[8] A. M. Turing, "A method for the calculation of the Zeta-function," London Math. Soc., *Proc.*, s. 2, v. 48, 1943, p. 180–197.
[9] H. G. Dawson, "On the numerical value of $\int_0^h e^{x^2}dx$," London Math. Soc., *Proc.*, s. 1, v. 29, 1898, p. 519–522.
[10] NBS, *Tables of Probability Functions.* V. 1, New York, 1941.

RECENT MATHEMATICAL TABLES

857[D].—J. K. Lynch, "Calculation of $(\sin x)/x$ tables," Postmaster-General's Department, *Research Laboratory Report No. 3238*, Melbourne, C. 1., (April 1949), C.E. 615/R, Case no. 2266. 7 mimeographed pages + drawing.

These tables of $(\sin x)/x$ were computed to facilitate the calculation of the harmonic components of periodic impulses. The function $(\sin x)/x$ is tabulated for $x = 0(1°)180°(5°)1800°$; 4D, with first differences. The drawing contains a graph of $(\sin x)/x$ for x ranging from 0° to 1600°, together with a straight line converting degrees to radians. The statement in the Introduction ". . . the interval of tabulations is such that linear interpolation will suffice," is not quite correct, since second differences are needed for full accuracy when x lies between 180° and 295°.

For another table of this function, see [*MTAC*, v. 4, p. 80].

H. E. Salzer

NBSCL

858[F, G, O].—R. E. Beard, "On the coefficients in the expansion of e^{e^t} and e^{-e^t}." Inst. of Actuaries, *Jn.*, v. 76, 1950, p. 152–163.

The expansions mentioned in the title are written

$$e^{e^t} = e \sum_{n=0}^{\infty} K_n t^n/n!$$

$$e^{-e^t} = e^{-1} \sum_{n=0}^{\infty} A_n t^n/n!$$

where the A's and K's are integers. They are tabulated for $n = 0(1)26$. [See the following review where K_n is denoted by $U(n)$ and a table is described for $n = 1(1)50$; the agreement for $n \leqslant 26$ is exact.] The author used the formulas

$$K_r = \sum_{k=1}^{r} \Delta^k 0^r/k!$$

and

$$A_r = \sum_{k=1}^{r} (-1)^k \Delta^k 0^r/k!$$

The values of the Stirling numbers of the second kind, $\Delta^k 0^r/k!$ were taken from Fisher & Yates [as corrected in *MTAC*, v. 4, p. 27] for $r \leqslant 25$. The necessary additional values of these numbers for $r = 26$ are appended on p. 163. The table of A_r appears to be new. Previous tables of K_n for $n \leqslant 20$ are mentioned in FMR, *Index* § 4.676.

D. H. L.

859[F, G, O].—H. GUPTA, "Tables of distributions," East Panjab University, *Research Bull.*, no. 2, 1950, p. 13–44.

The author denotes by $u(n, a)$ the number of ways that n unlike objects may be distributed into a boxes. If $a > n$ then $u(n, a) = 0$. The sum

$$U(n) = \sum_{a=1}^{n} u(n, a)$$

is the total number of distributions of n different objects into groups. It also represents the number of rhyming schemes in a stanza of n lines. In terms of generating functions we have

$$\sum_{n=0}^{\infty} u(n, a)x^n/n! = (e^x - 1)^a/a!, \qquad \sum_{n=0}^{\infty} U(n)x^n/n! = e^{-1} \exp (e^x).$$

The fundamental difference equation for $u(n, a)$ is

$$u(n + 1, a) = au(n, a) + u(n, a - 1).$$

By means of this relation the author has constructed a table (p. 17–43) of $u(n, a)$ for $1 \leqslant a \leqslant n \leqslant 50$, and has obtained, by summing, a table of $U(n)$ for $n = 1(1)50$. The function $U(n)$ may also be computed from the relation

$$(1) \qquad U(n + 1) = \sum_{k=0}^{n} U(k) \binom{n}{k}.$$

This was used to check the whole set of tables by computing the 48-digit number $U(50)$. The several terms of (1) for $n = 50$ are set forth on p. 44.

D. H. L.

860[F].—H. GUPTA, "A table of values of Liouville's function $L(t)$," East Panjab University, *Research Bulletin*, no. 3, 1950, p. 45–63.

