Now, as the means of M_{ij} and G_{ij} are the same, the inequality (1) is equivalent to $E[M_{ij}^2] \leq E[G_{ij}^2]$, i.e., by formulas (4) and (5) to

$$\frac{1 + \lambda_{jj}}{1 - \lambda_{jj}} \leq \frac{1}{p_j}.$$

Application of formula (3) transforms this into $p_j \leq \nu_j/(2 - \nu_j)$, which proves our statement.

The practical value of this theorem is limited, because ν_j is, in general, not known. But it shows that, at least in this special case, the answer to the question which method is preferable for the calculation of the element β_{ij} does not depend on the subscript i. It also confirms the intuitively plausible conjecture that M_{ij} is the better random variable to use whenever p_j is comparatively small.

W. R. WASOW

NBSINA

[1] G. E. FORSYTHE & R. A. LEIBLER, "Matrix inversion by a Monte Carlo method," *MTAC*, v. 4, 1950, p. 127–129.
[2] J. H. CURTISS, "Sampling methods applied to differential and difference equations," Seminar on Scientific Computation, *Proc.*, Nov. 1949, p. 87–109. IBM, New York.
[3] W. WASOW, "Random walks and the eigenvalues of elliptic difference equations," NBS, *Jn. Research*, v. 46, 1951, p. 65–73.

RECENT MATHEMATICAL TABLES

961[B, E].—R. C. SPENCER & G. E. REYNOLDS, *A Table of Normalized Parabolic Coordinates and Arc Lengths.* Air Force Cambridge Research Laboratories, publication E 4083, Cambridge 1951, 9 p., 21.9 × 27.3 cm.

The functions

$$y = \tfrac{1}{2}x^2, \quad s = \tfrac{1}{2}(\operatorname{arc\,sinh} x + x(1 + x^2)^{\frac{1}{2}})$$

are given to 5D for $x = 0(.01)2$. These give the ordinates and arc distances of points on the "normalized" parabola $y = \frac{1}{2}x^2$. From these tables two graphs are derived showing s as a function of x and s, x as functions of y. From the latter graph the excess of arc length over abscissa may be read off to facilitate the laying out of a parabolic antenna.

D. H. L.

962[F].—J. P. KULIK, L. POLETTI & R. J. PORTER, *Liste des Nombres Premiers du Onzième Million (plus précisément de* 10006741 *à* 10999997). [Amsterdam 1951, published by N. G. W. H. Beeger, Nicolaas Witsenkade 10, Amsterdam]. ii + 25 p., 20.3 × 27.6 cm. Price 3 Dutch florins.

This table is the result of collating three of the four existing manuscript factor tables of the eleventh million.[1] The arrangement is essentially that of LEHMER's[2] list of primes of which this table is a natural extension. Thus the rank of a prime occupying page P, column C, and line L is given by

$$2500P + 100C + L + 662400.$$

The number of primes in this million is 61938.

Much of the credit for the successful completion of this table goes to BEEGER and GLODEN who were responsible for collating and reconciling the manuscripts and for the preparation of the final clear handwritten manuscript from which the table was offset.

D. H. L.

[1] See *MTAC*, v. 4, p. 121, for a complete description of this project.
[2] D. N. LEHMER, *List of Prime Numbers from* 1 *to* 10006721. Washington 1914.

963[H].—B. M. SHUMĬAGSKIĬ, *Tablit͡sy dli͡a resheni͡ia kubicheskikh uravneniĭ metodom osnov* [Tables for the solution of cubic equations by the method of bases]. Moscow and Leningrad, 1950, 135 p., 12.9 × 20.1 cm. 3.90 rubles.

