[23] J. HENDERSON, *Bibliotheca Tabularum Mathematicarum.* Part I. Cambridge, 1926; p. 138, 178, three inaccurate statements: (i) about Wolfram's table; (ii) that Gray noted the error discovered by Gudermann (see note 13); (iii) that Wolfram calculated the common logarithm table, p. 259, only once (see note 8); p. 191 more than one misleading statement about the Thiele table.

[24] F. J. DUARTE, *Nouvelles Tables de Log n! à 33 Décimales depuis n = 1 jusqu'à n = 3000.* Geneva and Paris, 1927, p. iii; errors noted in Wolfram's table in connection with 829, 1087, 1409, 1900. On July 23, 1874, T. M. Simkiss reported the 829 case to J. W. L. Glaisher but his result was unpublished before 1928; earlier recordings 1087 (Peters & Stein), 1409 (Gray), 1900 (Wolfram and Kulik).

[25] F. J. DUARTE, *Nouvelles Tables Logarithmiques.* Paris, 1933, p. xxii; eleven Wolfram (1794 table) errors, including 3 of these listed in no. 24—the other 8 being in connection with 3571, 3967, 6343, 7247, 7853, 8837, 8963, 9623—earlier recordings being 3571 (Peters & Stein), 6343 (Steinhauser), 7247 and 8963 (Wolfram), 7853 (Kulik).

[26] NYMTP, *Table of Natural Logarithms.* V. 1–4, Washington, 1941, p. xi, xi, xi, xiii, respectively. W. errors are noted in 829, 1099, 1409, 1937, 1938, 2093, 3571, 4757, 6343, 7853, 8023, 8837, 9623, but the first announcement was made only in connection with 2093 and 8023.

[27] FMR, *Index.* 1946, p. 176–177, 432, 437, 440, 443; inaccurate contents descriptions for Thiele and Wolfram.

[28] J. H. LAMBERT, *Opera Mathematica*, ed. by A. SPEISER. Zürich, v. 1, 1946, p. xiv, xx, 123, 205; v. 2, 1948, p. ix, 70–71. We find in these v. various parts of *Beyträge*,[6] v. 1–3, and the essential parts of the *Zusätze*.[9]

[29] This information was furnished to us through the courtesy of Mr. Eugene Epperson, of Miami University, Oxford, Ohio.

[30] In no bibliography except the *British Museum Catalogue of Printed Books*, and the catalogue of the Royal Observatory Library, Edinburgh, could I find any reference to a Sarganeck: J. J. SCHMIDT, *Biblischer Mathematicus, Oder Erläuterung der Heil. Schrift aus den Mathematischen Wissenschaften . . . Als ein Anhang ist beygefüget Herrn Georg Sarganeck's Versuch einer Anwendung der Mathematic in dem Articul von der Grösse der Sünden-Schulden.* Zullichau, 1736, 27 plates, 11 ff + 672 p. + 16 ff.

[31] A. G. KÄSTNER published 10 volumes beginning with this word, hence it is not easy to determine which one refers to the Leibniz series; perhaps it was *Anfangsgründe der Analysis des Unendlichen.* Leipzig, 1760.

[32] Henderson's statement[23] concerning Sherwin may be recalled here: "No edition of Sherwin was stereotyped and so some of the later editions are less accurate than the earlier. The third edition in 1742, revised by Gardiner, is probably the most correct, although Hutton [Introduction to his *Tables*, p. 40] says it contains many thousands of errors in final figures. With regard to the fifth edition [1770] Hutton remarks, 'It is so erroneously printed that no dependence can be placed in it, being the most inaccurate book of tables I ever knew.'"

[33] The number 200 was undoubtedly suggested to Wolfram by the fact that in his 1726 edition of Sherwin's *Tables*, log 199 was the last entry in Sharp's table as given there.

[34] What I have written here is not very illuminating. Wolfram's complete statement in this regard, however, is as follows (p. 459): "Auf gleichem Grunde habe ich die Cubicwurzeln von Eins bis auf 125, die man in der Artillerie zum Caliberstabe nöthig hat, ohne wirkliche Ausziehung berechnet."

[35] The German passage on which the first of these statements is based is as follows (v. 5, p. 463): "Ich war schon 1776 auf den Einfall gekommen, durch die Perioden der Dezimalzahlen zu beweisen, dass die Quadratur des Zirkels durch keinen endlichen Werth, weder in Rational- noch Irrationalzahlen ausgedrukt werden könne." The second passage is of very similar construction.

RECENT MATHEMATICAL TABLES

794[B, F].—H. E. SALZER, *Table of Powers of Complex Numbers.* NBS, *Applied Math. Series*, no. 8, Govt. Printing Office, Washington, 1950, iv, 44 p. 18 × 26 cm. For sale by Superintendent of Documents, Washington, price 25 cents.

This short table gives the exact real and imaginary parts of $(x + iy)^n$ for $x = 1(1)10$, $y = 1(1)10$, $n = 1(1)25$. The last page gives x^n for $x = 2(1)9$ and $n = 1(1)25$.

The table is unnecessarily repetitive in that it gives powers of both $x + iy$ and $y + ix$. The essential information of the table can be drawn from that portion which treats $(x + iy)^n$ with $y \leqslant x$. If we write $(x + iy)^n = x_n + iy_n$ then the relations

$$x_{n+1} = x\,x_n - y\,y_n$$
$$y_{n+1} = x\,y_n + y\,x_n$$

were used to construct the table. Perhaps a simpler method would have been to use the fact that x and y are second order linear recurring series,

$$x_{n+1} = 2x\,x_n - (x^2 + y^2)x_{n-1}$$
$$y_{n+1} = 2x\,y_n - (x^2 + y^2)y_{n-1}.$$

The table will be of use for checking formulas involving powers of complex numbers. The numbers x_n, and y_n are examples of LUCAS' functions in the theory of numbers and possess a number of remarkable properties. The table serves the useful purpose of illustrating these properties.

D. H. L.

795[B].—H. S. UHLER, "Table of exact values of high powers of 2," *Scripta Math.*, v. 15, 1949, p. 247–251.

The author has computed, since 1947, a number of isolated powers of 2 of which nine are presented in this note. These are 2^n for $n = 778$, 889,971,1000,2000,2222,3000,3889,4001. The first, third and sixth of these numbers were also calculated by J. W. WRENCH, JR., and the agreement was exact. Fermat's theorem, $2^p - 2$ is divisible by the prime p, was used to check all nine values, although for composite n this required the derivation of "near by" values of 2^p. A simpler and more searching test could have been applied without regard to the character of the exponent n. In fact, if we choose some 10-digit number, quite at random, say 68584 07347 $= N$, we can find by successive squaring and reduction modulo N that $2^{4001} \equiv 47697\ 23697$ (mod N). This requires less than six minutes with any standard desk calculator. This means that if the author's value of 2^{4001} be divided by 68584 07347 the remainder should be exactly 47697 23697. No doubt it is!

The factors 32009, 224057 of $2^{4001} + 1$ and the factor 24007 of $2^{4001} - 1$, recent results of ALAN L. BROWN, are noted also. The first and last of these factors were used as additional verifications of 2^{4001}.

D. H. L.

796[C].—A. OPLER, "Spectrophotometry in the presence of stray radiation: A table of log $[(100 - k)/(T - k)]$," Optical Soc. Amer., *Jn.*, v. 40, 1950, p. 401–403.

The table mentioned in the title is a 4D table for $T = 2(1)99$, $k = 0(\frac{1}{2})$-$5(1)14(2)20$ with the obvious restriction that $T > k$. The quantities T and k are "percentages" so that the table is in reality a table of log $[(1 - x)/(y - x)]$ for $x < y < 1$. The table was calculated with IBM tabulator and summary punch. For a slightly larger table see UMT **105**.

797[C, O].—C. O. SEGERDAHL, "A table of the interest intensity function for interest intervals of 0.01% from 0% to 7%," *Skand. Aktuarietidskrift*, 1949, p. 15–20.

The table gives in 4 pages 9D values of $\delta = \ln(1 + i)$ for $100i = 0(.01)7$. The table is principally to 8 significant figures and is designed for use in IBM 600 type machines. Great pains were taken to insure the correct last decimal.

The author refers to a previous table of STEFFENSEN[1] which gives δ for $100i = [0(.05)10; 7D]$.

D. H. L.

[1] J. F. STEFFENSEN, "A table of the function $G(x) = x/(1 - e^{-x})$ and its applications to problems in compound interest," *Skand. Aktuarietidskrift*, 1938, p. 47–71.

