Comparison of the known behavior of $f(x)$ as $x \to 0$, namely

$$f(x) \sim -\ln x - \tfrac{1}{2}\gamma$$

with the fact that $Ei(x^2) \sim \ln x^2 + \gamma$ shows that the constant in equation (10) can be taken as zero if the lower limit of the integral is taken as $-\infty$. Equation (6) follows immediately.

This corresponds to the use of a limit of $i\infty$ when (7) is integrated so that it is not surprising that Dr. Goodwin missed the result.

I should like to express my thanks to Dr. Goodwin for permission to use his method in the latter part of this paper.

R. H. Ritchie

University of Kentucky
Lexington, Kentucky

[1] E. T. Goodwin & J. Staton, "Table of $\int_0^\infty e^{-u^2}du/(u+x)$," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 319–326.
[2] H. T. Carslaw & J. C. Jaeger, *Operational Methods in Applied Mathematics*, Oxford, 1941.
[3] L. A. Pipes, *Applied Mathematics for Engineers and Physicists*, McGraw-Hill, New York, 1946.
[4] R. V. Churchill, *Modern Operational Mathematics in Engineering*, McGraw-Hill, New York, 1944.
[5] FMR *Index*, p. 189–191, 220, 231.
* An ambiguity in this result will be discussed in the next issue—Ed.

RECENT MATHEMATICAL TABLES

723[A–Z].—Harold T. Davis & Vera Fisher, *A Bibliography of Mathematical Tables. Copyright, 1949, by Harold T. Davis*, xxii, 286, 27 leaves. For sale University Bookstore, Northwestern Univ., Evanston, Ill., bound \$4.00. Mimeographed one side of each sheet. 20.9 × 27.5 cm.

This copyright work has been issued in an edition of only 50 copies in order to test the possible demand for a revised and more complete volume of this kind. The title page states that it was "prepared under the direction of" Professor Davis "with the assistance of" Miss Fisher. We shall try, in a general way, to articulate clearly what is to be found in this tentative edition and to make some (among many possible) comments, which may serve as useful suggestions in any later revisions.

The volume is divided into four parts:

I. Introduction, leaves i–xxii.
II. Bibliography of Mathematical Tables, leaves 1–196 + 27.
III. Index of Tables by Classification of Functions, leaves 197–262.
IV. Index of Tables by Names of Authors, leaves 263–286.

We shall begin by first considering the 223 pages of Section II. There are here about 3,680 entries dated 1475–1948. Titles of almost all periodical articles are given, which is a highly praiseworthy feature, but there is a most undesirable total lack of uniformity of treatment of such titles, as well as of the abbreviated forms for titles of periodicals, and other details. In the case of pamphlets or books there is a similar lack; sometimes the total number of pages in the volume is given but more often not; names of places of publications are omitted in a number of entries, and generally unintelligible Latin forms are also to be found; there is no uniformity in the forms of names of authors (at least six of which are incorrectly spelled), and titles such as

"Major General" and "Sir" are given in some cases but wholly lacking in dozens of others; dates of publication are absent in a number of cases; in at least one case the library and location of a publication is given, and in a few cases some inkling is noted as to the contents of entries, but the exact pages in books, where material of interest in the bibliography appears, are practically never given; the date of a third edition of an important work is given as if it were the first, and similarly for a third tirage; there are many titles with errors, some of the most fantastic character. There are at least three cases where two works of each of three authors are each ascribed to two different people. A number of dates of publications are incorrect. Wrong tabular references to certain authors are to be found in at least two cases.

The arrangement of titles is alphabetical according to authors, except in such cases as reports of the Table Committee of the BAAS (where the authors of tables are almost completely known), Harvard University, Computation Laboratory, NBSCL, etc. For each of 20 letters of the alphabet, one to three extra pages (27 in all), numbered xa, xb, xc, with additional titles, are added.

The list gives evidence that the method of compilation included the copying (not always accurately) of practically every entry of published material in:

(a) FMR, *Index*, 1946 (with the addition of titles in the case of periodical references);
(b) D. H. LEHMER, *Guide to Tables in the Theory of Numbers*, 1941;
(c) *Scripta Mathematica*, v. 2 (1934), p. 91–93, 297–299, 379–380; v. 3 (1938), p. 97–98, 192–193, 282–283, 364–366; v. 4 (1936), p. 101–104, 198–201, 294–295, 338–340.
(d) J. HENDERSON, *Bibliotheca Tabularum Mathematicarum*, 1926;
(e) J. W. L. GLAISHER, *Report of the Committee on Mathematical Tables*, 1873; and some titles from:
(f) R. C. ARCHIBALD, *Mathematical Table Makers*, 1948;
(g) *MTAC*, nos. 1–26, 1943–1949;
(h) A. DE MORGAN, "Table," *The English Cyclopaedia*, *Arts and Science Sect.*, v. 7, 1861.

These sources certainly account for more than 90% of the titles. The reviewer has verified that of the 3,680 entries there are 1,085 in the letters A–F and that of these 94% of the entries were in (a)–(h). In the 6% of new titles 36 referred to tables, 21 to theory, and 6 to calculating machines (including slide rules). In the whole book there are references to over 130 authors of material on calculating machines, and 16 references to graphical aids. It will thus be seen that the title of Section II is a decided misnomer and that the title of the work under review is only partially descriptive.

Because of the nature of the compilation there are scores of titles referring to material of no possible current use and almost wholly inaccessible to nearly every reader. There is no such exclusion as was practiced in the FMR, *Index*. In its present form Section II is wholly unreliable as a historical or bibliographical guide. An enormous amount of labor will still be necessary to bring this fundamental Section into a form having appeal to scholars.

With Section II we next naturally associate the 24 pages of Section IV and the 12 pages of Subject Classification, p. xi–xxii, of I. In IV is an alphabetical list of authors followed by indications of the kind of tables, or other contents, in the entries of II, according to the useful scheme of classification, which is a great elaboration of the classification given by Professor Davis in his *Tables of the Higher Mathematical Functions*, v. 1, 1933, p. 5–12. In a general way the new classification was that adopted by the NRC Committee on Mathematical Tables and Other Aids to Computation.

In Section III (64 p.), under about 300 subject headings are references by authors to the entries in IV.

In practically no case is there any indication of the range of any table. No warning is anywhere given concerning wholly useless tables, because of their gross inaccuracies.

If, in the whole work, 6% of the titles are new items for consideration by the authors of the possible forthcoming new edition of FMR, *Index*, a valuable contribution to scholarship will have been made. Furthermore, the ready subject index of Section III may, with the aid of other works, be frequently useful to specialists. Misspelled "Napierian" on p. xi and 202 should be wholly eliminated.

In the Introduction is a table showing the "distribution by centuries of 3,410 contributions to tables and table-making," for the period 1500–1947. The difference between 3,410 and 3,680 appears to indicate the number of titles published in 1948. For the century 1500–1599 two titles are indicated, presumably for Rheticus and Otho (listed as a separate title and not as part of the work of Rheticus); Regiomontanus, 1475, is not counted. The numbers are indicated for the following quinquennial periods; for the century 1600–1700 the total number of titles is given as 70; 1701–1800, 84; 1801–1900, 985; 1901–1947, 2,262.

It has not seemed worth-while to give chapter and verse for every statement made in this review. In a work of this kind a "prepared-under-the-direction-of" prop is indeed a frail one whereon to lean.

R. C. A.

724[A, B, D, E, G, H, I].—FRITZ EMDE, *Tables of Elementary Functions. Second Edition. With 83 Figures. Tafeln elementarer Funktionen.*, Leipzig, Teubner, 1948, xii, 181 p. 16.3 × 24 cm. Price 11.60 German marks, bound in boards and canvas.

This work, first published in 1940, was a great elaboration of the 78 pages devoted to Elementary Functions in the third edition (1938) of JAHNKE & EMDE's *Tables of Functions*. Professor Emde has the following preface in the present edition:

"The Second Edition of the Tables of Elementary Functions (an almost unchanged reprint of the First Edition of 1940) should have been issued in 1944. But after having been printed all copies were destroyed at the book binder's by bombs and fire during the war. It is only now possible to reprint this edition from the same manuscript. Pretzfeld, January 1948"

An offset reprint of the 1940 edition was made at Ann Arbor in 1945, and this was reviewed in *MTAC*, v. 1, p. 384–385.

The only difference which the reviewer noticed, in comparing this edition with the work under review, was the substitution, on p. 10, of a fine relief (Fig. 3) of the function z^z, z being complex (instead of $z = x^y$ on logarithmic scale for $-1 < y < +1$) supplementing the altitude chart, Fig. 79, p. 158 [the function $z^z = (re^{ip})^{x+iv}$].