In a previous paper[1] the author discussed properties of the solutions of

$$z^N + Az - A = 0, \tag{1}$$

and their application to the solution of the more general trinomial equation

$$y^m + py^n + q = 0. \tag{2}$$

In particular when $m = 3$ and $n = 1$, then $N = 3$ and (2) is transformed into (1) by letting

$$y = -(q/p)z \quad \text{and} \quad A = p^3/q^2. \tag{3}$$

The resulting cubic equation (1) has three real roots $z_1 < z_2 < z_3$ when $A < -6.75$; as A decreases to $-\infty$, these vary monotonically over the ranges $z_1 = (-3, -\infty)$, $z_2 = (1.5, 1)$, $z_3 = (1.5, \infty)$. When $A > -6.75$, (1) has one real root, which is the continuation of z_1, and two complex roots

$$z_{2,3} = -(z_1/2) \pm iz_1\alpha; \tag{4}$$

z_1 increases from -3 to 1 and α from 0 to ∞ as A becomes infinite.

The booklet under review has a dozen pages of explanation and examples followed by four tables which contain for the most part values of A (and α in the complex case) for equally spaced z_1 (and z_2 when real). There are about ten pages with values of z_1 (or z_2) and α (when the roots are complex) for equally spaced A. Non-trivial Δ are given for all functions.

Table I contains A and α for $z_1 = .0020(.0001).010(.001).969$ and z_1 and α for $A = 30(1)300$. Table II gives A and α for $z_1 = -.0020(-.0001)-.010(-.001)-3$. Table IIIa has A for $z_1 = -3(-.001)-7$ and z_1 for $A = -40(-1)-30$. Table IIIb shows A for $z_2 = 1.5(-.001)1.030$ and z_2 for $A = -37(-1)-300$. For $|A| > 300$ the following formulas are suggested without comment by the author (p. 9):

$$A > 300, \quad z_1 = (A+2)/(A+3), \quad z_{2,3} = -(z_1/2) \pm \sqrt{-A - 3(z_1/2)^2},$$

$$A < -300, \quad z_2 = (A+2)/(A+3), \quad z_{3,1} = -(z_2/2) \pm \sqrt{-A - 3(z_2/2)^2}.$$

The bulk of the values of A, z_1, and z_2 are computed to 5S while most of those of α are given to 4S. On p. 6 the author writes, "The tables give accuracy to four decimal places for real roots and three decimal places for complex roots. The 4th and 5th places for the first case and the 4th place for the second case can be obtained by linear interpolation. In most cases interpolation will give 5 correct places of a real root. Pages where linear interpolation gives only four places (but not five) of a real root have an asterisk in the upper left corner."

NOGRADY[2] has tabulated to 6D the functions $A = z_1^3/(1 - z_1)$ for $z_1 = -3(.001)1$ and $A = z_2^3/(1 - z_2)$ for $z_1 = 1(.001)1.5$. [Actually he uses the variable $Z = -z_1$ or $-z_2$.] Consequently, his values of A are computed to more decimal places than those on p. 19–26 ($.400 < z_1 < .971$), p. 45–70 ($-1 > z_1 > -3$), and p. 126–131 ($1.500 > z_2 > 1.030$) of the present volume.

If equation (2) (with $m = 3$, $n = 1$) is to be transformed into a one-parameter equation, the substitutions (3) leading to the form (1) (with $N = 3$) require only rational calculations with p and q in contradistinction to the irrational computations needed to put (2) into the more popular forms

$$|z^3 \pm z| = a \quad \text{and} \quad z^3 + 2 = 3pz.$$

ZAVROTSKY[3] has tabulated explicitly both real and complex solutions of (2) to 5D for $p = -100(1)100$, $q = 0(1)100$.

R. R. REYNOLDS

NBSINA

[1] B. M. SHUMĨAGSKIĬ, "Ischislenie shoĭrosh," *Mat. Sbornik*, v. 40, p. 394–409, 1933.
[2] H. A. NOGRADY, *A New Method for the Solution of Cubic Equations*. Ann Arbor, 1936 [*MTAC*, v. 1, p. 441–442].
[3] A. ZAVROTSKY, *Tablas para la Resolución de las Ecuaciones Cúbicas*. Caracas, 1945 [*MTAC*, v. 2, p. 28–29].