The function of LIOUVILLE, despite the title of the paper under review, is usually denoted by $\lambda(n)$ and is $+1$ or -1 according as the number of prime factors of n, each reckoned with its multiplicity, is even or odd. The function $L(t)$ is then defined as the sum function

$$L(t) = \sum_{n \leqslant t} \lambda(n).$$

According to a conjecture of POLYA, the values of $L(t)$ for $t > 1$ are never positive. This conjecture, when proved, will imply the truth of the famous RIEMANN hypothesis. In 1940 the author constructed a table of $L(t)$ for $t = 1(1)20000$ [*MTAC*, v. 1, p. 201]. The present paper presents a condensation of this table from which with very little effort an isolated value in the original table may be found. More specifically, the present table gives $-L(t)$ for $t = 5(5)20000$ and, by a clever device, the values of $\lambda(t)$ when t is not a multiple of 5. ($\lambda(5k) = -\lambda(k)$.)

Two small tables (p. 47) give some idea of the range of values of $L(t)$ for $t \leqslant 20000$. The conjecture of Polya is verified but nevertheless $-L(t)$, though positive, is fairly small. A maximum of 150 occurs at $t = 15810$. The function $t^{-1}L^2(t)$ has a maximum at $t = 9840$. This would indicate that $L(t) = O(t^{\frac{1}{2}})$ which would also imply the Riemann hypothesis. The author mentions that isolated values of $L(t)$ up to $t = 60000$ may be found by another method. By still another method the reviewer has found that $L(4000000)$ is very close to -1100.

D. H. L.

861[F].—D. JARDEN, "On the numbers V_{5n} (n odd) in the sequence associated with Fibonacci's sequence," *Riveon Lemat.*, v. 4, 1950, p. 38–40 [Hebrew with English summary].

This note contains a factor table of the numbers

$$V_{2n} - 5U_n + 3 \qquad \text{and} \qquad V_{2n} + 5U_n + 3$$

whose product, in case n is odd, is V_{5n}/V_n. Here V_m is the mth term of the FIBONACCI sequence: 2, 1, 3, 4, 7, $\cdots$

$$V_m = V_{m-1} + V_{m-2}, \qquad 5U_m = V_m + 2V_{m-1}.$$

The table is for $n = 1(2)77$. Factorization is incomplete in many cases. For the function $V_{2n} - 5U_n + 3$ complete factorization into primes is given only for $n = 1(2)35, 39(2)45, 49, 51, 63, 75$; for the second function only for $n = 1(2)45$. Most of the results come from the table of KRAITCHIK.[1]

D. H. L.

[1] M. KRAITCHIK, *Recherches sur la Théorie des Nombres.* Paris, 1924, v. 1, p. 80.

862[K].—A. N. BLACK, "Weighted probits and their use," *Biometrika*, v. 37, 1950, p. 158–167.

The paper tabulates

$$P = (2\pi)^{-\frac{1}{2}} \int_{\infty}^{Y-5} \exp\left(-\tfrac{1}{2}u^2\right)du, \qquad w = Z^2(PQ)^{-1},$$

$$wy_0 = w(Y - PZ^{-1}), \qquad wy_m = w(Y + QZ^{-1})$$

all to 3D for $Y = 1(.02)9$, except that wy_0 is replaced by $100 - wy_0$ for $Y > 6.42$. As usual $P + Q = 1$ and $Z = (2\pi)^{-\frac{1}{2}}\exp\left(-(Y-5)^2/2\right)$. The purpose of the table is to minimize the arithmetic of the maximum likelihood estimation of the probit regression. There are detailed instructions for calculation.

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863[K].—F. N. DAVID, "Note on the application of Fisher's k-statistics," *Biometrika*, v. 36, 1949, p. 383–393.

This paper contains a table of the terms which must be added to $\kappa(r^h s^l)$, the product cumulant of order hl of the r-th and s-th k-statistics about the population r-th and s-th cumulants respectively in terms of population cumulants, in order to give the corresponding product moment. The table gives all terms for $h + l = 4$ and 5 and those for $h + l = 6$ involving n^{-3} only where n is the sample size. The paper illustrates the utility of these tables in reducing the algebra in finding the approximate moment characteristics of certain statistics.

C. C. C.

864[K].—F. N. DAVID, "Two combinatorial tests of whether a sample has come from a given population," *Biometrika*, v. 37, 1950, p. 97–110.

In connection with the tests herein described and illustrated, the author tabulates (Table 1a, p. 109) $N^{-N}\binom{N}{Z}\Delta^{N-Z}0^N$ for $N = 3(1)20$ and $Z = 0(1)19$ to 4D. This is the probability of Z zero groups when N sample units are randomly arranged in N groups of equal probability and is thus related to a classical distribution problem. Table 4a, p. 110 gives values of $(2N)^{-N}\binom{N}{Z}\Delta^{N-Z}(N^N)$, for $N = 1(1)10$ and $Z = 0(1)10$ to 4D, which is the probability that if N balls be dropped at random into $2N$ compartments, Z of a specified set of N compartments will be empty.