R. C. A.

725[**B, L**].—H. BREMMER, *Terrestrial Radio Waves. Theory of Propagation.* New York, Elsevier Publishing Co., 1949, x + 343 p. 17 × 24.5 cm. Price $5.50.

Page 44 contains 3D values of $\frac{1}{2}(3x)^{2/3}$, where x is either $S + \frac{1}{4}$ or $S + \frac{3}{4}$ and $S = 0(1)5$. Page 45 contains 3D values of $\frac{1}{2}(3x)^{2/3}$, where the x are the first six positive roots of $J_{2/3}(x) - J_{-2/3}(x) = 0$ or $J_{1/3}(x) + J_{-1/3}(x) = 0$. Page 69 is a relief diagram of $H_{1/3}(x + iy)$, for $-10 \leqslant x \leqslant 10$ and $-5 \leqslant y \leqslant 5$. The branch cut is on the negative real axis.

Chapter VI. "Numerical computations and results." The results are mostly given in the shape of careful diagrams of which there is a large number. Other similar diagrams can be found throughout the book.

A. E.

726[**C, L**].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 22. *Tables of the Function* $\frac{\sin \phi}{\phi}$ *and of its First Eleven Derivatives.* Cambridge, Mass., Harvard University Press, 1949, xviii, 241 p. 19.6 × 26.6 cm. Price $8.00.

This interesting volume was prepared as an aid to the application of FOURIER transforms. It gives to 9D the functions

$$g(\phi) = \phi^{-1} \sin \phi$$

and $g^{(n)}(\phi)$, for $n = 1(1)11$ and $\phi = 0(\pi/360)20\pi - \pi/360$.

The argument ϕ is expressed in degrees, the interval being $\frac{1}{2}°$. The table is conveniently arranged so that at one opening there are 14 columns, the extreme ones giving the argument and the 12 others giving $g^{(n)}(\phi)$ for $n = 0(1)11$. The functions are very moderate in their behavior. In fact the bulk of the table gives only 8 significant figures. Beyond p. 38 all values are less than a tenth. However, the functions are still very much alive at 20π.

Since the tables give successive derivatives, interpolation is most easily performed by TAYLOR's formula, the second-order formula being adequate for 8D work, linear interpolation giving accuracy to within 4 units of the 6-th decimal.

The table was computed on Harvard's Mark I from an 18D table of $\sin \phi$, the successive g's were computed from the recurrence formula

$$g^{(n)}(\phi) = \left[\frac{d^n \sin \phi}{d\phi^n} - ng^{(n-1)}(\phi)\right] \Big/ \phi.$$

The whole computation required less than two weeks of machine time.

An introduction (p. xiv–xviii) by R. C. SPENCER includes a few of the applications of the table to the theory of Fourier transforms, although no actual examples are worked out. Briefly, there are two main uses which facilitate the transformation of a function defined or given by a power series (really a polynomial of degree $\leqslant 12$) or a Fourier series (really a trigonometric polynomial of degree $\leqslant 20$). These methods apply to functions

$F(x)$ of the real variable x, which vanish outside a finite interval, say $-1 < x < 1$. Then the Fourier transform G of F is given by

$$G(u) = \int_{-1}^{1} F(x)e^{iux}dx.$$

In case

$$F(x) = \sum_{n=0} A_n x^n,$$

then

$$G(u) = 2 \sum_{n=0} A_n i^{-n} g^{(n)}(u),$$

and in case

$$F(x) = \sum_{n=0} A_n e^{\pi inx},$$

then

$$G(u) = 2 \sum_{n=0} A_n g(u + n\pi).$$

Thus, the table is used in the first case by rows and in the second case by its first column to obtain directly the values of the transformed functions.

D. H. L.

727[D, P].—JULES GAUNIN, L. HOUDAILLE, & A. BERNARD, *Tables pour le Tracé des Courbes de Chemins de Fer, Routes & Canaux. Nouvelle édition revue et corrigée. Nouveau tirage. Première Partie: Tables Trigonométriques . . . Deuxième Partie: Recueil de Coordonnées* . . . Paris, Dunod, 1948, xlvii, 181 p., xiv, 182 p. 13.7 × 21 cm.

This is a new tirage of a very old book; it seems to be practically identical with the edition of 1922, when the chief author Gaunin was already dead; there was another tirage in 1925. According to the *Catalogue* of the Bibliothèque Nationale there were a 1919 edition containing 426 p., a two-v. edition in 1911; and the second ed. in 1904, 2 parts in one v. The first edition of *Tables Trigonométriques pour le tracé des Chemins de Fer* . . . , was published in Paris, Dunod, 1862, xxxii, 181 p.; the second part, *Recueil de Coordonnées*, by Jules Gaunin, L. Houdaille, & A. Bernard, Paris, 1896, xxvi, 176 p.

The first part gives a 6D table of the six trigonometric functions, versed sine, and versed cosine, for $\alpha = 0(30'')90°$. There are also several other columns with values of 2α, $180° - 2\alpha$, $90° - \alpha$, 6D values of $\pi\alpha/180°$ and of $\pi(90° - \alpha)/180°$. The introductory pages deal with material of the tables, with trigonometry, and with some practical problems.

The second part is devoted to tables for determining coordinates of points on circular arcs. The tables on p. 1–120 are for finding ordinates corresponding to abscissae measured on a tangent to the arc to be determined. The tables on p. 121–154 are for solving similar problems with reference to a chord of the arc; and the tables, p. 155–156, for points corresponding to a prolongation of the chord.

Of the five remaining tables, A–E, in A, for $R = 100(5)600(100)$-$3000(500)4000$, 9D values are given for $2\pi R$; lengths of arcs corresponding to central angles, 1°, 1′, 1″, i.e., $2\pi R/360$, $2\pi R/21600$, $2\pi R/1296000$; and angles in seconds corresponding to an arc of 1^m; also the values of $1000/R$.

In B, for $R = 1^m$, corresponding to 1(1)100, interpreted as central degrees, or minutes, or seconds, are given the corresponding lengths of arcs in meters.

In C, for arc of .01(.01)1(1)10(10)100(100)1000 meters, and R = 100(50)-600(100)1000, 1.1(.1)3(.5)4(1)6, 10, are given the corresponding angles to the nearest thousandth of a second.

Tables D and E are for conversion of degrees to grades, and for grades to degrees.

We have elsewhere referred to such easement curves as the clothoid which are the ones of importance in modern times in laying out railways and other routes; see *MTAC*, v. 3, p. 146, 452.

R. C. A.

728[F].—H. CHATLAND, "On the euclidean algorithm in quadratic number fields," Amer. Math. Soc., *Bull.*, v. 55, 1949, p. 548–553.

For each prime p of the form $24x + 1$ less than $2^{14} = 16384$ with the exception of p = 73, 97, 193, 241, 313, 337, 457 and 601 there is given a representation

$$p = q_1m_1 + q_2m_2, \tag{1}$$

where q_1, q_2, m_1, m_2 are quadratic nonresidues of p, and where q_1 and q_2 are odd primes dividing q_1m_1 and q_2m_2 respectively to odd powers. No such representations are possible in the exceptional cases of p listed above.

No column headings indicate which numbers in the table are the q's. These seem to be the first and third numbers on the right of the equal sign, except in the case of p = 409 and 577.

ERDÖS & KO[1] showed that in case the prime p has a representation (1) the field $K(p^{\frac{1}{2}})$ has no euclidean algorithm. The reason for the upper limit 2^{14} is that according to a theorem of DAVENPORT,[2] no field $K(p^{\frac{1}{2}})$ is euclidean for $p > 2^{14}$.

D. H. L.

[1] P. ERDÖS & CH. KO, "Note on the euclidean algorithm," London Math. Soc., *Jn.* v. 13, 1938, p. 3–8.

[2] H. DAVENPORT, "Indefinite binary quadratic forms and Euclid's algorithm in real quadratic fields," to appear in London Math. Soc., *Proc.*

729[F].—S. D. CHOWLA & J. TODD, "The density of reducible integers," *Canadian Jn. Math.*, v. 1, 1949, 297–299.

A definition of a reducible integer is given in RMT **734**. An alternative definition is the following. An integer is reducible in case the prime factors of $1 + n^2$ are all less than $2n$. The authors in collaboration with J. W. WRENCH have examined the first 5000 numbers and find that about 30 per cent of them are reducible. This conjecture is unproved. The number of reducible integers in each of the 50 centuries is tabulated. The table also gives the number B_n of integers n whose prime factors are less than $2n^{\frac{1}{2}}$. These numbers are shown to have a density

$$1 - \ln 2 = .3068528.$$

The table has been completely recalculated by one of the authors and was found to contain 25 errata. Since the table contains only 12 lines it is more economical to reproduce it below in corrected form than to single out its errata. The number A_n of reducible integers $<n$ is given on the left, not on the right as stated in the paper.