798[F].—J. W. S. CASSELS, "The rational solutions of the diophantine equation $Y^2 = X^3 - D$," *Acta Math.*, v. 82, 1950, p. 243–273.

If the cubic curve Γ: $Y^2 = X^3 - CX - D$, (C, D given integers) is of genus one, the elliptic arguments of its rational points form an additive group U with a finite number of generators, so that all rational points on Γ may be obtained from a finite number of fundamental points by rational operations (MORDELL, WEIL). Upper limits for W, the number of infinite generators of the group U, have been obtained by BILLING[1] using classical algebraic number theory.

The present author confines himself largely to the equiharmonic case when $C = 0$, but by using deeper results of class-field theory, he is able to delimit more closely the dependence of W on D and the associated real cubic field $R(\delta)$, $\delta^3 = D$.

At the close of the paper, both the fundamental rational points on $Y^2 = X^3 - D$ are tabulated for $|D| \leqq 50$. The class number and fundamental unit of $R(\delta)$ are tabulated for $D = 2(1)50$. In this connection the paper of C. WOLFE[2] cited by Cassels does not tabulate the fundamental unit of $R(\delta)$, but merely a unit $x + y\delta + z\delta^2$ of the ring $R[\delta]$ for $D = 1(1)100$ with x, y, z non-negative.

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[1] G. BILLING, "Beiträge zur arithmetischen Theorie der ebenen kubischen Kurven vom Geschlecht eins," K. Vetenskaps Soc., Upsala, *Nova Acta*, s. 4, v. 11, no. 1, 1938.

[2] C. WOLFE, "On the indeterminate equation $x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1$," Univ. of Calif., *Publ. in Math.* v. 1, no. 16, 1923, p. 359–369.

799[F].—J. LEHNER, "Proof of Ramanujan's partition congruence for the modulus 11^3," Amer. Math. Soc., *Proc.*, v. 1, 1950, p. 172–181.

The congruence referred to in the title is

$$p(1331k + 721) \equiv 0 (\text{mod } 11^3)$$

where $p(n)$ denotes as usual the number of unrestricted partitions of n. The proof is made to depend upon certain modular functions whose Fourier series coefficients are tabulated. The various functions may be described as follows, in which a few liberties are taken with the author's notation:

Let
$$F = \prod_{m=1}^{\infty} (1 - x^m)$$

$$Q = 1 + 240 \sum_{m,n=1}^{\infty} n^3 x^{mn}$$
$$\theta = \sum_{m,n=-\infty}^{\infty} x^{m^2+mn+3n^2}$$
$$\psi = xF(x)F(x^{11})$$
$$G = (121Q(x^{11}) - Q(x))/120$$
$$\Phi = x^5F(x^{121})/F(x)$$
$$A = \psi^{-1}\theta^2$$
$$B = \psi^{-2}G$$
$$C = (A^2 - 10A - B - 22)/242.$$

The first 23 coefficients in the expansions of

$$\psi^{-1}, \theta^2, A, A^2, \psi^{-2}, G, B$$

are given, reduced modulo $2\cdot 11^4$. The next 5 or 6 coefficients are given modulo $2\cdot 11^3$. The first 23 coefficients of C are given modulo 11^2. The next 5 coefficients of C are given modulo 11. The first 30 coefficients of Φ are given modulo 11. The tables may be of use in investigations of the general conjecture of Ramanujan

$$p(n) \equiv 0 (\text{mod } 11^\alpha),$$
where
$$24n - 1 \equiv 0 (\text{mod } 11^\alpha).$$

D. H. L.

800[F].—H. S. Uhler, "A colossal primitive pythagorean triangle," *Amer. Math. Monthly*, v. 57, 1950, p. 331–332.

Exact values are given of

$$a = 2^{4000} + 2^{2001}, \quad b = 3\cdot 2^{3998} - 2^{2000} - 1,$$
$$c = 5\cdot 2^{3998} + 2^{2000} + 1.$$

As may be verified, $a^2 + b^2 = c^2$. This pythagorean triangle has almost exactly the same shape as the traditional 3, 4, 5 triangle, the tangent of half the angle A being $\frac{1}{2}(1 + 2^{-1999})$ instead of $\frac{1}{2}$.

D. H. L.

801[G].—Paul Lévy, "Sur quelques classes de permutations," *Compositio Mathematica*, v. 8, 1950, p. 1–48.

The principal results of this work were announced in two notes.[1] The author examines the permutation P_n, among the first n positive integers, definable by (a) $P_n(x) = 2x - 1$ $(2x - 1 \leq n)$, and (b) $P_n(x) = 2(n + 1 - x)$, $(2x - 1 > n)$. The author observes that 1 is invariant, as is $2(n + 1)/3$, if this latter represents an integer. The least common multiple of the order of the cycles of P_n, is the order of the cycle which contains 2. The type of a cycle is an expression such as $a^{\alpha_1}b^{\beta_1}a^{\alpha_2}b^{\beta_2} \cdots a^{\alpha_\nu}b^{\beta_\nu}$ which defines the succession of operations (a) and (b), which must be performed on x to reobtain this initial element. The order σ is the sum $\sum\alpha_i + \sum\beta_i$. The class of values of n (for which x is an integer) is designated, for given type and σ, by e_σ. It is an arithmetic progression, identified by its least member, n_0. Table I indicates the decomposition of P_n into cycles for $n = 2, 3, \cdots, 45$ and for certain larger values, notably all those for $n \leq 75$ and such that $2n - 1$ is prime, and for those of the form $2^q + \delta$, ($q \leq 11$, $\delta = 0$ or 1). The previous work is generalized: Consider a pack of n cards arranged initially in a certain order, the first being the top of the pack. Place the first on the table, the second under the pack, the next on the table, the next under the pack and so forth alternately, until the pack is reduced to a single card which is placed on the table following the others. The passage from the initial to the final order is the operation Q_n. Table II, (p. 48) gives the decomposition of Q_n into cycles for different values of n. The corresponding indicated order, $\sigma = \Omega(n)$, seems to bear a complicated relation to n, concerning which some partial results are obtained. One notes that $\Omega(127) = 52\,780$, $\Omega(128) = 420$, $\Omega(129) = 8$. In particular if $n = 2^q + 1$, then $\Omega(n) = q + 1$.

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[1] P. LÉVY, "Étude d'une classe de permutations," Acad. Sci. Paris, *Comptes Rendus*, v. 227, 1948, p. 422–423; "Étude d'une nouvelle classe de permutations," *ibid.*, p. 578–579.

EDITORIAL NOTE: The permutations considered above were introduced a decade ago by NARASIMHA MURTI, "On a problem of arrangements," Indian Math. Soc., *Jn.*, new series, v. 4, 1940, p. 39–43.

802[K].—F. J. ANSCOMBE, "Tables of sequential inspection schemes to control fraction defective," R. Stat. Soc., *Jn.*, v. 112A, 1949, p. 180–206.

For special conditions on certain parameters, discussion, examples, and comparison with closely related tables,[1] see ANSCOMBE, above. For background in British work in sequential sampling see BARNARD[2] and Anscombe.[3] Barnard's (and Anscombe's) scoring system is, count $+1$ for a good unit, $-b$ for a bad unit, starting score 0, sampling randomly one by one. Accept the batch if score $\geq +H_1$, reject if score $\leq -H_2$. Introduce $b + 1$ ($\equiv$ WALD[4] $1/s$); $R_1 = \dfrac{H_1}{b+1}$ ($\equiv$ Wald $-h_0$), $R_2 = \dfrac{H_2}{b+1}$ ($\equiv$ Wald h_1), p the batch fraction defective, P_p the probability of accepting a batch of fraction defective p, A_p the average sample size for a batch of quality p. In Table I, upper: p (though $p(b + 1)$ is tabled) is given to 5S for $P_p = .99, .90, .50, .10\ .01$ for $R_2 = R_1(R_1 = 1(\frac{1}{4})4)$; $R_2 = 2R_1$ for $R_1 = \frac{3}{4}(\frac{1}{4})\frac{3}{2}, 2, \frac{5}{2}$; $R_2 = 3R_1(R_1 = \frac{3}{4}(\frac{1}{4})\frac{3}{2}, 2)$ and three odd pairs of R_1, R_2: (1, 4), (1, 7), (2, 3). In Table II, upper: for each pair of R_1, R_2 above, ratios of p's to 4S for the following ratios of P_p(.99/.90, .99/.50, .99/.10, .99/.01, .90/.10) are given. Table III, upper, gives A_p to 3, 4S (though $A_p/(b + 1)$ is tabled) for each pair of R_1, R_2, for each P_p of Table I upper; also maximum A_p $\left(\text{though maximum } \dfrac{A_p}{b+1} \text{ is tabled}\right)$ for each pair of R_1, R_2 above. Tables I, II, III, lower, give values as above for 19 combinations of R_1, R_2, $K(2 \leq K \leq 12)$ with a truncating condition (if no decision is reached on inspecting $K(b+1)$ units, accept the batch if the score > 0, reject if the score < 0). "Rectifying inspection" is defined by. With defective units removed or replaced by good, let N be the batch size, αN the size of first sample, βN the size of each further sample, Y the initial number of defectives in the batch, Z the number of defectives in batch after inspection, ξ the proportion of batch inspected at any stage $(0 < \xi < 1)$, and y the number of defectives so far found $(0 < y < Y)$; then inspection ceases after first sample if 0 defectives are found, after second sample if one defective is found, after $(r+1)^{st}$ sample if r defectives are found. Required, the maximum probability ϵ that the number of defectives left in the batch be $\geq Z$ (small compared to N).

