) Ei(x^2) + \pi^{\frac{1}{2}} F(x),$$

where

$$Ei(y) = \int_{-\infty}^y u^{-1} e^u du \quad \text{and} \quad F(y) = \exp(-y^2) \int_0^y \exp(u^2) du$$

are well-tabulated functions.⁵ By expanding this result in an ascending series it can be identified term by term with equation (6) of Goodwin and Staton's paper.

I have communicated this result to Dr. Goodwin who informed me that although not aware of it he was within one step of deriving it from equation (4) of his paper. The following derivation is due to Dr. Goodwin:

Equation (4) gives

$$f'(x) + 2xf(x) = \pi^{\frac{1}{2}} - x^{-1}.$$

Thus

$$(7) \quad d[\exp(x^2)f(x)]/dx = \pi^{\frac{1}{2}} \exp(x^2) - x^{-1} \exp(x^2).$$

If this could be integrated between suitable limits it would give f in terms of $Ei(x^2)$ and $F(x)$ but, in fact, it is impossible to choose such real limits and it was at this point that Dr. Goodwin stopped. However, equation (7) can be written

$$(8) \quad \frac{d}{dx} \left\{ \exp(x^2)f(x) - \pi^{\frac{1}{2}} \int_0^x \exp(u^2) du \right\} = -x^{-1} \exp(x^2).$$

Taking the indefinite integral

$$(9) \quad \exp(x^2)f(x) - \pi^{\frac{1}{2}} \int_0^x \exp(u^2) du = - \int_0^x u^{-1} \exp(u^2) du + c$$

$$(10) \quad = -\frac{1}{2} \int_0^{x^2} t^{-\frac{1}{2}} e^t dt + c.$$

Comparison of the known behavior of $f(x)$ as $x \rightarrow 0$, namely

$$f(x) \sim -\ln x - \frac{1}{2}\gamma$$

with the fact that $Ei(x^2) \sim \ln x^2 + \gamma$ shows that the constant in equation (10) can be taken as zero if the lower limit of the integral is taken as $-\infty$. Equation (6) follows immediately.

This corresponds to the use of a limit of $i\infty$ when (7) is integrated so that it is not surprising that Dr. Goodwin missed the result.

I should like to express my thanks to Dr. Goodwin for permission to use his method in the latter part of this paper.

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¹ E. T. GOODWIN & J. STATON, "Table of $\int_0^\infty e^{-u^2} du/(u+x)$," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 319-326.

² H. T. CARSLAW & J. C. JAEGER, *Operational Methods in Applied Mathematics*, Oxford, 1941.

³ L. A. PIPES, *Applied Mathematics for Engineers and Physicists*, McGraw-Hill, New York, 1946.

⁴ R. V. CHURCHILL, *Modern Operational Mathematics in Engineering*, McGraw-Hill, New York, 1944.

⁵ FMR Index, p. 189-191, 220, 231.

* An ambiguity in this result will be discussed in the next issue—ED.

RECENT MATHEMATICAL TABLES

723[A-Z].—HAROLD T. DAVIS & VERA FISHER, *A Bibliography of Mathematical Tables*. Copyright, 1949, by Harold T. Davis, xxii, 286, 27 leaves. For sale University Bookstore, Northwestern Univ., Evanston, Ill., bound \$4.00. Mimeographed one side of each sheet. 20.9×27.5 cm.

This copyright work has been issued in an edition of only 50 copies in order to test the possible demand for a revised and more complete volume of this kind. The title page states that it was "prepared under the direction of" Professor Davis "with the assistance of" Miss Fisher. We shall try, in a general way, to articulate clearly what is to be found in this tentative edition and to make some (among many possible) comments, which may serve as useful suggestions in any later revisions.

The volume is divided into four parts:

- I. Introduction, leaves i-xxii.
- II. Bibliography of Mathematical Tables, leaves 1-196 + 27.
- III. Index of Tables by Classification of Functions, leaves 197-262.
- IV. Index of Tables by Names of Authors, leaves 263-286.

We shall begin by first considering the 223 pages of Section II. There are here about 3,680 entries dated 1475-1948. Titles of almost all periodical articles are given, which is a highly praiseworthy feature, but there is a most undesirable total lack of uniformity of treatment of such titles, as well as of the abbreviated forms for titles of periodicals, and other details. In the case of pamphlets or books there is a similar lack; sometimes the total number of pages in the volume is given but more often not; names of places of publications are omitted in a number of entries, and generally unintelligible Latin forms are also to be found; there is no uniformity in the forms of names of authors (at least six of which are incorrectly spelled), and titles such as