	0		1000		2000		3000		4000			
n	A_n	B_n	A_n	B_n	A_n	B_n	A_n	B_n	A_n	B_n		
1–100	30	57	31	43	29	43	35	41	28	42		
101–200	29	50	27	43	30	42	28	43	28	40		
201–300	27	47	33	44	23	42	24	43	28	41		
301–400	26	45	27	41	32	39	32	43	31	40		
401–500	31	45	31	45	27	44	28	41	27	42		
501–600	29	45	23	44	32	39	34	41	37	39		
601–700	29	44	27	40	26	43	24	40	33	41		
701–800	29	44	35	43	32	41	30	43	35	39		
801–900	27	44	27	45	27	43	29	40	30	43		
901–1000	23	42	31	39	29	42	20	41	38	41		
Totals	280	463	292	427	287	418	284	416	315	408	1458	2132

D. H. L.

730[F].—L. Gambelli, "Sui caratteri di divisibilita con una tabella dei coefficienti di divisibilita di tutti i numeri da 2 a 101," *Period. Mat.*, s. 4, v. 27, 1949, p. 109–116.

This note gives a table of the absolutely least remainder of 10^n on division by m for all values of n and each integer $m \leqslant 101$. In case this remainder is negative it is printed in boldface type. The purpose of the table is to give rules for finding the remainder on division of a given number N by m by replacing N by a linear combination of its decimal digits, the coefficients of this combination being tabulated. These coefficients are periodic, preceded by a preliminary nonperiodic part whenever m is not prime to 10.

As an example of the use of the table, for $m = 37$ the table gives the period

$$1, 10, -11, 1, 10, -11, \cdots.$$

The corresponding criterion for divisibility by 37 is as follows: Let the digits of N, beginning from the left, be d_0, d_1, d_2, $\cdots$. Then N is divisible by 37 if and only if

$$d_0 + 10d_1 - 11d_2 + d_3 + 10d_4 - 11d_5 + \cdots$$

is divisible by 37.

D. H. L.

731[F].—H. Gupta, "On a conjecture of Miller," Indian Math. Soc., *Jn.*, v. 13, 1949, p. 85–90.

The well-known function $\mu(n)$ of Möbius, which is $+1$ or -1 according as n is a product of an even or odd number of distinct primes and which is zero otherwise, plays an important role in the theory of distribution of primes. Stieltjes conjectured in 1885 that the function

$$M_1(x) = \sum_{n \leq x} \mu(n)$$

satisfies

$$|M_1(x)| \leqslant x^{\frac{1}{2}},$$

whose truth would imply the truth of the celebrated Riemann hypothesis. The conjecture mentioned in the title of the present paper involves the sum function

$$M_2(x) = \sum_{n \leq x} M_1(n)$$

and asserts that $M_2(x) \leqslant 0$ for $x \geqslant 3$. The paper verifies this conjecture for $x \leqslant 20\,000$. A table is given of $-M(x)$ for $x = 25(25)20\,000$. There is also a table of $-A(x) = -M(x)/x$ for $x = [100(100)20\,000; 5D]$.

The author conjectures that $A(x)/\log x$ is bounded. According to a heuristic argument of BRUN,[1]

$$A(x) = -2 + 12/x + \cdots.$$

The present table fails to support this result since $A(x) < -4$ for x near 18500.

D. H. L.

[1] VIGGO BRUN, "La somme des facteurs de Möbius," Den 10. Skandinaviske Matematiker Kongres, *Comptes Rendus*, Copenhagen, 1947, p. 40–53.

732[F].—M. PETROVICH, "Elementarna posmatrania o rasporedu omaniih̑k prostik̑h brojeva," [Elementary observations on the distribution of small prime numbers], Srpske Akad. Nauka, Belgrade, *Glas*, v. 189, 1946, p. 5–45.

Page 43 contains tables of the number of primes not exceeding x and the number of these which are of the forms $6m + 1$ and $6m - 1$ for $x = 50(50)1000$. These functions are compared with certain approximating functions.

733[F].—H. TIETZE, "Tafel der Primzahl-Zwillinge unter 300 000," Akad. d. Wissen., Munich, *math. nat.-Kl.*, *Sitz.*, 1947, p. 57–72 (1949).

If p and $p + 2$ are both primes, then $(p, p + 2)$ is called a prime pair or a pair of twin primes. The table in this paper lists the prime pairs $<300\,000$ by giving $p + 2$ in each case. This makes 2994 such pairs.

The author is apparently unaware of three recent papers by SUTTON,[1] CHERWELL[2] and SELMER & NESHEIM[3] [*MTAC*, v. 2, p. 210, 342]. This latter table agrees as far as it goes (200 000) with the one under review. The discrepancy noted between Sutton and Selmer & Nesheim in *MTAC*, v. 2, p. 342, is thus to be blamed on Sutton, who omits two prime pairs between 70 000 and 80 000, one between 90 000 and 100 000 and one between 120 000 and 130 000.

On p. 58 the author gives a rather incomplete list of errata in KRAITCHIK's list of primes to 300 000. Missing errata,[5] however, do not happen to influence the author's table.

D. H. L.

[1] C. S. SUTTON, "An investigation of the average distribution of twin prime numbers," *Jn. Math. Phys.*, v. 16, 1937, p. 1–42.

[2] LORD CHERWELL, "Note on the distribution of the intervals between prime numbers," *Quart. Jn. Math.*, Oxford s., v. 17, 1946, p. 46–62.

[3] E. S. SELMER & G. NESHEIM, "Tafel der Zwillingsprimzahlen bis 200 000," K. Norske Videnskabers Selskab, Trondhjem, *Forhandlinger*, v. 15, 1942, p. 95–98.

[4] M. KRAITCHIK, *Recherches sur la Théorie des Nombres*, v. 1, Paris, 1924.

[5] On page 156 of the reviewer's *Guide to Tables in the Theory of Numbers*, Washington, 1941, is given what was hoped to be a complete list of errata in Kraitchik's list of primes. Tietze, however, notes two additional errata, namely:

for	$p = 252141$	read	$p = 252143$	
for	$p = 297671$	read	$p = 297971$	[*MTAC*, v. 2, p. 313]

734[F].—J. TODD, "A problem on arctangent relations," *Amer. Math. Monthly*, v. 56, 1949, p. 517–528.

There are two tables giving the representation of arctangents of integers and rationals as linear combinations with integral coefficients of arctangents of integers. The numbers

$$\text{(S)} \qquad 3, 7, 8, 13, 17, 18, 21, 30, \cdots$$

have the property that, if m be one of them, arctan m can be expressed in terms of arctangents of integers $<m$ and not in (S). Thus

$$\arctan 18 = 3 \arctan 1 - 2 \arctan 2 + \arctan 5.$$

The utility of such representations has been pointed out by J. C. P. MILLER [*MTAC*, v. 2, p. 62–63, 147–148] in connection with the preparation of a table of the Gamma function of a complex variable. The study of such relations goes back to GAUSS. The numbers of the set (S) are called reducible.

Table I (p. 525) lists all reducible integers not exceeding 342 and gives for each of these 100 numbers n the expression of arctan n in terms of the arctangents of irreducible integers. There is also given an auxiliary integer c_n permitting an immediate passage from an arctangent to an arccotangent relation, where c_n is the new coefficient of arccot 1. Thus for $n = 18$, $c_n = 1$ and

$$\operatorname{arccot} 18 = \operatorname{arccot} 1 - 2 \operatorname{arccot} 2 + \operatorname{arccot} 5.$$

Two errata have been supplied by the author

$n = 183$	*for*	$c_n = 1$	*read*	$c_n = 0$
	for	$4(1) + (3)$	*read*	$7(1) - (2)$
$n = 307$	*for*	$c_n = 2$	*read*	$c_n = -2$.

Table II gives for each prime $p \leqslant 409$ of the form $a^2 + b^2$ (i.e., for 2 and for all primes of the form $4k + 1$) expansions of arccot (a/b), $(a > b)$ as a linear combination of arctangents of irreducible integers. In addition there are given the numbers c for converting to arccotangents. With each p is given also the least positive integer n_p for which $1 + n^2_p = mp$, the quotient m is given in terms of its prime factors.

D. H. L.

735[G].—ALBERT SADE. *Sur les Chevauchements des Permutations*. Published by the author, Marseille, 1949. 8 p.