Table IV: For $\epsilon = .1$, $Z = 5(5)30, 40, 50, 60, 80, 100$ the average sample size A/N is given to 3S for ten values of Y (varying) and for 10 combinations of α and β (varying, each to 4S). Also the AOQL to 2S for each pair (α, β) and the value of Y for which the AOQL is attained.

Table V: is the same as Table IV, for $\epsilon = .01$.

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[1] H. A. Freeman, M. Freedman, F. Mosteller, & W. A. Wallis, *Sampling Inspection*, New York, 1948.
[2] G. A. Barnard, "Sequential tests in industrial statistics," R. Stat. Soc., *Jn.*, Supplement, v. 8, 1946, p. 1–26.
[3] F. J. Anscombe, "Linear sequential rectifying inspection for controlling fraction defective," *ibid.*, p. 216–222.
[4] A. Wald, *Sequential Analysis*, New York, 1947.

803[K].—Alice A. Aspin, "Tables for use in comparisons whose accuracy involves two variances, separately estimated," *Biometrika*, v. 36, 1949, p. 290–296.

The tables are designed for use when the precision of a normally distributed estimate, y, of a population parameter, η, depends linearly on two population variances, σ_1^2 and σ_2^2, the sampling variance of y being therefore of the form $(\lambda_1^2\sigma_1^2 + \lambda_2\sigma_2^2)$ where λ_1 and λ_2 are known positive constants. If s_1^2 and s_2^2 are independent estimates of σ_1^2 and σ_2^2, based on f_1 and f_2 degrees of freedom, respectively, then the tables give, for the 5% and 1% probability levels, critical values of the ratio

$$v = (y - \eta)[\lambda_1 s_1^2 + \lambda_2 s_2^2]^{-\frac{1}{2}}$$

to 2D for f_1 and $f_2 = 6, 8, 10, 15, 20, \infty$. These tables can be used in testing the difference between two means of samples from two normal populations whose standard deviations cannot be assumed equal.

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804[K].—P. K. Bose, "Incomplete probability integral tables connected with Studentised D^2-statistic," Calcutta Stat. Assn. *Bull.*, v. 2, 1949, p. 131–137.

The D^2-statistic is employed as a measure of the distance between two p-variate normal populations. It is a function of the values in samples of sizes n and n' from the populations, its distribution depending on p, n, n', and the true distance Δ^2 between the populations. The author tables to 3D the upper 5% point of the statistic $C^2D^2/(N + C^2D^2)$ for $\beta = 0$, $p = 1(1)6$, $N = 1(1)50(10)90$ (Table 1), and for $\beta = 5$, 20, 50, 100, $p = 2$, 4, 6, $N = 3(2)49$ (Tables 2–5). Here $\beta = \frac{1}{2}C^2\Delta^2$, $N = n + n'$, $NC^2 = nn'p$. The computations employ recursion formulae previously found by the author.

It is well known that a close relation exists between D^2 and F. In fact, $C^2(N - 1 - p)D^2/Np$ has the distribution of F with p and $N - 1 - p$ degrees of freedom and parameter $\lambda = C^2\Delta^2$. Using this fact, the entries in Table 1 may be obtained at once from readily available tables. The reviewer has checked most of the entries in Table 1 without finding any error. Tables 2–5 seem to be new, adding, for fixed numbers of degrees of freedom, 4 new percentage points to the 8 to 16 points previously given by TANG[1] and the 4 points previously given by EMMA LEHMER.[2]

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[1] P. C. TANG, "The power function of the analysis of variance tests with tables and illustrations of their use," *Stat. Res. Mem.*, v. 2, 1938, p. 126–149.

[2] EMMA LEHMER, "Inverse tables of probabilities of errors of the second kind," *Annals Math. Stat.*, v. 15, 1944, p. 388–398.

805[K].—D. J. FINNEY, "The estimation of the parameters of tolerance distributions," *Biometrika*, v. 36, 1949, p. 239–256.

On page 252 there is a table of weights useful in certain estimation problems. The function tabulated is Z^2/Q, where Z is the ordinate of the normal distribution and Q is the area of the distribution to the right of Z. The table gives 1S or 5D, whichever is greater, for $x = 1.1(.1)9$, x being 5 greater than the argument of the normal distribution. The reviewer recalculated the table, and found no error.

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806[K].—EVELYN FIX, "Tables of noncentral χ^2," Univ. of Calif., *Publ. in Stat.*, v. 1, no. 2, 1949, p. 15–19.

These tables are for the power of certain "chi-square tests" at the 1% and 5% levels of significance. Let $x_1, \cdots, x_f$ denote independent standard normal deviates. For any real $a_1, \cdots, a_f$ the distribution of the non-central chi-square variable

$$\chi'^2_{f,\lambda} = \sum_{i=1}^{f} (x_i + a_i)^2$$

depends only on f and

$$\lambda = \sum_{i=1}^{f} a_i^2.$$

If $\chi_f^2(\alpha)$ is the upper α-point of a central chi-square variable with f degrees of freedom, that is, if

$$Pr\{\chi'^2_{f,0} > \chi_f^2(\alpha)\} = \alpha,$$

then the power of these tests against alternatives characterized by λ is

$$\beta(\lambda) = Pr\{\chi'^2_{f,\lambda} > \chi^2_f(\alpha)\}.$$

The tables give λ as a function of α, β, and f to 3D or 4S for $\alpha = .01, .05$; $\beta = .1(.1).9$, and $f = 1(1)20(2)40(5)60(10)100$. In the table heading on p. 17, $\alpha = .01$ should be changed to $\alpha = .05$, and the opposite change should be made on p. 19.

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807[K].—A. K. GAYEN, "The distribution of Student's t in random samples of any size drawn from non-normal universes," *Biometrika*, v. 36, 1949, p. 353–369.

Unity minus the cumulative density function of t is expressed in the EDGEWORTH series form $P_0(t) + \lambda_3 P_{\lambda_3}(t) - \lambda_4 P_{\lambda_4}(t) + \lambda_3{}^2 P_{\lambda_3{}^2}(t)$, where λ's are cumulants of population sampled. Values of the P's are listed on p. 361 (Table 1) for $t = [0(.5)4; 4D]$ and 1(1)6, 8, 12, 24, ∞ degrees of freedom. Four graphs of corresponding probability density function terms appear on p. 362–63.

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808[K].—J. M. HOWELL, "Control chart for largest and smallest values," *Annals Math. Stat.*, v. 20, 1949, p. 305–309.

Given L and S the largest and smallest values in a sample of size n from a normal universe; $\bar{L}$ and $\bar{S}$ their respective means for k such samples; $R = L - S$, $\bar{R} = \bar{L} - \bar{S}$, and $M = \frac{1}{2}(\bar{L} + \bar{S})$. Constants are provided for so-called upper 3-sigma control limits for L above M in the form: U.C.L. $= M + A_3\bar{R} = M + (0.5\bar{R} + 3\sigma_L) = \bar{L} + 3\sigma_L$; and in the form: $a + A_4\sigma_L$ in which $A_3 = 0.5 + 3d_4/d_2$, where $d_4 = \sigma_L = \sigma_S$, $d_2 = E(R)$ for samples of size n from a standard normal universe, $A_4 = d_2/2 + 3d_4$, and a is the mean of the normal universe sampled. Because of symmetry the constants apply to the lower 3-sigma control limit for S in the forms, L.C.L. $= M - A_3\bar{R}$ and $a - A_4\sigma_S$. In Table I values of d_2, d_4, A_2, A_3, A_4 are given for $n = 2(1)10$. (d_2 and $A_2 = 3/(d_2\sqrt{n})$ are available elsewhere.[1] d_4 for $n = 2, 5, 10$ was given first by TIPPETT,[2] and also, for $n = 2(1)10$ by GODWIN,[3] to 7S.) HOWELL's values, given to 3S, are in error in the third figure for $n = 2, 4, 5, 6, 7, 8, 10$. A_3 and A_4 are given to 3S.