C. C. C.

865[K].—F. E. GRUBBS, "Sample criteria for testing outlying observations," *Annals Math. Stat.*, v. 21, 1950, p. 27–58.

The author proposes the following statistics

$$\frac{S_n^2}{S^2} = \frac{\sum_{i=1}^{n-1}(x_i - \bar{x}_n)^2}{\sum_{i=1}^{n}(x_i - x)^2} = 1 - \frac{1}{n-1}T_n^2, \qquad T_n = \frac{x_n - \bar{x}}{S}$$

$$\frac{S_1^2}{S^2} = \frac{\sum_{i=2}^{n}(x_i - \bar{x}_1)^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = 1 - \frac{1}{n-1}T_1^2, \qquad T_1 = \frac{\bar{x} - x_1}{S}$$

where $x_, \leq x_2 \leq \cdots \leq x_n$, $\bar{x}_n = \frac{1}{n-1}\sum_{i=1}^{n-1} x_i$, $\bar{x}_1 = \frac{1}{n-1}\sum_{i=2}^{n} x_i$, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

for testing the greatest and lowest observations respectively in a sample of size n, from a normal population. Values of S_n^2/S^2 or S_1^2/S^2 to 4D are given at percentage points 1, 2.5, 5 and 10 for $n = 3(1)25$ in Table I (p. 28) and corresponding values to 3D of T_n or T_1 in Table IA (p. 29). Grubbs' values in Table IA calculated from the exact distribution concur with those of E. S. PEARSON & C. CHANDRA SEKAR[1] for common entries.

Table II (p. 31–37) contains values of $P(u_n \leq u)$ to 5D for $u = 0(.05)4.85$, $n = 2(1)17$, $u = .5(.05)4.90$, $n = 18$, 19 and to 4D for $u = .5(.05)4.90$, $n = 20(1)25$, where $u_n = (x_n - \bar{x})/\sigma$ in samples of size n drawn from a parent normal population with zero mean and standard deviation σ. These values were obtained from the cumulative distribution

$$F_n(u) = n^{\frac{1}{2}}(2\pi\{n-1\})^{-\frac{1}{2}}\int_0^u \exp\left(-\frac{x^2}{2}n/(n-1)\right)F_{n-1}(nx/(n-1))dx$$

where $F_n(u) = \int_0^u dF(u_n) = P(u_n \leq u)$. This distribution (obtained in 1945) was also independently arrived at by K. R. NAIR[2] and is equivalent to the result of A. T. MCKAY[3] arrived at by means of characteristic functions. Nair[2] also gave tables of this probability integral for $u = 0(.01)4.7$,

$n = 2(1)9$, to 6D which concur in corresponding values with Grubbs' Table II.

Table III (p. 45) contains values to 3D of $u_n = (x_n - \bar{x})/\sigma$ for percentage points 90, 95, 99, and 99.5, $n = 2(1)25$, also given by Nair[2] to 2D for percentage points .01, .5, 1, 2.5, 5, 10, 90, 95, 97.5, 99, 99.5, 99.9, $n = 3(1)9$.

Table IV (p. 46) contains the mean, standard deviation, α_3 and α_4 of u_n to 4D for $n = 2(1)15$, and to 3D for $n = 20$, 60, 100, 200, 500, and 1000. These values agree with corresponding values of TIPPETT[4] for the mean and those of McKay[3] for the standard deviation except for $n = 2$, which Grubbs gives correctly as 0.4263 (by McKay's formula) whereas McKay gives 0.6179.

Table V contains percentage points 1, 2.5, 5, and 10, $n = 4(1)20$ to 4D for $S^2_{n-1,n}/S^2$ or $S^2_{1,2}/S^2$ where

$$S^2_{n-1,n} = \sum_{i=1}^{n-2} (x_i - \bar{x}_{n-1,n})^2, \qquad \bar{x}_{n-1,n} = (n-2)^{-1} \sum_{i=1}^{n-2} x_i,$$

$$S^2_{1,2} = \sum_{i=3}^{n} (x_i - \bar{x}_{1,2})^2, \qquad \bar{x}_{1,2} = (n-2)^{-1} \sum_{i=3}^{n} x_i,$$

S^2 and $\bar{x}$ as above. These statistics are proposed for testing the significance of the two largest and the two smallest observations in samples of size n drawn from parent normal populations.

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[1] E. S. PEARSON & C. CHANDRA SEKAR, "The efficiency of statistical tools and a criterion for the rejection of outlying observations," *Biometrika*, v. 28, 1936, p. 308–320.