The tables (for $n \leqslant 12$) are of various classifications of the permutations of n distinct elements associated with the foldings of a linear strip of postage stamps which are characterized as being without "chevauchements." A chevauchement is an interlacing of number pairs a, $a + 1$, and b, $b + 1$ for a, b both odd or both even. Also, for $n \leqslant 8$, the permutations of n distinct elements with one element fixed are tabled according to the number of such interlacings.

J. RIORDAN

Bell Telephone Laboratories
New York, N. Y.

736[I].—R. E. Greenwood. "Numerical integration for linear sums of exponential functions," *Annals Math. Stat.*, v. 20, 1949, p. 608–611.

The author considers numerical integration over a finite range, using $n + 1$ evenly spaced abscissae. He introduces coefficients which would make the integration exact if the integrand were a linear combination of functions of the form $\exp(jx)$, where j runs from 0 to n (positive case), or from $-m$ to m, $2m = n$ (symmetric case). For both cases, for $n = 1(1)6$, the coefficients are given to from 5D to 9D; about 45 values in all. He compares the results with the usual Newton-Cotes method, for $n = 4$, and finds that his coefficients "compare favorably" with Newton-Cotes for the functions $1/(x + 3)$, $\exp(-x^2)$, $x \exp(x)$, x^6, and $\exp(2.2x)$.

J. L. Hodges, Jr.

Univ. of California
Berkeley, Calif.

737[I, L].—H. E. Salzer & Ruth Zucker, "Table of the zeros and weight factors of the first fifteen Laguerre polynomials," Am. Math. Soc., *Bull.* v. 55, 1949, p. 1004–12.

For the first fifteen Laguerre polynomials, the zeros $x_i^{(n)}$ are given to 12 decimal places, the weight factors

$$\alpha_i^{(n)} = [n!/L_n'(x_i^{(n)})]^2/x_i^{(n)}$$

and values of $\alpha_i^{(n)} \exp\{x_i^{(n)}\}$ to 12 significant places. The chief use of the table is for numerical integration in the semi-infinite range by means of the mechanical quadrature formula

$$\int_0^\infty f(x)dx = \sum_{i=1}^{n} \alpha_i^{(n)} \exp\{x_i^{(n)}\} f(x_i^{(n)})$$

which is exact when $e^x f(x)$ is a polynomial of degree not exceeding $2n - 1$.

A. E.

738[K].—H. J. Godwin, "On the estimation of dispersion by linear systematic statistics," *Biometrika*, v. 36, 1949, p. 92–100.

Let $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ be the order statistics of a random sample of n from a normal population with cumulative distribution function

$$F(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} \exp(-\tfrac{1}{2}t^2)dt,$$

and let $y_i = x_{i+1} - x_i$ $(i = 1, 2, \cdots, n - 1)$. Table 2 gives $E(y_i)$ and $E(y_i y_j)$ to 5 or more decimals for $n = 2(1)10$. The table of $E(y_i y_j)$ supersedes an equivalent table to 2 decimals by Hastings, Mosteller, Tukey, & Winsor [RMT **740**]. To calculate Table 2, $\psi(i)$ and $\psi(i, j)$ defined by

$$\psi(i) = \int_{-\infty}^{\infty} F^i(x)[1 - F(x)]^i dx$$

$$\psi(i, j) = \int_{-\infty}^{\infty} F^i(x) \int_x^\infty [1 - F(y)]^j dy dx$$

were used; these are tabled to 10 decimals in Table 1, for $i = 1(1)5$ and $j = i(1)10 - i$. Suppose next that the differences y_i are defined as above but with $F(x)$ replaced by $F_\sigma(x)$, where $F_\sigma(\xi) = F(\xi/\sigma)$.

Table 3 gives to 5 or more decimals the coefficients α_i which minimize the variance of $d = \sum_{i=1}^{n-1} \alpha_i y_i$ subject to the condition that $E(d) = \sigma$, for $n = 2(1)10$. Table 4 lists the efficiencies (calculated as inverse ratios of variances) of the unbiased estimates of σ formed from the following statistics: the "best" linear estimate d, the sample mean deviation from the mean, the sample mean deviation from the median, and $x_{n-k+1} - x_k$ for $k = 1(1)[\frac{1}{2}n]$, $n = 2(1)10$; the efficiencies are given to tenths or hundredths of a per cent.

Henry Scheffé

Columbia University
New York 27, N. Y.

739[K].—E. J. Gumbel, "Probability tables for the range," *Biometrika*, v. 36, 1949, p. 142–148.

The author is concerned with the asymptotic distribution of the range in a sample of n independent equally distributed random variables (that is, the difference between the largest and the smallest values in the sample). The desired distribution density is given by $\psi(R) = 2e^{-R}K_0(2e^{-1/2R})$, where $K_0(x)$ is the Bessel function. On p. 145 we find the values of $\psi(R)$ and

$$\Psi(R) = \int_{-\infty}^{R} \psi(x)dx.$$

The range is $-3.2(.1)10.6$. The number of decimals drops from 8 at the tail ends ($R < -2.9$ and $R > 9$) to 5 in the central part (about $-1.1 < R \leqslant 3$). [See also *MTAC*, v. 4, p. 21.]

W. Feller

Cornell Univ.
Ithaca, N. Y.

740[K].—Cecil Hastings, Jr., Frederick Mosteller, J. W. Tukey & C. P. Winsor, "Low moments for small samples: a comparative study of order statistics," *Annals Math. Stat.*, v. 18, 1947, p. 413–426.

This contains tables of the means, variances, and covariances of the order statistics in samples of size $\leqslant 10$ from a normal universe, a rectangular universe, and a special universe with very long tails. For the normal universe and the special universe values computed from asymptotic formulas are also given. Means are given to 5D in all cases and are believed to be accurate to one unit in the fifth decimal. For the normal universe standard deviations are given to 5D with a maximum error of 2 or 3 units in the fifth decimal, while variances and covariances are given to 2D, with a possible error of one unit in the second decimal (except in one or two cases in which the error may be two units). Variances and covariances for the other two cases are given to 5D and are thought to be accurate to the places given. In addition the correlation coefficients in all cases are given to 2D.

C. C. C.

741[K].—PALMER O. JOHNSON, *Statistical Methods in Research*, xvi + 377 p. New York, Prentice Hall, 1949, 15 × 22.7 cm. Price $5.00.

This book has five tables in an appendix. Tables I, II, III, and IV are standard tables of areas and percentiles for the normal, t, χ^2, and F distributions. These tables occur in most modern applied statistics books or tables.[1]

Table V contains first and fifth percentile values of the distribution of the NEYMAN-PEARSON L_1 statistic which is used to test the hypothesis that k normally distributed populations have equal variances. This test was designed for samples of sizes $n_1, n_2, \cdots, n_k$. These tables were constructed for the case $n_1 = n_2 = \cdots = n_k = n$, and have as arguments $k = 2(1)10(2)30$, $n = 2(1)10, 12, 15, 20, 30, 60, \infty$. They are reproduced from the *Statistical Research Memoirs* v. 1 edited by J. NEYMAN and E. S. PEARSON, and have not been widely reproduced elsewhere. These *Memoirs* are no longer easily available.

Examples of the use of several other tables are included in the text. References are made to those tables, but they have not been printed in the text. Some would not be easily available to most readers of this book. For example page 102, reference 22, is to a table in *Sankhya*, v. 4, 1938.

FRANK MASSEY

Univ. of Oregon
Eugene, Oregon

[1] For example, see:
R. A. FISHER & F. YATES, *Statistical Tables for Biological Agricultural and Medical Research*, Edinburgh, 3rd ed., 1948 [*MTAC*, v. 3, p. 360–361] from which Tables II and III were reproduced.
G. W. SNEDECOR, *Statistical Methods Applied to Experiments in Agriculture and Biology*, Ames, Iowa, 1940, from which Table IV was reproduced. For errata see *MTAC*, v. 1, p. 85–86.

742[K].—P. B. PATNAIK, "The non-central χ^2- and F-distributions and their applications," *Biometrika*, v. 36, 1949, p. 202–232.