964[K].—NILS BLOMQVIST, "Some tests based on dichotomization," *Annals Math. Stat.*, v. 22, 1951, p. 362–371.

Consider a random sample of n vectors for an m-dimensional population with unknown distribution. The only allowable values for a component of a vector are 0 and 1. Thus the vectors can be represented in the form $(x_{i1}, x_{i2}, \cdots, x_{im})$ $(i = 1, \cdots, n)$, where each x_{ij} is either 0 or 1. Let

$$n_j = \sum_{i=1}^{n} x_{ij}, \quad p_j = n/n_j, \qquad (j = 1, \cdots, m)$$

$$y_i = \sum_{j=1}^{m} x_{ij}, \qquad (i = 1, \cdots, n).$$

The y_i are considered to be random variables but the n_j (consequently, also the p_j) are considered to be fixed. Let

$$P = \sum_{j=1}^{m} p_j, \quad S = \sum_{i=1}^{n} (y_i - P)^2,$$

while all the x_{ij} are considered to be statistically independent. Also let $(u, v) \equiv (n = u, m = v)$. Table I presents values of $P(S \geqq S_0)$ for: (4, 3), $S_0 = 1(2)9$; (4, 4), $S_0 = 0(2)10$, 16; (4, 5), $S_0 = 1(2)13$, 17, 25; (4, 6), $S_0 = 0(2)20$, 26, 36; (4, 7), $S_0 = 1(2)13$, 17, (2)21, 25(2)29, 37; (4, 8), $S_0 = 0(2)26$, 30(2)40, 50; (6, 3), $S_0 = 1.5(2)9.5$, 13.5; (6, 4), $S_0 = 0(2)18$, 24; (6, 5), $S_0 = 1.5(2)25.5$, 29.5; (8, 3), $S_0 = 0(2)14$, 18; (8, 4), $S_0 = 0(2)26$; (10, 4), $S_0 = 0(2)30$; (10, 3), $S_0 = 2.5(2)18.5$; (12, 3), $S_0 = 3(2)23$; (14, 3), $S_0 = 3.5(2)25.5$; (16, 3), $S_0 = 0(2)28$. Values given have either 3D or 4D. Number of significant figures $\leqq 3$ (except for 1.000); 4D used when number of significant figures $\leqq 2$.

J. E. WALSH

U. S. Naval Ordnance Test Station
China Lake, California

965[K, P].—E. BROCKMEYER, *Sandsynlighedsregningens Anvendelse i Telefonien* [Applications of the Calculus of Probability to Telephony]. Copenhagen, 1949, Copenhagen Telephone Company. 54 p., 16.8 × 25.0 cm.

This work contains a table of solutions y to 2D of the equation

$$B \sum_{k=0}^{n} y^k/k! = y^n/n! \tag{1}$$

for $B = .001(.001).005$, $.01$, $.02$, $.05$ and for $n = 1(1)260$. The table is intended to indicate the traffic intensity y which n circuits will handle with the probability B of loss of a call. The relation (1) is known as Erlang's formula [*MTAC*, v. 3, p. 98–99]. This table appears also in RMT **966.**

D. H. L.

966[K, P].—E. BROCKMEYER, H. L. HALSTRØM & ARNE JENSEN, "The life and works of A. K. Erlang," Akad. Tekn. Videns., Copenhagen, *Trans.*, 1948, no. 2, 277 p.

There are small tables related to the Poisson distribution included in this work. Among those of some general use are

(a) 6D values of $m^x e^{-m}/x!$ for $x = 0(1)14$ and $m = -2(.1)0$. This table is on p. 137.
(b) 6D values of $(-m)^x e^m/x!$ for $x = 0(1)20$, $m = 2.1(.1)4$ on p. 195–196.
(c) 6D values of the real and imaginary parts of the solutions of
$$z \exp(z) = \alpha \exp(-\alpha) \exp(2\pi ik/40)$$
for $\alpha = 0(.1)1$, $k = 0(1)20$ (p. 197–198).
(d) BROCKMEYER's table, described in RMT **965.**

D. H. L.

967[K].—M. B. COOK, "Bi-variate k-statistics and cumulants of their joint sampling distribution," *Biometrika*, v. 38, 1951, p. 179–195.