[2] K. R. NAIR, "The distribution of the extreme deviate from the sample mean and its studentized form," *Biometrika*, v. 35, 1948, p. 118–144.

[3] A. T. MCKAY, "The distribution of the difference between the extreme observation and the sample mean in samples of n from a normal universe," *Biometrika*, v. 27, 1935, p. 466–471.

[4] L. H. C. TIPPETT, "On the extreme individuals and the range of samples taken from a normal population," *Biometrika*, v. 17, 1925, p. 364–387.

866[K].—A. HALD, "Maximum likelihood estimation of the parameters of a normal distribution which is truncated at a known point," *Skandinavisk Aktuarietidskrift*, v. 32, 1949, p. 119–134.

This paper includes four short tables (p. 130–134) of certain functions for use in estimating the mean and variance of normal populations from one-sided truncated samples with known points of truncation, both for the simple truncated case in which the number of missing (unmeasured) observations is unknown and for the case which the author labels as "censored" in which the number of unmeasured observations is known. Tables I and II pertain to the simple truncated case. Tables III and IV furnish similar information for the "censored" case. All four tables were computed from Fisher's I or Hh function.[1]

Table I of $z = f(y)$ is an inversion of Table XVI from the B.A.A.S. Tables[1] where z is the abscissa of truncation in standard units of the complete distribution and y is a function of moments of the truncated distribution. In practice, one sets $y = n \sum x_i^2/2(\sum x_i)^2$ and reads z directly from this table. Entries of z are given to 3D for $y = .55(.001).91$. In the notation of PEARSON & LEE,[2] $y = (\psi, -1)/2$, and in the notation of FISHER[3] $y = I_0I_2/I_1^2$. Table II contains $g(z)$, a function employed in estimating the variance, together with its first differences as an aid in interpolation. Entries are tabulated to 4D(4 or 5S). This table also includes elements of the variance-covariance matrix, $\mu_{11}(z)$, $\mu_{12}(z)$, and $\mu_{22}(z)$ tabulated to 4S, and $\rho(z)$, the correlation coefficient between estimates of population parameters, to 3D. The argument, z, ranges over $-3(.1)2$.

Table III of $z = f(h, y)$ was also computed by Statistika I/S for $h = .55(.05).8$. It permits the abscissa of truncation, z, for the "censored" case to be read directly for given values of h and y, where h is the ratio of the number of unmeasured to measured observations, and y is the counterpart of the similarly labeled function for the simple truncated case. Entries for z are given to 3D with $y = .5(.001)1.5$ and $h = .05(.05).8$. The functions of Table IV for the "censored" case correspond to those of Table III for the simple truncated case. The range and the interval of the argument, z, are the same in both tables, and the number of significant digits is the same in most instances.

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[1] B.A.A.S., *Math. Tables*, v. 1, 2nd ed., Cambridge, 1946, Tables XV and XVI.
[2] KARL PEARSON & ALICE LEE, "On the generalized probable error in multiple normal correlation," *Biometrika*, v. 6, 1908, p. 59–68.
[3] See 1st ed. of above B.A.A.S. tables, 1931, p. xxvi–xxxv.

867[K].—H. O. HARTLEY & E. S. PEARSON, "Table of the probability integral of the *t*-distribution," *Biometrika*, v. 37, 1950, p. 168–172.

This table gives the cumulative probability integral of *Student's t* for γ degrees of freedom, namely

$$P(t, \gamma) = \left\{\Gamma\left(\frac{\gamma+1}{2}\right)\Big/\left[\Gamma(\tfrac{1}{2}\gamma)\sqrt{\pi\gamma}\right]\right\}\int_{-\infty}^{t}(1 + x^2/\gamma)^{-(\gamma+1)/2}dx$$

to 5D for $\gamma = 1(1)20$; $t = 0(.1)4(.2)8$ and $\gamma = 20(1)24, 30, 40, 60, 120, \infty$; $t = 0(.05)2(.1)4, 5$. Also values of t and α such that $1 - P(t, \gamma) = \alpha$ are given for $\alpha = 10^{-3}, 10^{-4}, 10^{-5}, 5(10)^{-6}$; $\gamma = 1(1)10$. For $\gamma = 11(1)20$ the values of α to 10^{-5}, as well as values of α to $3(10)^{-5}$ and less for larger γ, are covered in the main part of the table. Formulas for both single-entry and double-entry interpolation are given.

This table may be particularly useful in problems where it is necessary to interpolate for fractional degrees of freedom. It is also useful in cases where distributions of other statistics can be expressed in terms of the t-distribution. Previous tables[1] were to 4D for $\gamma = 1(1)20$.