A noncentral chi-square variable, denoted by χ'^2, with n degrees of freedom (d.f.) and parameter χ may be defined as follows: Let x_i $(i = 1, \cdots, n)$ be n independent standard normal deviates, let a_i $(i = 1, \cdots, n)$ be n constants, and define $\chi'^2 = \sum_{i=1}^{n} (x_i + a_i)^2$. The distribution of χ'^2 is known to depend only on n and $\lambda = \sum_{i=1}^{n} a_i^2$. Noncentral F, denoted by F', with ν_1 and ν_2 d.f. and parameter λ may be defined as a random variable distributed like $(\nu_2\chi'^2)/(\nu_1\chi^2)$, where χ'^2 is a noncentral chi-square variable with ν_1 d.f. and parameter λ, χ^2 is a (central) chi-square variable with ν_2 d.f., and χ'^2 and χ^2 are statistically independent. Various approximations to the χ'^2 and F'-distributions are considered. There are seven small tables, six of which compare these approximations with exact values, one of which (Table 6) gives the power of a chi-square test at the 5% significance level, calculated from one of the approximations to the χ'^2-distribution. Tables similar to Table 6 but more extensive, for the 1% and 5% levels, and not based on approximations to the distributions, have been published recently by EVELYN FIX.[1] The following approximations are considered for χ'^2 with n d.f. and parameter λ: (i) χ'^2 is approximated by χ^2/K, where χ^2 is a (central) chi-square variable with ν d.f. (in general, fractional), and the constants ν and K are determined by fitting the first two moments; (ii) normal approximation; (iii) and (iv) two different series for the cumulative distribution function of χ'^2 in each of which (i) contributes the leading term. F' with ν_1 and ν_2 d.f. and parameter λ is approximated as F/K, where F has the (central) F-distribution with ν and ν_2 d.f., K and ν being fitted by the first two moments. The result is the same as though the above approximation (i) were used for the χ'^2 in the numerator of F'.

HENRY SCHEFFÉ

[1] EVELYN FIX, "Tables of noncentral χ^2," University of California, *Publications in Statistics*, v. 1, 1949, no. 2, p. 15–19.

743[L].—W. R. ABBOTT, "Evaluation of an integral of a Bessel function," *Jn. Math. Physics*, v. 28, 1949, p. 192–194.

The integral $v(t, n) = 2n \int_0^t u^{-1} J_{2n}(\gamma u) du$ occurs in the theory of transmission lines.[1] The expression of v as a finite combination of Bessel functions is known, but the two expansions obtained in this paper,

$$v = 1 - \frac{2}{\gamma t} \sum_{k=1}^{n} (2k - 1) J_{2k-1}(\gamma t)$$

and

$$v = 1 - \sum_{p=0}^{n-1} \frac{(-)^p (2n - p - 1)!}{p!(n - p - 1)!(n - p)!} \Lambda_{n-p}(\gamma t)$$

are better suited for numerical computation. The coefficients of the latter expansion are given numerically for $n = 1(1)9$. For $n > 9$ the first expansion is more useful.

A. E.

[1] See for example, M. F. GARDNER & J. L. BARNES, *Transients in Linear Systems, Studied by the Laplace Transformation*, v. 1, New York, 1942, p. 310–317.

744[L].—MILTON ABRAMOWITZ, "Asymptotic expansions of Coulomb wave functions," *Quart. Appl. Math.*, v. 7, 1949, p. 75–84.

The author considers the differential equation

$$\frac{d^2\eta}{d\rho^2} + (1 - 2\eta/\rho) y = 0,$$

and gives series-expansions of its solutions for different relative magnitudes of ρ and η (both real), which may be used for computational purposes. Starting with WHITTAKER'S integral representations of the confluent hypergeometric function, he defines two solutions y_1 and y_2 of the differential equation which are related to Whittaker's confluent hypergeometric function by the relations:

$$-2i\Gamma(1 + i\eta) y_1 = W_{i\eta, -\frac{1}{2}}(2i\rho)$$
$$2i\Gamma(1 - i\eta) y_2 = W_{-i\eta, -\frac{1}{2}}(-2i\rho).$$

He obtains asymptotic series for these solutions in the case that η is bounded and $\rho \to \infty$. These series are used to compute a table of the first three zeros of the function

$$\rho\Phi_0(\rho, \eta) = e^{\pi\eta}y_1 + y_2$$

for $\eta = 0(.5)3$. Three other expansions for $\rho\Phi_0(\rho, \eta)$ are obtained, one for the case $\rho < 2\eta$, one for $\rho > 2\eta$, and one for $2\eta < 1$ and ρ bounded. For bounded ρ and $\eta \to \infty$, an approximation in terms of Bessel functions is given.

MARIA A. WEBER

California Institute of Technology
Pasadena, California

745[L].—C. G. DARWIN, "On Weber's function," *Quart. Jn. Mech. Appl. Math.*, v. 2, 1949, p. 311–320.

WEBER'S equation is the differential equation

$$\frac{d^2u}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)u = 0.$$

The case $z = x\sqrt{i}$, $n = -\tfrac{1}{2} + ia$ (x, a real) arises in various wave problems, and is studied here. Instead of utilizing the theory of confluent hypergeometric functions, of which WEBER'S function is a particular instance, the author develops his theory *ab initio*. The even and odd solutions of the equation are unsuited because they are nearly proportional to each other for large x. The solutions selected for numerical tabulation now in progress at Scientific Computing Service, Ltd. (London) on behalf of the (British) National Physical Laboratory are

$$u_I = 2^{-\frac{1}{2}}\{(G_1/G_3)^{\frac{1}{2}}u_0 + (2G_3/G_1)^{\frac{1}{2}}u_1\}$$
$$u_{II} = 2^{-\frac{1}{2}}\{(G_1/G_3)^{\frac{1}{2}}u_0 - (2G_3/G_1)^{\frac{1}{2}}u_1\},$$

where

$$G_1 = |\Gamma(\tfrac{1}{4} + \tfrac{1}{2}ia)|, \qquad G_3 = |\Gamma(\tfrac{3}{4} + \tfrac{1}{2}ia)|,$$

and u_0 and u_1 are the even and odd solutions normalized to values 1 and x for small x. For positive a, u_I resembles an exponential function near the origin, and becomes oscillatory when $|x| > 2a^{\frac{1}{2}}$; for negative a, the function is oscillatory throughout. Convergent power series for small x, and asymptotic series for large x are given, both valid for fixed a. Other asymptotic expansions, valid for large a, are also developed.

Weber's equation, with real x and a, has also been investigated by MAGNUS,[1] and by CHERRY.[2]

A. E.

[1] W. MAGNUS, Deutsche Math.-Ver., *Jahrsb.*, v. 50, 1940, p. 140–161.
[2] T. M. CHERRY, Edinburgh Math. Soc., *Proc.*, s. 2, v. 8, 1948, p. 50–65.

746[L].—G. H. GODFREY, "Diffraction of light from sources of finite dimensions," *Australian Jn. Sci. Res.*, s. A, v. 1, 1948, p. 1–17.

The paper contains tables concerned with the diffraction of light at rectangular and circular apertures. In particular Table 1 (p. 7) is a table of

$$I(x) = [\mathrm{Si}(2x) - x^{-1}\sin^2 x]/\pi$$

for $x = [0(.1)15(.5)34.5; 5D]$.

There are also tables of the differences

$$I(x + 2.4\pi) - I(x) \quad \text{for} \quad x/\pi = -1.2(.1)1.3$$
$$I(x + 3.4\pi) - I(x) \quad \text{for} \quad x/\pi = -1.7(.1)0$$

to 5D. Table 4 (p. 13) gives 4D values of

$$I_1(x) = \int_0^x [H_1(2t)/(\pi t^2)]dt$$

for $x = 0(.1)15$, where

$$H_1(U) = \pi^{-1}\left\{2 + \int_0^\pi \sin(u \sin\theta - \theta)d\theta\right\}$$

is STRUVE'S function of order unity.

Table 5 (p. 15) gives the difference

$$I_1(x + 8.4) - I(x) \quad \text{for} \quad x = [-4.2(.1) - 2.8; 4D].$$

D. H. L.

747[L].—E. T. GOODWIN & J. STATON, "Table of $J_0(j_{0n}r)$," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 220–224.

Five decimal values of $J_0(j_{0n}r)$ are tabulated for $n = 1(1)10$ and $r = 0(.01)1$; j_{0n} being the nth positive zero of the BESSEL function of the first kind J_0. These values were calculated to seven decimal places by interpolation from the B. A. *Math. Tables*, v. vi, of Bessel functions; they were then differenced in the r direction on the National Accounting Machine as a check, and all doubtful roundings off were examined. The tables should be useful in the numerical discussion of potential problems with axial symmetry.

A. E.

748[L].—D. R. HARTREE, "The tabulation of Bessel functions for large argument," Cambridge Phil. Soc., *Proc.*, v. 45, 1949, p. 554–557.