In Table II are given values of P_1 and P_2, the power of the conventional 3-sigma control charts for sample range R, and sample mean $\bar{x}$, respectively, for a standard normal universe. Entries are given for $n = 3, 5$, universe mean $a = 0.5, 1.0, 2.0$, and universe standard deviation $\sigma = 1.2, 1.5, 2.0$, where the sizes of the respective 3-sigma regions are determined for $a = 0$, $\sigma = 1$. The power P_3 of the so-called largest and smallest value charts is calculated from $P_3 = Pr(-c < S, L < c)$ where c is determined so that $P_1P_2 = P_3$ for $a = 0$, $\sigma = 1$. Besides P_1P_2, values are also given for N_1, the smallest integer for which $(P_1P_2)^{N_1} \leq .01$ and for N_2, the smallest integer for which $(P_3)^{N_2} \leq .01$.

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[1] E.g., in *Control Chart Method for Controlling Quality During Production.* Z1–3, Amer. Standards Assn., New York, 1942.

[2] L. H. C. TIPPETT, "On the extreme individuals and the range of samples taken from a normal population," *Biometrika*, v. 17, 1925, p. 364–387.

[3] H. J. GODWIN, "Some low moments of order statistics," *Annals Math. Stat.*, v. 20, 1949, p. 279–285. [See *MTAC*, v. 4, p. 20.]

809[K].—NBS, *Tables of the Binomial Probability Distribution.* NBS Applied Math. Series, no. 6. x + 388. Washington, Govt. Printing Office, 1950. 21.3 × 26.9 cm. Price $2.50.

One of the fundamental distributions in mathematical statistics is the BERNOULLI probability function. Let p be the probability of success in a single trial, q the probability of failure, then the probability P_x of exactly x successes in n independent trials, the probability of success being constant from trial to trial, is $\binom{n}{x}p^xq^{n-x}$, and the probability of m or fewer successes is

$$\sum_{x=0}^{m} \binom{n}{x}p^xq^{n-x}.$$

The volume consists of a foreword by CHURCHILL EISENHART, an introduction (p. v–x) explaining the distribution, scope of the tables, method of preparation, interpolation, applications, and a listing of other tables, mostly unpublished, of the same function. This is followed by two tables, Table 1, (p. 1–195) and Table 2, (p. 197–387). Table 1 gives $\binom{n}{x}p^xq^{n-x}$ for $p = .01(.01).5$, $q = 1 - p$, $n = 2(1)49$, $x = 0(1)n - 1$, to seven decimal places, and Table 2, $\sum_{x=r}^{n} \binom{n}{x}p^xq^{n-x}$, $p = .01(.01).50$, $q = 1 - p$, $n = 2(1)49$, $r = 1(1)n$, to seven decimal places. It is a simple matter to find P_x, $p > .50$, and $\sum_{x=0}^{m} \binom{n}{x}p^xq^{n-x}$ from the results already tabulated. These tables were prepared from tables of the incomplete beta function[1] by BETTY ELSEN, AMY NORMAN, and BONNIE THOMAS of the personnel of the Department of the Army, and were issued in mimeographed form for limited distribution at the close of World War II. In present form, a photographic reproduction of the mimeographed tables, the preparation of the tables for publication was principally carried out by LOLA S. DEMING and CELIA S. MARTIN of the Statistical Engineering Laboratory of the NBS. Rarely will difficulties of reading the entries occur. Two instances of such difficulty are the entries for $n = 12$, $r = 6$, $p = .23$, (p. 209) and $n = 30$, $r = 8$, $b = .41$, (p. 272).

For Table 1 an accuracy of ± 1 in the seventh decimal place is claimed and for Table 2 an accuracy of ± 0.5 in the seventh decimal place. As a spot check the values of $n = 25$, $p = q = .5$ were calculated. P_1 should be .0000007, $P_3 = .0000685$, $P_6 = .0052780$, $P_8 = .0322334$, all within the claimed limits of accuracy. The results for $\sum_{i=x}^{n} \binom{n}{x}p^xq^{n-x}$ are correct as tabulated.

The tables will have many uses. One may mention the operating characteristic function in acceptance sampling and the power function in the testing of hypotheses in statistics. The National Bureau of Standards should take pride in this volume published at such nominal cost. It is desirable that the other tables of the Bernoulli probability function, still unpublished, should see the light of day and that the tables should be extended to high values of n, where the normal curve is a poor approximation in the extreme tails of the distribution.

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[1] K. PEARSON, *Tables of the Incomplete Beta-Function*, Cambridge, 1934.

810[K].—C. R. RAO, "On some problems arising out of discrimination with multiple characters," *Sankhya*, v. 9, 1949, p. 343–366.

The statistic $D_p^2 = \sum\sum_{i,j=1}^{p} s^{ij}d_id_j$, based on p characters, is used to estimate the squared difference between two populations, $\Delta_p^2 = \sum\sum \sigma^{ij}\theta_i\theta_j$. Samples of n_1 and n_2 for each character are drawn from the two populations. The variance-covariance matrix of the sample is s_{ij} with inverse s^{ij}, and the differences between the sample means are indicated by d_i. These are estimates of the population parameters, σ_{ij}, σ^{ij} and θ_i. An example is given for $p = 4$.

The only table given in this article presents the power function for D^2 to 2D when $\phi = 1$, 1.5 and 2, $N = n_1 + n_2 = 16(4)28$, and $p = 1(1)8$, where

$$\phi^2 = n_1n_2\Delta_p^2/N(p + 1).$$

Extensive tables are being prepared of (i) the probability integral of the conditional distribution of R, the statistic used to compare D^2 with p and $(p + q)$ characters

$$R = M_p/M_{p+q}, \text{ where } M_p = 1 + \frac{n_1n_2}{N(n_1 + n_2 - 2)} D_p^2,$$

and (ii) the percentage points of the null distribution of

$$W = \frac{M_{p+q} - M_p}{1 + M_{p+q} - M_p}.$$

It is proposed to compare the relative efficiencies of W and R.

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811[K].—MARJORIE THOMAS, "A generalization of Poisson's binomial limit for use in ecology," *Biometrika*, v. 36, 1949, p. 18–25.

The author introduces a "double Poisson distribution" as follows. Let $X_1, \cdots, X_r$ be independent random variables depending on a Poisson distribution with parameter λ, and let r itself be a random variable depending on a Poisson distribution with parameter m. Finally let $Z = X_1 + \cdots + X_r + r$ (for example, r may represent the number of clusters, $X_j + 1$ the number of points in the jth cluster). The author studies the distribution

$$Pr\{Z = k\} = \sum_{r=1}^{k} \frac{m^r e^{-m}}{r!} \sum_{\alpha} \prod_{j=1}^{r} \frac{\lambda^{\alpha_j - 1} e^{-\lambda}}{(\alpha_j - 1)!}$$

where $\alpha_1 + \cdots + \alpha_r = k$.

She calculates the mean and the variance and discusses various problems of statistical estimation. These lead to certain elementary equations and a few tables illustrate the practical procedure. Thus Table 5 gives the value of $1 + \lambda$ for given $e^{-m} = .05, .1(.1).9$ and $me^{-m-\lambda} = .05, .1, .2, .3$ to 3D. Other tables of roughly the same size pertain to more complicated functions.

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812[K].—J. W. WHITFIELD, "Intra-class rank correlation," *Biometrika*, v. 36, 1949, p. 463–467.