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[1] *Student*, "New tables for testing the significance of observations," *Metron*, v. 5, no. 3, 1925, p. 105–120.

868[L].—C. J. Bouwkamp, (I) "On the characteristic values of spheroidal wave functions," *Philips Res. Rep.*, v. 5, 1950, p. 87–90. (II) "On the theory of spheroidal wave functions of order zero," Nederl. Akad. Wetensch., *Proc.*, v. 53, 1950, p. 931–944.

I. The differential equation for spheroidal wave functions is taken in the form

$$(1) \qquad (1 - z^2)y'' - 2zy' + (\lambda - m^2(1 - z^2)^{-1} + k^2z^2)y = 0.$$

Corresponding to a countably infinite set of characteristic values λ, for given m and k, equation (1) admits a solution which is regular at $z = \pm 1$. It is known that the form for λ is

$$\lambda_n^m(k) = \sum_{i=0}^{\infty} p_i k^{2i}.$$

The author gives the coefficients p_i, $i = 0(1)4$, for general n and m, along with numerical values of the coefficients (in both fractional and decimal form) for $m = 1$, $n = 1(1)7$.

Three brief tables are given:

Table 1: Prime factors in the denominators of the coefficients p_i, for $n = 1(1)7$; $m = 1$.

Table 2: Decimal form of the coefficients p_i for $n = 1(1)7$; $m = 1$, to 12D. Meixner (1944) gave a more complete table, to 10D. [*MTAC*, v. 3, p. 524–6.]

Table 3: Numerical values of $\lambda_n^1(k)$, as calculated from power series, for $k^2 = \pm 1, \pm 4$, $n = 1(1)7$; to 8D for $k^2 = \pm 1$, and 5D for $k^2 = \pm 4$.

II. In this paper the author treats various series representations for the solutions of the first, second, and third kinds, associated with the characteristic values of the equation. In particular he derives a set of coefficients c_n, such that two independent solutions are of the form

$$y(z) = \exp(-kz) \sum_{n=0}^{\infty} c_n P_n(z); \qquad Y^*(z) = \exp(-kz) \sum_{n=0}^{\infty} c_n Q_n(z),$$

where $P_n(z)$ and $Q_n(z)$ are the Legendre functions. Here $y(z)$ is the regular solution. These solutions are valid in the domain of regularity of the function (outside the branch cut joining $z = 1$ and $z = -1$). Interesting relations are obtained between the eigenvalues $\mu_n(k)$ of the integral equation

$$y(z) = \mu \int_{-1}^{1} \exp(kzt) y(t) dt$$

and the coefficients c_p. Although the series involving c_p are important theoretically they do not converge as rapidly as the classical series involving Bessel functions.

Gertrude Blanch

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869[L].—B. W. Conolly, "A short table of the confluent hypergeometric function $M(\alpha, \gamma, x)$," *Quart. Jn. Mech. Appl. Math.*, v. 2, 1950, p. 236–240.

Tables of confluent hypergeometric functions are listed in section 22.56 of the FMR *Index*, and some have also been reviewed in *MTAC*. In the present paper the author tabulates

$$M(\alpha, \gamma, x) = \sum_{r=0}^{\infty} \Gamma(\gamma)\Gamma(\alpha + r)x^r/(r!\Gamma(\alpha)\Gamma(\gamma + r)) = \sum_{r=0}^{\infty} a_r x^r$$

to 11D for $x = .1, .2(.2)1$, $\alpha = -1(.2)1$, and $\gamma = .2(.2)1$. The tabulated values are believed to be correct to within 2 or 3 units of the last decimal place.

The coefficients a_r, and the values of $M(\alpha, \gamma, 1)$ were provided by J. C. P. MILLER.

A. E.

870[L].—A. COOMBS, "The translation of two bodies under the free surface of a heavy fluid." Cambridge Phil. Soc., *Proc.*, v. 46, 1950, p. 453–468.

In the course of the work a 6D table is constructed of

$$\{\mathrm{li}(e^\alpha) - \pi i\}\exp(-\alpha(1 - \tfrac{1}{2}i\beta)) - \sum_{n=0}^{\infty} \alpha^{-n-1}n! \sum_{s=n+1}^{\infty} (\tfrac{1}{2}i\alpha\beta)^s/s!$$

for $\beta = -1, 1$ and $\alpha = 1, 2, 2.5, 3(1)6, 8$. More extensive tables are provided for the forces acting upon two cylinders in the same vertical or horizontal plane.