The use of auxiliary functions in tabulating, to simplify interpolation, is well known. The author pleads for the use of auxiliary independent variables for the same purpose. As an illustration, he shows that linear interpolation in a suitably constructed table of only 41 entries should be sufficient to give $x^{\frac{1}{2}}J_0(x)$ and $x^{\frac{1}{2}}Y_0(x)$ from $x = 5$ to ∞ with an uncertainty of one unit in the seventh decimal. The auxiliary variable is x^{-2}, and the auxiliary functions are either $P(x)$ and $Q(x)$ defined by

$$J_0 + iY_0 = (2/\pi x)^{\frac{1}{2}}(P + iQ)e^{i(x-\pi/4)},$$

or $R(x)$ and $x\psi(x)$ defined by

$$J_0 + iY_0 = (2/\pi x)^{\frac{1}{2}}Re^{i(x-\pi/4+\psi)}.$$

Tables are given for $(2/\pi)^{\frac{1}{2}}P$ and $(2/\pi)^{\frac{1}{2}}Q$, and also for $(2/\pi)^{\frac{1}{2}}R$ and $x\psi$ for $x^{-2} = 0(.01).05$.

A. E.

749[L].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 13: *Tables of the Bessel Functions of the First Kind of Orders Sixty-four through Seventy-eight*. Cambridge, Mass., Harvard University Press, 1949, x, 566 p. 20 × 26.6 cm. Offset print. Price $8.00.

A summary of reviews in *MTAC* of earlier published volumes of the *Annals* is given in RMT **711**. The volume under review is the eleventh, in the monumental Harvard series of tables of BESSEL functions of the first kind, giving $J_n(x)$ for $n = 64(1)78$, $x = [0(.01)99.99;\ 10D]$, but prior to $x = 37.16$ all values of $J_n(x)$ to 10D are zero. $J_{78}(48.78)$ is the first significant value of this function.

All values in this table have not been previously published. Two more volumes in preparation will deal with the next 22 orders, and give also the values of $J_n(100)$ for $n = 0(1)100$.

R. C. A.

750[L].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 21: *Tables of the Generalized Exponential-Integral Functions*. Cambridge, Mass., Harvard Univ. Press, 1949, xxv, 416 p., 20 × 26.6 cm., $8.00.

The functions tabulated in this useful volume are

$$E(a, x) = \int_0^x (1 - e^{-u})u^{-1}dt, \qquad Es(a, x) = \int_0^x e^{-u}u^{-1} \sin u\, dt,$$

and

$$Ec(a, x) = \int_0^x (1 - e^{-u} \cos u)u^{-1}dt,$$

where $u = (a^2 + t^2)^{\frac{1}{2}}$. The integrals

$$\bar{E}(a, x) = \int_x^\infty e^{-u}u^{-1}dt \qquad \text{and} \qquad \int^\infty u^{-1} \cos u\, dt$$

can be evaluated in terms of elementary functions and the functions tabulated in this volume. All these functions are related to the generalized sine- and cosine-integral functions tables of which appeared in two earlier volumes of the same series[1] [*MTAC*, v. 3, p. 479–482]; they are also related (in special cases) to the exponential integral function by means of the relation

$$E(0, x) + \mathrm{Ei}(-x) = \log x + \gamma,$$

where log denotes the natural logarithm and γ is EULER's constant. These functions may also be regarded as incomplete (modified) BESSEL functions, for instance

$$\bar{E}(a, \infty) = K_0(a).$$

Asymptotic representations (for large x) can be obtained by the usual method of integration by parts.

Generalized exponential integral functions occur in the solution of the wave equation when line sources are immersed in a dissipative medium, and were encountered in particular in antenna theory. Their tabulation was undertaken, on the Automatic Sequence Controlled Calculator, at the request of RONOLD W. P. KING. First the integrands were tabulated, and then the integrals computed by numerical integration, partly by Weddle's rule, and partly by two rules based on fifth degree polynomials. An estimate of the error in the integrands and of that due to quadrature, together with a study of the difference sheets, led to an estimate of 3×10^{-8} for the maximum error in the integrals. The values, including differences, were rounded off to 6 decimals; an accuracy of 5.1 in the seventh decimal place is claimed for all numbers printed.

The volume contains a Preface by HOWARD AIKEN, an Introduction of four sections of which I "The generalized exponential-integral functions," II "Computation of the tables," III "Interpolation" were written by J. ORTEN GADD and THEODORE SINGER who also collaborated in preparing the coding and control tapes and supervising the calculation, and IV "Applications" by R. W. P. King and C. T. TAI. There are 8 tables each for a different value of the interval h. In each table both a and x run from 0 to $49h$ at steps of h. Each table consists of 50 pages of fifty lines each; each page contains, for a fixed value of a and for x running from 0 to $49h$, 6 decimal values of $E(a, x)$, $Es(a, x)$, and $Ec(a, x)$ together with first forward differences in both the a- and x-direction, except that differences are omitted when they can be found in the earlier parts of the volume. In the eight tables, h is respectively 0.001, 0.002, 0.005, 0.01, 0.02, 0.05, 0.1, 0.2.

All in all, this is a very useful and excellently arranged volume.

A. E.

[1] HARVARD UNIVERSITY COMPUTATION LABORATORY, *Annals*, v. 18, 19: *Tables of the Generalized Sine- and Cosine- Integral Functions*, Parts I and II, Cambridge, Mass., 1949.

751[L].—C. W. JONES, "On a solution of the laminar boundary-layer equation near a position of separation," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 385–407.

This paper is mainly concerned with coordinating the investigations of the problem mentioned in the title by S. GOLDSTEIN on the one hand, and by D. R. HARTREE on the other hand. Since rigorous proofs seemed too difficult, the discussion is largely computational.

Goldstein[1] expands the potential of the flow downstream in the form

$$\psi = \xi^3[f_0(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \cdots].$$

The f_r satisfy differential equations of the form

$$f_r''' - \tfrac{1}{2}\eta^3 f_r'' + (\tfrac{1}{2}r + 2)\eta^2 f_r' - (r + 3)\eta f_r = G_r,$$

where G_r depends on $f_0, \cdots, f_{r-1}$, and suitable boundary conditions.

$$f_0 = \eta^3/6, \qquad f_1 = \alpha_1\eta^2, \qquad f_2 = \alpha_2\eta^2 - \alpha_1^2\eta^5/15.$$

The differential equation for f_r, for $r \geqslant 3$, cannot be integrated explicitly, and so Jones uses for f_3 a step-by-step numerical integration, starting with the asymptotic solution for large η. His Table 1 gives f_r/α_1^r, and the derivative of this function, for $r = 0, 1, 2, 3$ and $\eta = 0(.1)4$, to a varying number of decimal places. f_4 and f_5 contain an unknown constant, and in these cases only those parts of the functions, and of their derivatives, which are independent of that constant are tabulated, for the same η, in Table 2. There is also a Table 3 giving $\sum \xi^r f_r'$ as function of η for several values of $\alpha_1\xi$, but this table, as the author points out, is of less general application.

A. E.

[1] S. GOLDSTEIN, "On laminar boundary-value flow near a position of separation," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 43–69.

752[L].—GEORGETTE DE NOCKERE, *Tables Numériques des Polynômes de Legendre* $P_{n,0}(\cos\theta)$ *et des Fonctions Associées* $P_{n,j}(\cos\theta)$ *ainsi que de leurs Intégrales* P^* *jusqu'a* $n = 15$ *et* $j = 4$, *pour l'argument* θ (*colatitude*) *variant de degré en degré. Tableaux des latitudes et longitudes divisionnaires et valeurs des multiplicateurs pour le calcul des coefficients du développement en série de polynômes de Laplace par la méthode des compartiments équivalents, d'une fonction de deux variables indépendantes.* Acad. r. de Belgique, *Cl. d. Sciences, Mémoires*, v. 24, fasc. 4, publ. no. 1592, 1949, 166 p. 16.1 × 25 cm. Price 150 Belgian francs.

The first table, 5D, Δ (p. 12–43), is of $P_{n,0}$ and of $P^*_{n,0} = \int P_{n,0}(\cos\theta)d\cos\theta$, for $n = 0(1)15$, $\theta^\circ = 0(1)90$. Here $P_{0,0} = 1$; $P_{1,0} = \cos\theta$, $P^*_{1,0} = \frac{1}{2}\cos^2\theta$; $P_{2,0} = \frac{1}{4}(3\cos 2\theta + 1)$, etc.