In analogy with the definition of intra-class correlation for numerical data, it is suggested that an appropriate measure when only ranks are available is the mean of KENDALL'S τ coefficient extended over all possible arrangements within classes. In case the classes consist of pairs, the following device affords a compact computation of the mean value. Arrange the pairs as $(a_1, b_1), (a_2, b_2), \cdots, (a_{n/2}, b_{n/2})$, so that each $a_i < b_i$ and $a_1 < a_2 < \cdots < a_{n/2}$. Compute a "score," S, by accumulating the differences for each individual of the numbers of values on his right greater than and less than his own (making no comparisons within pairs). Then, taking $S_p = S - n(n-2)/4$, the mean value of Kendall's τ is $\tau_p = 4S_p/(n^2 - 2n)$. A table is given for $Pr(S_p \geq S_p')$ to 5D for $n = 6(2)20$ and $S_p' = 0(2)90$ for the case of an uncorrelated universe. S_p is symmetrically distributed about 0 with variance $n(n-2)(n+2)/18$. Since $\beta_2 = 3 - 4.32n^{-1}$, a normal test of significance is indicated for large samples.

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813[K].—JOHN WISHART, "Cumulants of multivariate multinomial distributions," *Biometrika*, v. 36, 1949, p. 47–58.

The univariate BERNOULLI and PASCAL multinomial distributions are first considered. Using cumulant distribution functions recurrence relations are obtained from which cumulants to order four are recorded.

Bivariate cumulants to order four are found by recurrence formulae paralleling the univariate case and are also recorded.

Extension to the multivariate case follows from the more simple univariate and bivariate cases. Of importance is the fact that a notation is used which makes the corresponding cumulants of the Bernoulli and the Pascal distributions greatly resemble each other. A complete list of the auxiliary patterns and the cumulants to the fourth order is given for a particular case ($5 \times 4 \times 3 \times 2$) of the 4-variate multinomial Bernoulli distribution.

There are misprints on p. 52–3.

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814[K].—HERMAN WOLD, *Random Normal Deviates*. Tracts for Computers, No. XXV, Dept. of Statistics, Univ. College, Univ. of London. Cambridge, Cambridge Univ. Press. xiii, 51 p. 16×23.2 cm., Price 5 s.

This table contains 25,000 random normal deviates with mean zero, variance one, to 2D. The table was obtained by normalizing the KENDALL-SMITH table of random numbers row by row. Four tests for normality and randomness were applied to each of the 50 pages of the table, to each of the five blocks of 10 pages, and to the entire table. The results showed agreement with normality.

An introduction describes construction of the table and gives techniques for the construction of samples from multivariate normal distributions having prescribed parameters.

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815[L].—A. R. CURTIS, "The velocity of sound in general relativity with a discussion of the problem of the fluid sphere with constant velocity of sound," R. Soc. London, *Proc.*, v. 200A, 1950, p. 248–261.

The functions of Table 1 (p. 261) occur in a static and spherically symmetric metric of space-time. The coefficients e^{ν} and e^{λ} of this metric[1] are derived from a function $V(z)$ where z is a radial coordinate in suitable units. Denoting differentiation with respect to z by dots, $V(z)$ satisfies the nonlinear differential equation

$$3(2\ddot{V} - \dot{V} - V)(1 - V) + (2\dot{V} + 4V - 4e^{z})(2\dot{V} + V - e^{z}) = 0.$$

The pressure is

$$P = \tfrac{1}{12}e^{-z}(2\dot{V} + V) - \tfrac{1}{3}.$$

The table gives 5D values of V and of e^{ν}, together with 4D values of e^{λ} and P for $x = e^{\frac{1}{2}z} = 0(.02).38$, and .38791, the last value corresponding to the first zero of P.

A. E.

[1] A. S. EDDINGTON, *Mathematical Theory of Relativity*. Cambridge University Press, 1923, p. 72.

816[L].—I. FEISTER, "Numerical evaluation of the Fermi beta-distribution function," *Physical Review*, v. 78, 1950, p. 375–377.

The computation of the "Fermi distribution function"

$$f(Z, \eta) = \eta^{2s}e^{\pi y}|\Gamma(s + iy)|^{2},$$

where $s = (1 - \gamma^{2})^{\frac{1}{2}}$, $\gamma = Z/137$, and $y = (\gamma/\eta)(1 + \eta^{2})^{\frac{1}{2}}$ is now in progress at the Computation Laboratory of the NBS. The author gives here 3D values of f for $Z = 0(10)90$ and $\eta = .6, 1(1)5$, and compares these values with various approximations used in theoretical physics. Percentage errors of the approximations are also given. The so-called non-relativistic approximation is rather poor, the Bethe-Bacher approximation much better, while the Nordheim-Yost approximation rates between the two. The author emphasizes that "The table of Fermi functions, when completed, will in most cases make unnecessary the use of an approximation for the numerical evaluation of $f(Z, \eta)$."

A. E.

817[L].—R. B. DINGLE, "The electrical conductivity of thin wires," R. Soc. London, *Proc.*, v. 201A, 1950, p. 545–560.

The principal quantities occurring in the author's tabulation are

$$\frac{j(r)}{j_0} = \frac{3}{4\pi}\int_0^\pi d\theta \cos^2\theta \sin\theta \times \int_0^{2\pi} d\phi \left[1 - \exp\left\{-\frac{r\sin\phi + (a^2 - r^2\cos^2\phi)^{\frac{1}{2}}}{\lambda\sin\theta}\right\}\right]$$

and

$$\frac{\sigma}{\sigma_0} = \frac{2}{a^2}\int_0^a \frac{j(r)}{j_0} r dr, \qquad k = \frac{2a}{\lambda}.$$

Table 1, p. 553, gives approximate values for very large k of $j(r)/j_0$ to various number of decimal places for $k(1 - r/a) = 0, .2, .5, 1, 2, 5, 10, \infty$.

Table 2, p. 554, gives 3D values of $j(r)/(kj_0)$ for very small (positive) k and $r/a = 0(.25)1$.

Table 3, p. 554, gives 3D values of $j(r)/j_0$ and of σ/σ_0 for $k = .5, 1, 2$ and $r/a = 0(.25)1$.

Table 4 is more intimately connected with the physical problem in hand.

A. E.

818[L].—G. E. FORSYTHE, "Solution of the telegrapher's equation with boundary conditions on only one characteristic," NBS *Jn. of Research*, v. 44, 1950, p. 89–102.

The boundary value problem alluded to in the title presents itself in meteorological theory. Its solution is obtained by means of the GREEN's function

$$G(x, z) = \sum_{n=1}^{\infty} n^{-1}\sin(nx + n^{-1}z).$$

This function is tabulated in the paper to five decimal places for the following 14 values of z: 0, 3.15012, 4.14543, 6.30024, 8.29086, 8.45505, 9.00000, 12.60048, 16.91010, 18.00000, 18.90072, 25.20096, 27.00000, and 36.00000; in each case for $x = -\pi(\pi/36)\pi$. Since there is a discontinuity at $x = 0$, the table gives $G(-0, z)$, $G(0, z)$, and $G(+0, z)$.

The first 18 terms of the series were summed as they stand. The tail was expanded in powers of z, and the first 11 coefficients of this expansion were computed for all x (BERNOULLI polynomials enter in this computation): it is shown that the remainder then is less than 5 units of the 6th decimal place, and another 5 units were set apart for rounding off errors.

A. E.

819[L].—F. C. Frank, "Radially symmetric phase growth controlled by diffusion," R. Soc. London, *Proc.*, v. 201A, 1950, p. 586–599.

Table I on p. 589 gives values to varying accuracy of

$$F_n(x) = \int_x^\infty t^{1-n} \exp(-\tfrac{1}{4}t^2)dt$$

and

$$f_n(x) = \tfrac{1}{2}x^n F_n(x) \exp(\tfrac{1}{4}x^2)$$

for $n = 3, 2$ and $x = 0(.1).4(.2)4(1)6$. There are also graphs of some related functions. The integrals involved can be expressed in terms of the error function and the exponential integral function.

820[L].—W. Hodapp, "Über die Hermiteschen Funktionen zweiter Art von reellem und rein imaginärem Argument," *Arch. d. Math.*, v. 2, 1949/50, p. 186–191.

Two solutions of the differential equation $y'' - xy' + ny = 0$, $n = 0, 1, 2, \cdots$ are

$$H_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} e^{-\frac{1}{2}x^2}, \qquad h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} \int_0^x e^{-\frac{1}{2}(x^2-u^2)}du.$$

H_n is the Hermite polynomial, and the author calls h_n the Hermite function of the second kind. (It can be expressed in terms of the parabolic cylinder function.) He gives for h_n convergent expansions in ascending powers of x, explicit forms for $n = 0(1)3$, asymptotic expansions for large x, and indicates briefly some approximations for the real zeros of $h_n(x)$. A numerical table, to 2D, gives upper and lower bounds, approximations, and the exact values of the positive zeros of $h_n(x)$ for $n = 1(1)6$.