A. E.

871[L].—OTTO EMERSLEBEN, "Die elektrostatische Gitterenergie eines neutralen ebenen, insbesondere alternierenden quadratischen Gitters." *Zeit. für Physik*, v. 127, 1950, p. 588–609.

In the study of crystal lattices there appear the EPSTEIN zeta functions, defined for $R(s) > 2$ by

$$Z\begin{vmatrix} 0 & 0 \\ h_1 & h_2 \end{vmatrix}(s) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty}{}' \frac{\exp(2\pi i(k_1h_1 + k_2h_2))}{(k_1^2 + k_2^2)^{s/2}}.$$

For various integer values of s the author evaluates these functions in the four cases $(h_1, h_2) = (0, 0), (\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}), (\frac{1}{4}, \frac{1}{4})$. The last three cases can be computed directly from the first with the help of functional equations. For $s = 1$ the formula

$$Z\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}(1) = -4 + 2\sum_{n=1}^{\infty} n^{-\frac{1}{2}}r_2(n)[1 - F((\pi n)^{\frac{1}{2}})]$$

is used, where $r_2(n)$ is the number of ways n can be written as a sum of two squares and

$$F(x) = 2\pi^{-\frac{1}{2}} \int_0^x e^{-t^2}dt.$$

Eight terms of the series suffice to yield the value

$$Z\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}(1) = -3.900\,264\,92.$$

For $s = 2$ the values have been previously given by the author.[1] For $s = 3$

and $s = 4$ use is made of the product representation

$$Z\begin{vmatrix}0 & 0\\0 & 0\end{vmatrix}(s) = 4\zeta(\tfrac{1}{2}s)L(\tfrac{1}{2}s)$$

and tables of $\zeta(s)$, $L(s)$. Here

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \quad \text{and} \quad L(s) = \sum_{n=1}^{\infty} (-1)^{n-1}(2n-1)^{-s}.$$

An alternative method is also given in the case $s = 4$. For negative integer values of s the author uses the functional equation which gives the value at s in terms of the value at $2 - s$ and gamma functions.

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[1] OTTO EMERSLEBEN, "Einige Identitäten für Epsteinsche Zetafunktionen," *Math. Ann.*, v. 121, 1949, p. 103–106.

872[L].—B. FRIEDMAN, "Theory of underwater explosion bubbles," *Comm. on Pure and Applied Math.*, v. 3, 1950, p. 177–199.

Tables are given to 4S of the functions

$$f(x) = 2x \sum_{n=0}^{\infty} (-1)^n[(2n+1)^2 - x^2]^{-1}$$

$$F(x) = (1 - x)[\tfrac{1}{2}x^{-1}f(x) + \tfrac{1}{2}\ln 2 - (1 + x)^{-1}].$$

The range for x is 0(.05)1 for f and -1(.05)1 for F.

873[L].—E. L. KAPLAN, "Multiple elliptic integrals," *Jn. Math. Phys.*, v. 29, 1950, p. 69–75.

The author shows that the evaluation of integrals of the form

$$\int K\, k^n k'^{n'} dk, \qquad \int E\, k^n k'^{n'} dk$$

where n, n' are integers and k, k', K, E are the standard notations, and the evaluation of some similar integrals can be reduced to the computation of the eight integrals

$$\int K\, dk/k, \quad \int K'\, dk/k, \quad \int K\, dk/k', \quad \int E\, dk/k',$$

$$\int K\, dk, \quad \int K'\, dk, \quad \int K\, dk/k'^2, \quad \int K'\, dk/k'^2,$$

and that the last four can be dispensed with if Landen's transformation is used in the process of reduction. The values of the 8 basic integrals taken between the limits 0 and 1, and modified if necessary to produce convergence, can be expressed in terms of known constants and are given in the paper to 10D. There is, moreover, a 10D table of

$$(n + 1) \int_0^1 k^n K\, dk$$

for $n = -.9(.1)2$.

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874[L].—A. LEITNER & R. D. SPENCE, "The oblate spheroidal wave functions." Franklin Institute, *Jn.*, v. 249, No. 4, 1950, p. 299–321 [see also ref. in *MTAC*, v. 3, 1948, p. 99–101].

The differential equations for the "angular" and "radial" spheroidal wave functions of order m respectively are written in the form

(1) $d\{(1 - \eta^2)du/d\eta\}/d\eta - \{m^2(1 - \eta^2)^{-1} - \alpha + \epsilon^2(1 - \eta^2)\}u = 0,$

(2) $d\{(1 + \xi^2)dv/d\xi\}/d\xi - \{m^2(1 + \xi^2)^{-1} - \alpha + \epsilon^2(1 + \xi^2)\}v = 0.$

Here $\epsilon = ka$ (k is a given parameter and a is the radius of the focal circle) and α is a separation constant whose eigenvalues α_{lm} are such that a solution u_{lm} belonging to α_{lm} is finite at $\eta = \pm 1$.