The second main table, 5D, Δ, (p. 44–151), is of $P_{n,j}$ and of

$$P^*_{n,j} = \int P_{n,j}(\cos\theta)d\cos\theta$$

for $n = 1(1)15$, $j = 1(1)4$, $\theta^\circ = 0(1)90$. Here

$$P_{1,1} = \sin\theta,\ P^*_{1,1} = \tfrac{1}{4}(\sin 2\theta - 2\theta) + \tfrac{1}{4}/\pi;\ \text{etc.}$$

The constants of integration are always chosen such that for $\theta = 90^\circ$, $P^*_{n,j} = 0$. Because of this condition the values for $\int P_{n,j}(\cos\theta)d\cos\theta$ given by G. PRÉVOST, *Tables de Fonctions Sphériques et de leurs Intégrales. . . .* Bordeaux and Paris, 1933, p. 153*–155*, have to be checked for appropriate constants of integration. In calculating the integrals, computations were made either directly, or with the aid of the following recurrence relations of LIÉNARD:[1]

$$(n+2)(n+1-j)\int P_{n+1,j}(\cos\theta)d\cos\theta = (n-1)(n+j)\int P_{n-1,j}(\cos\theta)d\cos\theta - (2n+1)\sin^2\theta P_{n,j}(\cos\theta).$$

$$(j-1)\int P_{n,j+1}(\cos\theta)d\cos\theta = (j+1)(n+j)(n+1-j)\int P_{n,j-1}(\cos\theta)d\cos\theta + 2j\sin\theta P_{n,j}(\cos\theta).$$

The original contributions of Miss Nockere consist in computing:

(i) the values of $P_{n,j}(\cos\theta)$ for $n = 9(1)15$, $j = 1(1)4$, by the formula,

$$P_{n+1,j}(\cos\theta) = (2n+1)\sin\theta\, P_{n,j-1}(\cos\theta) + P_{n-1,j}(\cos\theta)$$

(ii) the values of $P^*_{n,0}$ for $n = 0(1)15$, by means of the classical formula,

$$(2n+1)\int P(\cos\theta)d\cos\theta = P_{n+1}(\cos\theta) - P_{n-1}(\cos\theta) + \text{const.}$$

(iii) the values of $P_{n,j}$, $n = 1(1)15$, $j = 1(1)4$.

The values of $P_{n,j}(\cos\theta)$ for $n = 0(1)8$, $j = 0(1)4$, were adapted from the 7–10D tables of TALLQVIST, (a) "Tafeln der Kugelfunctionen $P_n(\cos\theta)$," 1905; (b) "Tafeln der abgeleiteten und zugeordneten Kugelfunctionen erster Art," 1906, Finska Vetenskaps-Societeten, *Acta*, v. 33, nos. 4; 9.

Since the Preface to the volume was written by Prévost, and dated 1938, it seems probable that these tables were computed some years ago. On the title page is the statement: "Impression décidée 4 mai 1948."

The final section (p. 153–166) is devoted to the subject matter referred to in the latter part of the title and is an elaboration of ideas developed in Prévost's work mentioned above.

R. C. A.

[1] A. Liénard, "Formules de récurrence pour les intégrales des fonctions adjointes des polynômes de Legendre," Acad. d. Sciences, Paris, *Comptes Rendus*, v. 196, 1933, p. 1773–1778 (there's a slip in the page reference here).

753[L].—J. Pachner, "Pressure distribution in the acoustical field excited by a vibrating plate." Acoust. Soc. Amer., *Jn.*, v. 21, 1949, p. 617–625.

Notations: z_{mn} is the n-th root of the equation

$$J_m(iz)J_m'(z) - J_m(z)J_m'(iz) = 0,$$

$A_{mn} = e_m J_m{}^2(z_{mn})J_m{}^2(z_{mn}i)i^{2m}$, where $e_0 = 2$ and $e_m = 1$ for $m = 1, 2, \cdots$,
$B_{mn} = 2i^m J_m(iz_{mn})J_{m-1}(z_{mn})/z_{mn}$
$C_{mn} = J_m(z_{mn})/(z_{mn}J_{m-1}(z_{mn}))$.

Table I: z_{mn} to 1 to 3 decimal places for $m = 0$ and $n = 0(1)4$, $m = 1$ and $n = 0(1)3$, $m = 2$ and $n = 0(1)3$, $m = 3$ and $n = 0(1)2$, $m = 4$ and $n = 0, 1$, $m = 5(1)7$ and $n = 0$.

Table II: A_{0n} to 4 significant figures for $n = 0(1)3$.

Table III: A_{mn} to 4 significant figures for $m = 1$ and $n = 0(1)3$, $m = 2$ and $n = 0, 1$, $m = 3(1)5$ and $n = 0$.

Table IV: B_{mn} to 4 significant figures for $m = 0, 1$ and $n = 0(1)3$, $m = 2$ and $n = 0, 1$, $m = 3(1)5$ and $n = 0$.

Table V: C_{mn} to 4 decimal places for $m = 0, 1$ and $n = 0(1)3$.

Table VI: $2mC_{mn}$ to 4 decimal places for $m = 0$ and $n = 0(1)3$, $m = 1$ and $n = 0, 1$, $m = 3(1)5$ and $n = 0$.

Table VII: Table of values where the partial acoustic pressure falls to zero.

There are also various graphs representing numerical values.

E. A.

754[L].—Charles H. Papas & Ronold King, "Input impedance of wide-angle conical antennas fed by a coaxial line," IRE, *Proc.*, v. 37, no. 11, 1949, p. 1269–1271.

This paper introduces the auxiliary functions $\zeta_n(x) = g_n(x) + ib_n(x)$, which are related to the spherical Hankel function of the second kind $h_n{}^{(2)}$ by the equation:

$$\zeta_n(x) = \frac{h_n^{(2)}(x)}{h_{n-1}^{(2)}(x) - nx^{-1}h_n^{(2)}(x)}$$

and gives values for the real and imaginary parts of $\zeta_n(x)$ in the form of curves covering the range $1 < x < 15$ and for g_1 to g_{29} and b_1 to b_{17}.

P. T. Nims

Chrysler Corporation
Detroit 31, Michigan

755[L].—M. ROTHMAN, "Table of $\int_0^x I_0(x)dx$ for 0(0.1)20(1)25," *Quart. Jn. Mech. Applied Math.*, v. 2, 1949, p. 212–217.

The function $f(x) = \int_0^x I_0(t)dt$, where $I_0(t)$ is the Bessel function of imaginary argument, is tabulated for x = [0(.1)20; 8S], and the function $e^{-x}f(x)$ is given to 9S for x = 15(1)25. In both tables modified second and fourth differences are given.

These tables extend considerably those previously published [*MTAC*, v. 1, p. 250, v. 3, p. 308].

D. H. L.

756[L].—V. V. SOLODOVNIKOV, "O primenenii trapetsoidalnykh chastotnych kharakteristik k analizu kachestva sistem avtomaticheskogo regulirovaniîâ" [On the application of trapezoidal frequency characteristics to the analysis of the behavior of systems of automatic regulation], *Avtomatika i Telemekhanika*, v. 10, 1949, p. 362–376.

To aid in the treatment of the transform

$$g(t) = (2/\pi) \int_0^\infty x^{-1}f(x) \sin tx \, dx$$

the author tabulates (p. 370–371) the function

$$(2/\pi)\{\mathrm{Si}(kt) - (1 - k)^{-1}[\mathrm{Si}(t) - \mathrm{Si}(kt) + t^{-1}(\cos t - \cos kt)]\}$$

for [k = 0(.05)1, t = 0(.5)26; 3D]. This function, which is the transform of a standard "trapezoidal function," is also graphed for k = .8, .9, 1. [cf. RMT **726**].

D. H. L.

757[L].—A. ULRICH, "Die ebene laminare Reibungschicht an einem Zylinder." *Arch. d. Math.*, v. 2, 1949, p. 37–41.

The functions appearing in the integration of the boundary layer equations according to the method of BLASIUS[1] and HOWARTH[3] satisfy certain nonlinear differential equations, of which the author gives explicitly those for the functions tabulated in his paper.

Table 1. Tables of f_1 and f_3 with first and second derivatives to 4 decimal places for η = 0(.1)4.

Table 2. Tables of g_5 and h_5 with first and second derivatives to 4 decimal places for η = 0(.2)4.

Table 3. Tables of g_7, h_7, k_7, g_9, h_9, k_9, j_9, q_9 with first and second derivatives to 3 decimal places for η = 0(.1)4.

Tables of f_1 and f_3 have been first given by HIEMENZ.[4] Howarth[3] improved the values of f_3 and gave first approximations for g_5 and h_5, which were later improved by FRÖSSLING.[2] The functions with subscripts 7 and 9 are tabulated here for the first time. They were obtained by numerical integration, by Adams' method, of the nonlinear ordinary differential equations which they satisfy.

A. E.

[1] H. BLASIUS, "Grenzschichten in Flüssigkeiten mit kleiner Reibung," *Zschr. f. Math u. Phys.*, 56, 1908, p. 1.