This paper has for its purpose the derivation of "(a) formulae giving bivariate population moment-coefficients in terms of cumulants, and cumulants in terms of moment-coefficients up to the sixth order; (b) formulae for the cumulants of the joint distribution of the simpler bivariate k-statistics." In pursuance of (a) the author indicates three methods of obtaining the results but he actually uses Kendall's operational method[1] to put on record the bivariate moments about zero and about the means through order 6 in terms of cumulants and conversely the bivariate cumulants through order 6 in terms of moments about the means. In the second part of the paper he turns to the calculation of the cumulants of the joint distribution of bivariate k-statistics.[2] This is carried out by an extension of the operator method used in the first part. He lists results through weight 9. The results, being largely new, were checked by calculation in two different ways.

J. J. LIVERS

The Boeing Aircraft Company
Seattle, Wash.

[1] M. G. KENDALL, "The derivation of multivariate sampling formulae from univariate formulae by symbolic operation," *Annals Eugenics*, 10, 1940, p. 392–402.
[2] R. A. FISHER, "Moments and product moments of sampling distributions," London Math. Soc., *Proc.*, s. 2, v. 30, 1929, p. 199–238.

968[K].—S. T. DAVID, M. G. KENDALL & A. STUART, "Some questions of distribution in the theory of rank correlation," *Biometrika*, v. 38, 1951, p. 131–140.

Let $(X_1, X_2, \cdots, X_n)$ be a random ordering of the integers $(1, 2, \cdots, n)$, and let $S(d^2) = \sum_{i=1}^{n} (X_i - i)^2$. The authors give (Tables 1, 2) the exact probability distribution of $S(d^2)$ for $n = 9, 10$. [The distribution for $n = 1(1)8$ has been published earlier, e.g. by KENDALL.[1]] These tables permit exact tests of the significance of the Spearman rank correlation coefficient for samples of size n. By obtaining algebraic expressions for the moments and cumulants through order 8, the authors provide approximate distributions for $S(d^2)$ which are sufficiently accurate for practical purposes when $n = 10$.

J. L. HODGES, JR.

University of Chicago
Chicago, Ill.

[1] M. G. KENDALL, *The Advanced Theory of Statistics*. v. 1. London, 1948, p. 396.

969[K].—(a) J. DURBIN & G. S. WATSON, "Testing for serial correlation in least squares regression. II," *Biometrika*, v. 38, 1951, p. 159–178. (b) G. S. WATSON & J. DURBIN, "Exact tests of serial correlation using noncircular statistics," *Annals Math. Stat.*, v. 22, 1951, p. 446–451.

Tables in (a) facilitate the test for serial correlation of error terms, ϵ, of a regression model $y = \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \epsilon$ where y and the x_i are observed and the x_i regarded as "fixed variables." Let z denote the residual from regression and Δz denote the successive first differences of z. The statistic $d = \sum(\Delta z)^2/\sum z^2$ is used for the test. Define $k' = k - 1$ and set $x_k = 1$. Upper and lower bounds (d_U and d_L) are tabulated for d_α where $P(d > d_\alpha) = \alpha$ for $\alpha = .05, .025, .01$ for sample size $n = 15(1)40(5)100$, for $k' = 1(1)5$ and assuming normality of residuals. The statistic d is always between 0 and 4 and tabled values are given to 2D. The use of these tables causes the test to be inconclusive if $d_L < d < d_U$ and approximate procedures are recommended for this case. Examples of the use of the above tables are given for one independent and two dependent observations, two-way classification and polynomial regression.