After a summary of the methods for representing the eigenvalues α_{lm} and the functions $u_{lm}(\eta)$ and $v_{lm}(\xi)$, expansion formulas for spherical and plane waves in terms of the spherical wave functions are given. The numerical results consist of tables and curves [p. 314–320] and contain for $\epsilon = 1(1)5$ the following quantities:

The eigenvalues α_{lm} for $m = 1$, $l = 4(1)8$; $m = 2$, $l = 2(1)9$; $m = 3$, $l = 4(1)10$; $m = 4$, $l = 4(1)11$.
The norm N_{lm} for $m = 0$, $l = 0(1)5$; $m = 1$, $l = 1(1)6$.
The norm N_{lm} is defined as $\int_{-1}^{1} u_{lm}u_{l'm}d\eta = \delta_{ll'}N_{lm}$.
q_{lm} for $m = 0$, $l = 0(1)5$; $m = 1$, $l = 1(1)6$
Q_{lm} for $m = 0$, $l = 0(1)5$.
The q_{lm} and Q_{lm} are constants associated with the radial functions.
Finally *the angular function* $u_{lm}(\eta)$ for $m = 0$, $l = 0(1)5$; $m = 1$, $l = 1(1)6$; $m = 2$, $l = 2$ and 3, and for values of the argument $\eta = 0(0.1)1$.

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875[L].—BRIGITTE RADON, "Sviluppi in serie degli integrali ellittici," Accad. Naz. Lincei, *Atti, Mem., Cl. Sci. Fis. Mat. Nat.*, s. 8, v. 2, 1950, p. 69–109.

The usual notations of elliptic integrals,

$$F(\varphi, k) = \int_0^{\varphi} \Delta^{-1}(\psi, k)d\psi, \qquad E(\varphi, k) = \int_0^{\varphi} \Delta(\psi, k)d\psi,$$

$$\pi(\varphi, \rho, k) = \int_0^{\varphi} [(1 + \rho \sin^2 \psi)\Delta(\psi, k)]^{-1}d\psi$$

are used, where

$$0 \leq \varphi \leq \tfrac{1}{2}\pi, \qquad 0 \leq k^2 < 1, \qquad \text{and} \qquad \Delta(\psi, k) = (1 - k^2 \sin^2 \psi)^{\frac{1}{2}}.$$

The author gives the expansion of both complete ($\varphi = \frac{1}{2}\pi$) and incomplete (φ general) integrals in powers of k^2, also in powers of $k'^2 = 1 - k^2$ (in the latter case there are also logarithmic terms). In addition, for the incomplete integrals there are expansions in powers of $\Delta(\varphi, k) - \cos \varphi$, in powers of $1 - \Delta(\varphi, k)$, and for F and E also as a trigonometric series. For the integrals of the third kind expansions in powers of $k^2 + \rho$ are also provided.

In each case the first few coefficients are given explicitly, for the others either a general form or a recurrence relation. The region of convergence is noted, as is the region of "practical convergence" where the series converges at least as well as an infinite geometrical progression with ratio $\frac{1}{2}$.

A. E.

876[L].—WASAO SHIBAGAKI, *Table of the Modified Bessel Functions*, Kyushu University, Physics Department, April 1946, 132 p. 25.4 × 18.3 cm.

This table tabulates the modified Bessel functions $I_n(x)$, $K_n(x)$ or $\sqrt{x}\, I_n(x)$, $\sqrt{x}\, K_n(x)$.

The functions $I_0(x), I_1(x), \cdots, I_{21}(x)$ are tabulated for $x = 0(.01)1(.02)5$; $I_{22}(x)$ for $x = 1(.02)5$; the functions $x^{\frac{1}{2}}I_n(x)$ are tabulated for $x = 5(.04)25$, $n = 0(1)22$.

The functions $K_n(x)$ are tabulated for $x = 0(.01)1(.02)5$, $n = 0(1)23$; the functions $x^{\frac{1}{2}}K_n(x)$ are tabulated for $x = 5(.04)25$, $n = 0(1)23$.

All functions are given to mostly five significant figures, and to facilitate interpolation the first differences, divided by 100 h, where h denotes the tabular interval in x, are tabulated alongside the functions. The purpose is to avoid the computation of $p = \Delta x/h$ in the interpolation.