[2] N. FRÖSSLING, "Verdunstung, Wärmübertragung und Geschwindigkeitsverteilung bei zweidimensionaler und rotationssymmetrischer laminaren Grenzströmung," Lund, Sweden, Univ., *Acta*, Afd. 2, *Årsskrift*, v. 36, 1940, p. 1–32.

[3] L. HOWARTH, "On the calculation of steady flow in the boundary layer near the surface of a cylinder in a stream," ARC *Report* no. 1632, 1934.
[4] K. HIEMENZ, "Grenzschichten an einem in einen gleichförmigen Flüssigkeitsstrom eingetauchten geraden Kreiszylinder," *Dinglers Polytechn. Jn.*, v. 326, 1911, p. 321.

758[L, V].—G. N. WARD, "The approximate external and internal flow past a quasi-cylindrical tube moving at supersonic speeds," *Quart. Jn. Mech. Appl. Math.*, v. 1, 1948, p. 225–245.

In the course of the work indicated in the title, the following two functions are used:—$W(z)$ which is the inverse LAPLACE transform of $1 - K_0(p)/K_1(p)$, and $V(z)$ which is the inverse Laplace transform of $-p^{-1}K_1(p)/K_1'(p)$, where K_n is the modified BESSEL function of the third kind. Five decimal values of both functions, for $z = 0(.2)10$ have been computed by the Admiralty Computing Service of Great Britain and are reproduced here together with values of $W(z)$ for $z = -1.8(.2)-.2$ computed by the author and Miss ROUSSAK at Manchester, England.

Cf. also British Admiralty Report SRE/ACS 89, 1945 [*MTAC*, v. 2, p. 294–295].

A. E.

759[V].—ZDENĚK KOPAL, *Tables of Supersonic Flow Around Cones of Large Yaw*. Technical Report no. 5, Massachusetts Institute of Technology, 1949. xviii + 125 p., 19.5 × 26.8 cm.

The present volume is the third in the well-known series of computations of supersonic flow past cones.[1] The title of these tables is somewhat misleading. The title "Tables of Supersonic Flow Around Cones of Large Yaw" compared with the previous volume "On Supersonic Flow Past Slightly Yawing Cones" suggests two limiting cases, one for large and one for small angles of yaw. This, however, is not the case and the present tables deal with the second order approximation, the previous one with the first order approximation in a development in powers of the yaw angle ϵ. Both tables are based on partly unpublished results of STONE.

The radial and normal velocity components u, v and the pressure and density p and ρ are written in the general form

$$f = f_0 + \epsilon \sum_{n=0}^{\infty} f_{1,n} \cos n\phi + \epsilon^2 \sum_{n=0}^{\infty} f_{2,n} \cos n\phi + \cdots,$$

where ϵ denotes the angle of yaw, ϕ the circumferential angle. The axes are fixed with respect to the flow; the origin is at the apex of the cone. The circumferential velocity component w, i.e., the component in the direction of increasing ϕ is given in the form

$$g = \epsilon \sum_{n=0}^{\infty} g_{1,n} \sin n\phi + \epsilon^2 \sum_{n=0}^{\infty} g_{2,n} \sin n\phi + \cdots.$$

In these series f_0; $f_{1,n}$ and $g_{1,n}$; $f_{2,n}$ and $g_{2,n}$ stand for the zero, first order and second order terms respectively. The cos and sin series are due respectively to the boundary conditions which also lead to the result that only $f_{1,1}$ and $g_{1,1}$ differ from zero in the first order and $f_{2,0}$, $f_{2,2}$ and $g_{2,0}$, $g_{2,2}$ in the second order terms.

Hence, the tables consist essentially of tabulated values for the $f_{2,0}$, $f_{2,2}$; $g_{2,0}$ and $g_{2,2}$ corresponding to the five variables u, v, w, p, ρ. The velocities are referred to C, the maximum velocity obtainable by isentropic expansion.

Values are given for 7 cone opening angles in the range from 10° to 50° total included apex angle. An additional short survey of the main results is included. The representation, print, arrangement, etc., is similar to the previous volumes and of very high quality.

In general four significant figures are given in the numerical results. It is stated that this required an accuracy of six digits in the zero approximation (TAYLOR-MACCOLL solution) and five digits in the first order (STONE) approximation. It is pointed out that this accuracy is greater than present-day experimental methods warrant but kept for future improvements in experimental technique. On this point the reviewer does not agree: The necessary accuracy of computations of perfect fluid solutions has a natural limit due to viscous effects. Viscosity is bound to have an effect for flow past yawing cones especially at larger angles of yaw. Hence, this point together with limitations due to possible convergence difficulties in the series in ϵ seem to govern the desired accuracy rather than experimental errors. The reviewer feels that the accuracy in these tables is very probably higher than necessary. Still it may be argued that a standard solution like the cone case should be carried out with too great an accuracy just in order to enable an observer to evaluate viscous effects. The main justification of computing the cone flow in such great detail is, of course, the fact that here one standard solution is provided for comparison with more approximate theoretical methods on one hand and for experimental exploration of viscosity effects on the other.

H. W. LIEPMANN

California Institute of Technology
Pasadena, California

[1] See *MTAC*, v. 3, p. 37–40, 197–198.

760[V].—T. Y. THOMAS, "Calculation of the curvature of attached shock waves," *Jn. Math. Phys.*, v. 27, 1949, p. 279–297.

In a previous paper[1] the author considered the well-known problem of supersonic flow past a pointed, curved, two-dimensional object to obtain an expression for the curvature of the attached shock-wave in terms of the curvature of the stream-lines immediately behind the shock. The present paper contains a numerical application of the previous result to obtain the ratio of curvature of shock-wave to curvature of stream-line at the vertex of the body. The expression for the curvature then involves only the standard ratios characterizing an oblique shock—density on the two sides of the shock, etc. The set of tables and graphs expressing the results of the calculation, therefore, are simply those for the much-discussed flow against a wedge with the one additional quantity, the curvature ratio, included. Consequently, it is surprising to be told in §3 of the paper that to the semi-vertex angle zero we must associate the shock inclined at 90° rather than at the well-known Mach angle arccsc M. One is presumably to infer therefore that the author considers either the larger shock-inclination branch, of the well-known double solution to the problem, to apply to physical reality—or that one has a discontinuous jump from one branch to the other in the neighborhood of vertex angle zero. (So far as the reviewer is aware, neither of these phenomena has been observed although attempts have been made at devising experimental conditions under which the former might occur.) The mathematical error in the paper arises from the author's statement of the compression condition as:

(density behind shock)/(density before shock) > 1 rather than $\geqslant 1$.

The worst physical error would seem to be that of publishing the discussion accompanying the tables without first looking at some of the experiments to which presumably they are meant to be applied.

If one ignores the discussion of §3 of the paper, and substitutes for it the caution that for a given semivertex-angle one should choose the smaller of the two shock-angles, the tables and graphs of the paper should be useful for obtaining the initial curvature of the shock in the case of a curved, pointed, two-dimensional object such as an air-foil. The tabulation is in the form of 13 tables, each for a single Mach number ($M = 1.05$, 1.08, 1.12, 1.18, 1.25, 1.35, 1.47, 1.63, 1.83, 2.12, 2.56, 3.24, 4.45) with argument within the table being the shock-angle. There is included the additional useful table giving the relation between vertex-angle, shock-angle, and Mach number for the greatest vertex-angle with attached shock at given Mach number. The accompanying graphs exhibit curvature ratio and vertex-angle as functions of shock-angle.

RICHARD N. THOMAS

University of Utah
Salt Lake City, Utah

[1] T. Y. THOMAS, "On curved shock waves," *Jn. Math. Phys.*, v. 26, 1947, p. 62–68.

MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in RMT **729** (Chowla & Todd), **733** (Tietze), **734** (Todd), **741** (Johnson); UMT **93** (Fukamiya).

168.—F. S. CAREY, "Notes on the division of the circle," *Quart. Jn. Math.*, v. 26, 1893, p. 332–371.

Table IV, giving the coefficient of the 6-nomial sextic, has the following errata. This list is the result of a recalculation of the table.

p	*Coefficient of*	*For*	*Read*
61	x^0	−27	27
109	x^2	39	135
181	x^0	13565	1685
193	x^0	−5182	−5184
	x	1936	1744
229	x^0	−2103	187
241	x	594	580
373	x^3	381	380
397	x	4960	−5040
433	x^0	−130032	−1728
457	x	3561	3461
103	x^0	1773	1373
127	x	−977	−972
151	x^0	6547	6543
163	x^0	21323	5023
223	x	−3276	5644
	x^0	−71228	4592
331	x^0	84429	84427

EMMA LEHMER

942 Hilldale Ave.
Berkeley 8, Calif.