A. E.

821[L].—M. Kotani & H. Takahashi, "Numerical tables of functions useful for the calculation of resonant frequencies of a cavity magnetron," Phys. Soc. Japan, *Jn.*, v. 4, 1949, p. 73–77.

The authors tabulate three functions

$$f(x) = \frac{J_0(x)}{2xJ_0'(x)} + \sum_{m=1}^{\infty} \left\{\frac{J_m(x)}{xJ_m'(x)} - \frac{1}{m}\right\}$$

$$f(\zeta, \nu; x) = \tfrac{1}{2} \sum_{m=0}^{\infty} \left\{\frac{\nu J_{\zeta+m\nu}(x)}{xJ'_{\zeta+m\nu}(x)} - \frac{1}{m+1}\right\} + \tfrac{1}{2} \sum_{m=0}^{\infty} \left\{\frac{\nu J_{\nu-\zeta+m\nu}(x)}{xJ'_{\nu-\zeta+m\nu}(x)} - \frac{1}{m+1}\right\}$$

$$f(0, \nu; x) = \frac{\nu J_0(x)}{2xJ_0'(x)} + \sum_{m=1}^{\infty} \left\{\frac{\nu J_{m\nu}(x)}{xJ'_{m\nu}(x)} - \frac{1}{m}\right\}$$

$$g(\xi, \nu; x) = \frac{\nu}{2} \sum_{m=-\infty}^{\infty} (-1)^m \frac{J_{|\xi+m\nu|}(x)}{J'_{|\xi+m\nu|}(x)}.$$

The tables are to 4S, mostly, and are as follows:

$f(x)$ for $x = .2(.01)2$
$f(\xi, \nu; x)$ for $x = .2(.1)2.3$ for the following pairs

ν	ξ	ν	ξ
4	0(1)2	8	0(1)4
5	0(1)2	10	0(1)5
6	0(1)3	12	0(1)6

$g(\xi, \nu, x)$ for $x = .2(.1)2.3$ and for
$\nu = 4, \quad \xi = 0, 1$
$\nu = 5, 6 \quad \xi = 0, 1, 2.$

822[L].—A. R. Low, *Normal Elliptic Functions*. Univ. of Toronto Press, Toronto, 1950, 32 p., 15.5 × 23.5 cm, Price $1.25.

Elliptic functions are inversions of integrals whose integrands are rational in the variable of integration and in the square root of a quartic polynomial. Various theories of elliptic functions differ in the standardization which they adopt for the quartic radicand. The most symmetric theory is WEIERSTRASS' where one of the zeros of the quartic is at infinity, and the sum of the other three is zero. The most highly standardized is JACOBI's theory in which two of the zeros are assumed at 1 and -1, while the other two (at $\pm k^{-1}$) are symmetric with respect to the origin. The author's "normalized form" assumes three of the zeros at the standard positions 0, 1, ∞: the fourth zero is called the parameter and denoted by m: it is the k^2 of the Jacobian theory. $m' = 1 - m$ is the complementary parameter, and the case of principal practical importance is $0 < m, m' < 1$.

The standard cubic is $P = p(p - m)(p - 1)$, and it is proved that every elliptic integral can be reduced to a form in which the integrand is a rational function of p and $P^{\frac{1}{2}}$. The elliptic function $p_1 = p_1(u, m)$ is defined by the relation

$$u = \frac{1}{2}\int_{p_1}^{\infty} P^{-\frac{1}{2}}dp$$

and is clearly $\mathrm{ns}^2(u, k)$ of the Jacobian theory. Three other elliptic functions are defined by the relations

$$p_1(u, m)\cdot p_3(u, m) = m,\ p_2(u, m') + p_3(u, m) = 1,$$
$$- p_4(u, m') + p_1(u, m) = 1,\ m' = 1 - m.$$

Values of p_1 to p_4 were computed by the aid of these relations from MILNE-THOMSON's tables of elliptic functions.[1] There are six tables, each to 5D, for $m = 0(.1)1$ and $u/K = .1(.1)1$.

Table I: K, K/K', u. Table II: $\mathrm{sn}(u, k)$. Table III: $p_1(u, m)$. Table IV: $p_2(u, m)$. Table V: $p_3(u, m)$. Table VI: $p_4(u, m)$.

A. E.

[1] L. M. MILNE-THOMSON, *Die elliptischen Funktionen von Jacobi*. Berlin, 1931.

823[L].—R. S. Scorer, "Numerical evaluation of integrals of the form $I = \int_{x_1}^{x_2} f(x)e^{i\phi(x)}dx$ and the tabulation of the function $Gi(z) = \frac{1}{\pi}\int_0^\infty \sin (uz + \frac{1}{3}u^3)du$," *Quart. Jn. Mech. Appl. Math.*, v. 3, 1950, p. 107–112.

Integrals of the form

$$I = \int_{x_1}^{x_2} f(x)e^{i\phi(x)}dx$$

often occur in calculating the wave form due to a source in a dispersive medium, and are frequently evaluated by the method of stationary phase. If the approximation of $\phi(x)$ by its Taylor series, in the vicinity of a point of stationary phase, up to and including cubic terms is adequate, the answer can be expressed in terms of Airy integrals (for which tables already exist[1]) and of the two functions

$$Gi(z) = \frac{1}{\pi}\int_0^\infty \sin (uz + \tfrac{1}{3}u^3)du, \qquad Hi(z) = \frac{1}{\pi}\int_0^\infty \exp (uz - \tfrac{1}{3}u^3)du.$$

$Gi(z)$ and $Hi(-z)$ were computed, on the EDSAC in Cambridge, England, by numerical integration of the differential equations which they satisfy. The computation was performed to 8D, with the interval .02 in z. The tables given in the paper are to 7D for $z = 0(.1)10$: modified second differences are also given. Outside of the tabulated range the asymptotic formulae recorded in the paper are valid to the accuracy contemplated.

In an accompanying note by Miller & Mursi[2] it is shown how the functions tabulated here together with the Airy integrals can be used to solve the differential equation $y'' - xy = f(x)$ numerically when f is a polynomial.

A. E.

[1] B. A. Math. Tables Committee, Pt.-V.B. *The Airy Integral.* By J. C. P. Miller, Cambridge Univ. Press, 1946, 56 p.

[2] J. C. P. Miller and Zaki Mursi, "Notes on the solution of the equation $y'' - xy = f(x)$," *Quart. Jn. Mech. Appl. Math.*, v. 3, 1950, p. 113–118.

824[L].—E. Sauvenier-Goffin, "Les fonctions $\Gamma(x)$ correspondant, pour les Naines blanches, aux exposants adiabatiques Γ_i des configurations gazeuses," Soc. Roy. Sci., Liége, *Bull.*, 1950, p. 47–54.

4D tables are given of

$$\frac{8x^5}{3(x^2+1)^{\frac{1}{2}}f(x)} \qquad \frac{4x^2+5}{3(x^2+1)} \qquad \text{and} \qquad \frac{8x^5(x^2+1)^{\frac{1}{2}}}{8x^5(x^2+1)^{\frac{1}{2}} - (x^2+2)f(x)}$$

where

$$f(x) = x(2x^2-3)(x^2+1)^{\frac{1}{2}} + 3 \operatorname{arc\,sinh} x,$$

for $x = 0(.1)3(.5)10$. Exact values are given for $x = 0, \infty$.

825[L].—Dorothy A. Strayhorne, *A Study of an Elliptic Function*, Thesis, Chicago, 1946. Air Documents Division T-2, AMC, Wright Field, Microfilm No. R c-734 F 15000.

The mathematical part of this paper repeats results which are well known. The numerical part consists of two tables. The first one gives the numerical values to 4D of the WEIERSTRASS elliptic function $\wp(u)$ for the values of the invariants $g_2 = 37$, $g_3 = -42$ corresponding to the periods $2\omega = 2.2772$ and $2\omega' = 1.3674i$. The argument is imaginary between $.04i$ and $1.36i$ in steps of $.04i$. The second table gives the corresponding JACOBI elliptic functions sn$u$, cn$u$, dn$u$, likewise to 4D, for $u = 0(.05)1$. The value of the modulus is $k = .9585$ corresponding to the choice of the invariants mentioned above.

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826[L].—MARCEL TOURNIER & MARC BASSIÈRE, "Une solution des équations de la couche limite obtenue par la considération de phénomènes transitoires," *La Recherche Aéronautique*, 1948, no. 4, p. 67–72.