Two small tables appear in (b). The first is 95th percentile values for the distribution of c_1 where

$$\sum_{i=1}^{n} x_i^2 c_1 = \sum_{i=2}^{n} x_i x_{i-1} - x_m x_{m+1}$$

for $n = 2m$ and where $x_{m+1} \equiv 0$ for $n = 2m + 1$ (effectively making $n = 2m$) given to 3D for $n = 10(2)22$. The second table gives 95th percentile values for the distribution of d_3 where

$$\sum_{1}^{n} (x_i - \bar{x})^2 d_3 = \sum_{i=2}^{n} (x_i - x_{i-1})^2 - (x_{m+1} - x_m)^2$$

for $n = 2m$ and samples of $2m + 1$ are reduced to samples of size $2m$ by omitting the middle observation. Values to 3S are given for $n = 12(2)30$.

Much of this table appears in Table 7 in (a) which also considers upper and lower bounds for the 95th percentile of a comparable statistic for least squares regression for $k' = 1(1)5$ and $n = 13(2)23$.

W. J. DIXON

University of Oregon
Eugene, Ore.

970[K].—A. K. GAYEN, "The frequency distribution of the product-moment correlation coefficient in random samples of any size drawn from non-normal universes," *Biometrika*, v. 38, 1951, p. 219–247.

The author's summary includes the following description of part of the content of the paper and of Table 1 of the paper. "The mathematical form of the distribution of r in non-normal samples is obtained, the parent population being specified by the bivariate Edgeworth surface including terms as far as those in $\lambda_{30}^2, \lambda_{30}\lambda_{21}, \cdots, \lambda_{03}\lambda_{12}$ and $\lambda_{03}^2 \cdots$. The tables of the ordinates of the corrective functions due to fourth-order and the square and product of third-order semi-invariants . . . are obtained for $N = 11, \rho = 0.0$, and 0.8, and $N = 21, \rho = 0.8$."

These corrective functions provide multipliers for certain functions of the semi-invariants λ_{ij}. In terms of formulas, Table 1 gives values to 5D of

$$\psi(r, \rho) = \frac{N-2}{\pi}(1-\rho^2)^{\frac{1}{2}(N-1)}(1-r^2)^{\frac{1}{2}(N-4)}\int_0^\infty \frac{dt}{(\cosh t - \rho r)^{N-1}},$$

$$\psi_{4,i}(r, \rho) = \frac{N-1}{8N(N+1)}\frac{\partial^i}{\partial \rho^i}\psi(r, \rho) \quad \text{for} \quad i = 1, 2, \text{and}$$

$$\psi_{6,i}(r, \rho) = \frac{N-2}{12N(N+1)(N+3)}\frac{\partial^i}{\partial \rho^i}\psi(r, \rho) \quad \text{for} \quad i = 1, 2, 3.$$

$\psi(r, \rho)$ is the ordinate of the distribution of the correlation coefficient in samples from a bivariate normal population and its values are taken from F. N. David's table.[1] The derivatives of ψ with respect to ρ are expressible simply in terms of ψ's. Values are given for $N = 11$, $\rho = 0.6$, $r = -1.00\,(.05)\,1.00$; $N = 11$, $\rho = 0.8$, $r = -.75\,(.05)\,.60\,(.025)\,1.000$; and $N = 21$, $\rho = 0.8$, $r = -.25\,(.05)\,.60\,(.025)\,1.000$. In the last two cases the entries are zero to 5D for smaller values of r than those given.

The author finds the semi-invariants of several bivariant samples which exist in the literature and discusses, in part with the aid of tables and graphs, the effect on the distribution of r of taking into account semi-invariants of order greater than two. The effects of non-normality in a bivariate population of the distribution of $z = \frac{1}{2}\log_e[(1+r)(1-r)^{-1}]$ is also discussed.

K. J. ARNOLD

University of Wisconsin
Madison, Wis.

[1] F. N. DAVID, *Tables of the ordinates and probability integral of the distribution of the correlation coefficient in small samples*. London, 1938.

971[K].—D. G. C. GRONOW, "Test for the significance of the difference between means in two normal populations having unequal variances," *Biometrika*, v. 38, 1951, p. 252–256.