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877[L].—TOKYO NUMERICAL COMPUTATION BUREAU, *Report no.* 3. *Tables of Spherical Wave Functions*, 1950. VI. 1. (Keisuh-Renkyukjo) 466, Kyohdoh, Setagaya, Tokyo, Japan. 1950, iv, 77 p. 25.4 × 36.2 cm.

The text (p. i–iv) of this publication is in Japanese although the title page is in English. Various tables occupy p. 2–77. The tables are primarily concerned with the Riccati Bessel functions arising in connection with the wave equation, as follows:

$$F_n(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_{n+\frac{1}{2}}(x) = x^{n+1}f_n(x)/(2n+1)!!$$
$$G_n(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_{-n-\frac{1}{2}}(x) = x^{-n}g_n(x)(2n-1)!!$$
$$F_n'(x) = (n+1)x^n\bar{f}_n(x)/(2n+1)!!$$
$$G_n'(x) = nx^{-n-1}\bar{g}_n(x)(2n-1)!!$$

where $(2n+1)!! = 1\cdot3\cdot5\cdots(2n+1)$.

The first tables are of F_n, G_n, F_n', G_n' for $n = 0(1)6$, $x = [0(.05)10; 6D]$, δ^2. In limited ranges where F_n, F_n' have small significant numbers and G_n, G_n' are too irregular to be interpolated, 6D values of ϕ_n, $\bar{\phi}_n$, ψ_n, $\bar{\psi}_n$ are also given, where

$$\phi_n = \log f_n(x) - \log (2n+1)!!$$
$$\bar{\phi}_n = \log \bar{f}_n(x) + \log [(n+1)/(2n+1)!!]$$
$$\psi_n = \log g_n(x) + \log (2n-1)!!$$
$$\bar{\psi}_n = \log \bar{g}_n(x) + \log [n(2n-1)!!].$$

Recurrence formulas may be used for $n \geqslant 7$. Earlier tables of this kind, by DOODSON and others [*MTAC*, v. 1, p. 233], were at interval never less than .1.

The above mentioned tables occupy 53 pages. Then come 6D tables of values of C_n, θ_n, $\bar{C}_n$, $\bar{\theta}_n$, P_n, Q_n, R_n, S_n where

$$\begin{aligned} G_n + iF_n &= C_n \exp\left(i\{x - \tfrac{1}{2}\pi n + \theta_n\}\right) \\ F_n' - iG_n' &= \bar{C}_n \exp\left(i\{x - \tfrac{1}{2}\pi n + \bar{\theta}_n\}\right) \\ F_n - iG_n &= (-Q_n + iP_n) \exp\left(i\{x - \tfrac{1}{2}\pi n\}\right) \\ G_n' + iF_n' &= (R_n - iS_n) \exp\left(i\{x - \tfrac{1}{2}\pi n\}\right). \end{aligned}$$

These functions are given for $y = 1/x$ in the range 0 to .1 at varying intervals .01, .005, .002, and δ^2.

Finally, there are 6D tables of f_n, $\bar{f}_n$, g_n, $\bar{g}_n$ for $n = 1(1)6$, and $x = [0(.05)m, 6D]$, δ^2, m varying from 2 to 5.35.

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MATHEMATICAL TABLES—ERRATA

In this issue references have been made to Errata in RMT **865** (GRUBBS).

181.—R. L. ANDERSON & E. E. HOUSEMAN, *Tables of Orthogonal Polynomial Values Extended to N* = 104 [*MTAC*, v. 1, p. 148–150].

A recalculation of this table reveals the following complete list of errata

Page	Line	n	*For*	*Read*
615	14	31	broken type	−585
618	22	39	496388	4496388
	14	40	−3583	−2583
625	2	57	+42481	+32481
629	17	61	−2648	+2648
642	38	74	13505	135050
649	14	81	+701925	+701935
653	40	85	−88686	+88686
663	44	95	+2107	−2107
665	45	97	−110308	+110308
669	24	101	−26593	−26592

The last of these was reported by W. F. BROWN JR. in *MTAC*, v. 4, p. 222. The values given for $n = 72$ on p. 640, line 36 are really for $n = 71$. The correct values are: 124392, 966 52584, 15878 63880, 39 89066 92520, 3436 29622 27080.

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182.—N. G. W. H. BEEGER, "On the congruence $(p - 1)! \equiv -1 \pmod{p^2}$," *Messenger Math.*, v. 51, p. 149–150, 1922.

On p. 150, $p = 239$ *for* $w = 147$ *read* 107.

This error was discovered as the result of recalculation and extension of Beeger's table of Wilson's quotient to $p < 1000$ by use of the SEAC.

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