On p. 72 are two tables. Table 1 is of $E_3(z) = \Gamma(\frac{4}{3})\int_0^z e^{-t^3}dt$, $z = [0(.02)1.68; 4D]$, Δ. The values were calculated by the development into powers of z up to $z = 1$. For values of $z > 1$, interpolations were made in the table of PEARSON.[1] A short table of $E_3(.611x)$ $(x = 0(.1)2.8)$ is also given to 4D and compared with a "curve of BLASIUS."

R. C. A.

[1] K. PEARSON, *Tables of the Incomplete Γ-Function*. London, 1922.

827[L, P].—GEORG VEDELER, "Basic function for beams with arbitrary constraint," K. Norske Videnskabers Selskab, *Forhandlinger*, v. 22, 1949, p. 171–177.

The author compares a vibrating beam of length l pinned and having fixations f_A, f_B, respectively, at the two ends with a beam whose ends are clamped and subjected to static shears and moments of such a nature that the two beams have the same frequency spectra. The nth deflection mode of the first beam is

$$F_n = A_n(\cosh \alpha_n x - \cos \alpha_n x) - B_n(\sinh \alpha_n x + \sin \alpha_n x).$$

For the case $f_A = f_B = f$, for $n = 1(1)6$ a table of $\alpha_n l$, A_n, and B_n is given for $f = 0(.05).7(.02)1$.

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828[L].—F. G. TRICOMI, "Sul comportamento asintotico dei polinomi di Laguerre," *Ann. di Mat.*, ss. 4, v. 28, 1949, p. 263–289.

A set of four formulae which describe completely the asymptotic behavior of

$$L_n^{(\alpha)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!}$$

as $n \to \infty$. The four formulae are valid, respectively, in the following four cases: (i) x is in the neighborhood of the origin, (ii) $0 < x < \nu$, (iii) x is in the neighborhood of ν, (iv) $x > \nu$, where $\nu = 4n + 2\alpha + 2$. These formulae are numerically tested on $l_{10}(x) = e^{-\frac{1}{2}x}L_{10}^{(0)}(x)$ for which 4D values are computed both from the exact equation and from the approximation. These values are given together with the absolute error in case (i) for $x = 0(.1)1$, in case (ii) for $x = .5(.5)3(1)6(2)36$, in case (iii) for $x = 34(2)50$, and in case (iv) for $x = 46(2)52$. There is also a table comparing errors of the various formulae in the regions where they overlap.

Asymptotic formulae are also given for the zeros, and they are tested numerically on the zeros of $l_{10}(x)$.

P. 289 is an auxiliary table for the roots of the transcendental equation $x + \sin x = a$, when a and x are expressed in degrees and decimal parts of degrees. If $0 \leq a \leq 90°$, TRICOMI puts $x = \frac{1}{2}a + f(a)$, and if $90° \leq a \leq 180°$ he puts $a_1 = 180° - a$ and $x_1 = 180° - x = 10^{1.4314665}a_1^{\frac{1}{3}} + f_1(a_1)$. 3D tables, with first differences, are given for $f(z)$ and $f_1(z)$ for $z = 0°(1°)90°$.

A. E.

829[L].—CHANG WEI, "Der Spannungszustand in Kreisringschale und ähnlichen Schalen mit Scheitelkreisringen unter drehsymmetrischer Belastung," Nat. Tsing Hua Univ. *Sci. Rep.*, s. A, v. 5, 1949, p. 289–349.

Table I, p. 345, gives the real and imaginary parts of $J_{\frac{1}{3}}(r\sqrt{i})$ and $H_{\frac{1}{3}}^{(1)}(r\sqrt{i})$; Table II, p. 346, the real and imaginary parts of the derivatives with respect to r of the functions tabulated in table I; Table III, p. 347, the real and imaginary parts of $J_{\frac{1}{3}}(-r\sqrt{-i})$ and $H_{\frac{1}{3}}^{(1)}(-r\sqrt{-i})$; and Table IV, p. 348, the real and imaginary parts of the derivatives of the functions tabulated in table III. All four tables are to varying degrees of accuracy, for $r = 0(1)30$. There are several other tables in the paper, but they are of less universal interest.

A. E.

830[M].—W. M. STONE, "A list of generalized Laplace transforms," Iowa State College, *Jn. of Science*, v. 22, 1948, p. 215–225.

This paper presents a table of "generalized Laplace transforms," which is in fact a list of 75 functions $f(s)$, each corresponding to a numerical function $F(k)$, by means of the relation $sf(s) = \sum_{k=0}^{\infty} F(k)s^{-k}$ so that f is essentially the generating function of F. The functions F are chosen from rational functions and combinations of sines and cosines. The functions f are all elementary.

D. H. L.

831[U].—J. C. LIEUWEN, *Kortbestek Tafel*, being v. II of *Zeevaartkundige Tafels uitgegeven op last van het Ministerie van Marine*, The Hague Staatsdrukkerij-en Uitgeverijbedrijf, 1949, ii, 160 p., cloth, 20.7 × 29.5 cm. No price stated.

This collection of navigation tables adds one more to the long list of short methods for the reduction of astronomical sights; it contains many points of interest. Before the main tables there are an arc-time conversion table and four short tables of more or less standard form. Table I is a straightforward traverse table giving d'lat and departure to $0'.1$ for each degree of bearing and for distances of 10(10)490 and 1(1)9 minutes of arc. Table II is for converting departure into d'long and vice versa and gives the first nine integral multiples of secant (middle latitude) and cosine (middle latitude) for $1°(1°)13°(30')24°(20')34°(10')53°(5')70°(4')72°28'$; the full multiplication has to be done by adding the products of successive integers. The curious choice of intervals is evidently dictated by the desire to limit the relative error, without interpolation, to a minimum of 1 in 500. The third table, comprising Tables III a, b, and c, is a collection of altitude correction tables; from these it transpires that

(a) the interpolation table c_1, for latitude, has been "faked" by the incorporation of the second-difference correction on the assumption that the altitude is 70°;
(b) the altitude correction tables have been correspondingly adjusted for a mean value of the latitude difference.

This very considerable complication arises from the curvature of position lines derived from observations at high altitudes; the error due to curvature is precisely that arising from neglect of second differences in interpolations and only those who have striven to find a way of incorporating the corrections in, say, triple-entry tables can fully appreciate the ingenuity of this device. In this case interpolation to the exact D. R. longitude offers no similar difficulties since the interval (1^m) is so small that second differences do not arise; neither do the cross-terms, which are so very difficult to deal with.

Table IV is a collection of small tables for ex-meridian sights; it is ingeniously arranged with one of the arguments in the body of the table in a manner typical of many of the tables to be described in detail later.

The three main tables, A, B, and C, for the calculation of altitude and azimuth from an assumed position are based on a modification of SOUILLAGOUËT'S and DREISONSTOK'S methods, though the table for obtaining the second azimuth angle is new. The astronomical spherical triangle is divided into two right-angled triangles by a perpendicular, length a, from the zenith to the opposite side, meeting this in a point whose declination is K. The first, or time-triangle, is solved directly by double-entry tables. Table A thus gives K to $0'.1$, T_1 to $0°.1$, (the angle at the zenith, contributing towards the azimuth) and $A = 10^5 \log \sec a$ to the nearest unit, or in some cases the nearest 10 units, for the page heading degrees of latitude (b) from 0° to 71° and with horizontal and vertical arguments hours and minutes of hour angle (P); this table is identical in scope with many others, in particular with Table I of *Hughes' Tables for Sea and Air Navigation.*

The second triangle can be solved in many ways but only by direct double-entry table if the greatest care is taken to avoid difficulties of interpolation and loss of accuracy. Practically all modern tables (with the exception of those of DE AQUINO) solve this triangle by logarithms. Here, however, direct values are tabulated for the second azimuth angle T_2, with the usual (HUGHES-DREISONSTOK) logarithmic solution for the altitude. Table B consists of two parts. On the left-hand pages is given a straight table of $B = 10^5 \log \sec (K \pm d)$ with argument $K \pm d$, where d is the declination, for the range $0°(0'.5)20°(1')89°59'$. The interval appears to be determined by the requirements of the facing page; at intervals of 1′ the half-difference to give the value at the following 0′.5 is also tabulated. The corresponding right-hand pages contain an ingeniously arranged double-entry table for the angle T_2; the horizontal argument is integral degrees of $K \pm d$, covering the same range as on the facing page, and the second argument is $A = \log \sec a$ for which values are given in the body of the table, corresponding to integral degrees of T_2. There is also tabulated T_2, the variation of T_2 with $K \pm d$. The table is based on the formula:

$$\tan T_2 = \tan (K \pm d) \csc a$$

and so is difficult to interpolate when $K \pm d$ is small. Generally, however, the table is satisfactory for its purpose, though the unfamiliar form of entry may confuse users; it is reasonable to ask whether it would not have been better to have accepted the rapidly changing intervals and given T_2 directly with argument A; the present table looks neater but interpolation to tenths of a degree is in places difficult.