The significance of the difference is to be tested by one of the two criteria

$$u = (\bar{x}_1 - \bar{x}_2)\left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\left(\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}\right)\right]^{-\frac{1}{2}}$$

or

$$v = (\bar{x}_1 - \bar{x}_2)\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^{-\frac{1}{2}}$$

where $s_t^2 = \sum_{i=1}^{n_t} (x_{ti} - \bar{x}_t)^2/(n_t - 1)$, $t = 1, 2$. Only two-tailed tests are considered. One sample of n_1 observations is taken from a normal population with mean $\xi_1 = \kappa_1$ and variance κ_2 and one sample of n_2 observations is taken from a normal population with mean $\xi_2 = \kappa_1'$ and variance κ_2'. The tables in the paper give probabilities that each of u and v falls beyond the 5% level and that each falls beyond the 1% level of t (two-sided) for $n_1 + n_2 - 2$ degrees of freedom, when $n_1 = n_2 = 10$ and when $n_1 = 15$, $n_2 = 5$ for various combinations of $\delta/\sqrt{\kappa_2}$ and κ_2'/κ_2. $\delta = \xi_1 - \xi_2$. Probabilities are given for $n_1 = n_2 = 10$ (in which case $u = v$), $\delta/\sqrt{\kappa_2} = 0(.5)3$, $\kappa_2'/\kappa_2 = 1(.5)3.0$, and for $n_1 = 15$, $n_2 = 5$, $\delta/\sqrt{\kappa_2} = 0(.5)3.0$, $\kappa_2'/\kappa_2 = 1/3$, 1/2, 1, 2, 3. 4D are given for $\delta = 0$, 3D otherwise.

As the author states, "this problem has already been solved in principle by Hsu,[1] but his solution is not in a form which readily allows of numerical calculation." The author has therefore approximated the first four moments of the distribution of each of his statistics for each case considered, fitted "the appropriate Pearson or Gram-Charlier Type A curve," and used this approximation to the distribution of the statistics to obtain the tabled probabilities. Checking by using Hsu's results in a few cases in which they can be simplified, no serious discrepancy was found although it appears likely to the reviewer that the 7 in 0.276 for $n_1 = n_2 = 10$, 1% level, $\delta/\sqrt{\kappa_2} = 1$, $\kappa_2'/\kappa_2 = 1$ is a misprint. When δ/κ_2 appears in a heading or in the text it should be $\delta/\sqrt{\kappa_2}$.

K. J. Arnold

University of Wisconsin
Madison, Wis.

[1] P. L. Hsu, "Contribution to the theory of 'Student's' t-test as applied to the problem of two samples," *Stat. Res. Mem.*, v. 2, 1938, p. 1–24.

972[K].—E. S. Keeping, "A significance test for exponential regression," *Annals Math. Stat.*, v. 22, 1951, p. 180–198.

The author considers the two regression equations

$$(1) \quad Y = be^{px} \qquad \text{and} \qquad (2) \quad Y = a + be^{px}$$

under the assumptions that the values of x are in arithmetic progression and that the standard deviation of the observed y is constant for all x. The null hypothesis being tested is that $be^{px} = 0$ for all x while the alternative hypotheses are specified by $b \neq 0$, $p \neq -\infty$. In Table III the critical values of R, the sample correlation coefficient (uncorrected for means) between the observed values of x and y in the case of (1), are given for samples of 3 (1) 10, 12, 15 at the 5%, 1%, .1% and .01% levels of significance to 3D at least.

In Table IV for the range $n = 4\ (1)\ 12$, and the same significance levels as in Table III, the author provides the critical values of R to 3D at least, for (2), where R in this case is computed in the usual way.

L. A. Aroian

Hughes Aircraft Co.
Culver City, California

973[K].—L. R. Packer, "The distribution of the sum of n rectangular variates I," Inst. Actuaries Students' Soc., *Jn.*, v. 10, 1951, p. 52–61.