Table C is a straight table of $10^5 \log \csc (h - 1°)$ with argument h for the range 0°(0′.5)88°. It is identical in principle with Table II in Hughes' tables, though it does not include the additional decimal for high altitudes. It is entered inversely with argument $A + B$ to give the calculated altitude corresponding to the assumed position.

The second innovation is now introduced in the shape of two tables c_1 and c_2 for the interpolation of the altitude to the D. R. position. The first of these is for interpolation for the minutes of latitude and gives, to 0′.1, the corrections for each minute and for azimuths at intervals of 1°, except within 30° of the meridian when larger intervals are used. This would be apparently a straightforward table of $b \cos$ (Az.), but it is modified in two ways: firstly (as mentioned earlier) by the second-difference correction appropriate to an altitude of 70°, and secondly by the addition of 30′. Thus, all the entries are positive. In using this table the integral degree *nearest* to the D. R. latitude *must* be used; for the exact 30′ the next larger value is the appropriate one. The second table c_2 gives the corresponding interpolation for longitude, but here no second-difference correction is required and it is necessarily triple-entry. The table is in two parts, the one on the left-hand page being essentially a table of the rate of change of altitude with time, $-\cos$ (lat.) $\sin$ (Az.). The horizontal argument is latitude and the body of the table gives the azimuth corresponding to certain approximately equally-spaced values of the rate of change, which are however not specifically given. This page therefore determines a particular line, for which the facing page gives the appropriate multiples (with the addition of 30′) at intervals of 3^s. This is again an ingenious and carefully considered arrangement. It will be noted that the correction is always positive and in fact, is always greater than 15′. Moreover, the indication is that interpolation is always to be done in a forward direction, as opposed to the backward and forward method for the latitude; even so the theoretical error is negligible owing to the smallness of the interval to be covered.

There is also a two-page table and a chart for drawing position lines.

The interpolation tables described are nominally to an accuracy of 0′.1, but this accuracy can only be obtained with some care in use. Direct entry in the interpolation tables, without mental interpolation or adjustment, will suffice to a reduced standard of 0′.5 in altitude and 0°.5 in azimuth, which the author clearly regards as adequate for most navigational purposes.

Sufficient has been said in the above description of the contents of these tables to show that they have been devised with extraordinary care and skill. The arrangement, layout, printing, paper, binding are all excellent; the figures have heads and tails, are well spaced and are easy to read. The only real criticism is that the ingenuity by which the interpolations to the D. R. position are designed makes it essential to use the tables precisely as instructed. These tables must stand very high among those using the so-called "short-methods."

These tables are the product of prolonged research not only by the author but by a representative committee of Dutch experts. All concerned deserve great credit for an achievement combining ingenuity, practical insight and fine execution. An English edition is contemplated.

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832[V].—BALLISTIC RESEARCH LABORATORIES, *Tables of Ballistic Functions* $\xi(\theta)$, $c(\xi)$, $s(\xi)$, $X(\beta, b)$, $Y(\beta, b)$, $T(\beta, b)$. Aberdeen Proving Ground, 1949. Approx. 270 leaves, 21.6 × 27.9 cm, tabulated from punched cards.

These are tables of ballistic functions describing the motion of an object subject to the acceleration of gravity and to a resistance proportional to the square of the velocity. The equations describing such motion are solvable by quadrature, as observed by EULER. The approximation (square law drag and constant density) is usable primarily for mortars and for certain rockets, and is now, save for variations of the SIACCI method, the only widely used method which avoids numerical integration of the normal equations. These tables give complete trajectories, rather than terminal data only, as given in the tables of OTTO and LARDILLON.[1] The data have been put in a compact and convenient form and should prove useful.

Table 1 is a repetition of one given by C. CRANZ,[1] and corrects a large list of errors (29) occurring in the latter.

The tables were computed on IBM equipment under the direction of I. SCHOENBERG.

Specifically, the tables comprise:

1. $\xi(\theta) = \int_0^\theta \sec^3 t dt$ to (approximately) 7S for $\theta = 0°(1')87°$.
2. $s(\xi) = \sin\theta(\xi)$ for $[\xi = 0(.01)50; 9D]$ where $\theta(\xi)$ is the function inverse to ξ.
3. $c(\xi) = \cos\theta(\xi)$ to 8D for the same range of ξ as in 2.
4. $X(\beta, b) = \int_1^\beta c(b(1 - t))t^{-1}dt$.
5. $Y(\beta, b) = \int_1^\beta s(b(1 - t))t^{-1}dt$.
6. $T(\beta, b) = \int_1^\beta c(b(1 - t))t^{-\frac{1}{2}}dt$.

The last three functions have been tabulated to 8D, for $b = .1(.1)2$ and $\beta = 0(.02)3$, with upper limits of these ranges restricted by $\beta(b - 1) \leq 2$. Save in Table 1, differences through the third order are given. The physical meaning of the parameters is:

$$b = g/(2kV_s^2),$$

where g is the acceleration due to gravity, k is the usual "resistance" coefficient, V_s is the summital velocity, and $\beta = V_s/(\text{horizontal component of velocity})^2$. Thus, along a single trajectory, b = constant, and the change in horizontal distance, vertical distance, and time is given parametrically with β, as the change in X, Y, and T divided by $2k$, $2k$, and $2kV_s$, respectively.

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[1] C. CRANZ, *Lehrbuch der Ballistik*, v. 1, Berlin, 1925.

833[V].—A. VAN WIJNGAARDEN, "Écoulement potentiel autour d'un corps de révolution," Centre National de la Recherche Scientifique, *Colloques Internationaux*, XIV: *Méthodes de Calcul*, Paris, 1949, p. 72–87.

The paper tabulates functions used in calculating approximate potential flow about a body of revolution with or without angle of attack by a method corresponding to an improvement of VON KÁRMÁN's source doublet distribution method. The body shape is approximated by a finite number of distributed sources, the ith source distributed along the axis of revolution between $x = (i - 1)a$ and $x = (i + 1)a$. The value of the stream function induced by the ith source at any radius r from the axis and at $x = ka$ is $\psi_i = - Q_i c_{ik}/4\pi$ where Q_i is the source strength and

$$c_{ik} = 2 - \frac{1}{4}\frac{r_k}{a} (\delta_x^2 \tan^2 \tfrac{1}{2}\theta + \delta_x^2 \ln \tan^2 \tfrac{1}{2}\theta).$$

The δ_x^2 indicates the second order central difference in the x direction and $\tan\theta = r_k\{a(k - i)\}^{-1}$. Table 1 (p. 78, 79) gives 4D values of c_{ik} for $r_k/a = 0(.02)2$ and for $k - i = 0(1)9$. The corresponding velocity components are $u_r = (2\pi a^2)^{-1}Q_i u_{rik}$ normal to the axis and $u_x = (2\pi a^2)^{-1}Q_i u_{xik}$ parallel to the axis where $u_{rik} = \frac{1}{2}\delta_x^2 \tan^2 \frac{1}{2}\theta$ and $u_{xik} = \frac{1}{2}\delta_x^2 \ln \tan^2 \frac{1}{2}\theta$. The u_{xik} and u_{rik} are given in Tables 2 and 3, respectively (p. 80–83), for the same range of parameters. The velocities induced by similarly distributed doublets are needed when the body moves other than parallel to its own axis of revolution. If ϕ is the angle about the axis of revolution, the axial velocity induced by a doublet of moment M_i is $(4\pi a^3)^{-1}M_i u'_{xik} \cos\theta$ while the radial velocity is $(4\pi a^3)^{-1}M_i u'_{rik} \cos\phi$, where

$$u'_{xik} = ar^{-1}\delta_x^2 \cos\theta;\ u'_{rik} = ar^{-1}\delta_x^2 (\cot\theta \cos\theta).$$

These two functions are given in Tables 4 and 5, respectively (p. 84–87), for the range of parameters given above.

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