The author rederives, using difference methods, the well-known distribution function for the sum of n randomly drawn values from the continuous rectangular universe on the interval (0, 1). The cumulative distribution function, $F(x, n)$ is tabulated to 6D for $n = 1(1)12$ and $x = .5(.5)6$. As an application he gives a table of the exact probabilities that the error in the sum of 12 numbers, each rounded off to the nearest unit, will lie in the interval $(-x, x)$ for $x = .5(.5)6$.

C. C. C.

974[K].—E. S. Pearson & Maxine Merrington, "Tables of the 5% and 0.5% points of Pearson curves (with argument β_1 and β_2) expressed in standard measure," *Biometrika*, v. 38, 1951, p. 4–10.

As an alternative to using the normal distribution as an approximation to the exact distribution of a statistic, the authors propose that one might use the distribution of the Pearson system corresponding to the moment ratios,

$$\beta_1 = \mu_3{}^2\mu_2{}^{-3}, \qquad \beta_2 = \mu_4\mu_2{}^{-2}$$

of the statistic. This proposal could be adopted if one can compute the first four sampling moments of the distribution, and if tables of areas for the corresponding Pearson curve are available.

As a start towards the satisfying of the latter requirement, the authors have tabulated the values of

$$u = (x_\alpha - \mu)\sigma^{-1} \quad \text{where} \quad \int_{b_1}^{x_\alpha} f(x)\,dx = \alpha$$

for $\alpha = 0.005, 0.05, 0.95$, and 0.995, in which μ, σ, β_1 and β_2 are the moment ratios of the statistic whose distribution one is interested in, and $f(x)$ is the solution of the corresponding Pearson differential equation,

$$y^{-1}\,dy/dx = -\,(x + c_1)(c_0 + c_1x + c_2x^2)^{-1}.$$

The values of $(x - \mu)\sigma^{-1}$ are tabulated to 2D for $\beta_1 = 0, .01(.02).05(.05).2$ $(.1)1$, and for $\beta_2 = 1.8(.2)5$.

By using these tables one can approximate the 5% and 0.5% critical points of a statistic's distribution. Tables which will yield additional critical points are in process of being computed. The appropriateness of using the Pearson system for such approximations is briefly discussed but no criterion which would help one determine when he should use this method is given.

C. F. Kossack

Purdue University
Lafayette, Ind.

975[K].—K. C. S. Pillai, "On the distribution of an analogue of Student's *t*," *Annals Math. Stat.*, v. 22, 1951, p. 469–472.

Let $\bar{x}$ be the mean and w the sample range in a sample of n from a normal population with mean a and variance unity; further define $G = \dfrac{\bar{x} - a}{w}$. The author lists in Table I for $n = 3\ (1)\ 8\ (2)\ 14\ (3)\ 20$, (1) the confidence interval for a based on the G test, (2) the confidence interval for a based on Student's t test, and (3) the efficiency of the G test, defined as the ratio, (1)/(2), all to 4D. For this range the efficiency of the G test is always greater than 96%. The levels of significance, α, are .1, .05, .01 and .001 for $n = 3$; .1, .05, and .01 for $n = 4$ and 5; and .05 and .01 for the other values of n.

L. A. Aroian

Hughes Aircraft Co.
Culver City, California

976[K].—M. S. Raff, "The distribution of blocks in an uncongested stream of automobile traffic," Am. Stat. Assn., *Jn.*, v. 46, 1951, p. 114–123.

Cars passing through a fixed location on a highway are observed; the assumption is made that the number k of car passages during a time interval of length t has the Poisson distribution $e^{-Nt}(Nt)^k(k!)^{-1}$. A car arriving at this location on a side road is assumed not to enter the highway until there is a "gap" of duration $\geqq L$ between consecutive passages of cars on the highway. Intervals of time containing no gap of duration $\geqq L$ are called "blocks" of traffic, while each gap of duration $\geqq L$ is called an "antiblock." Various probability distributions relating instants of arrival of cars on the side road (assumed to be uniformly distributed), blocks, antiblocks and their durations are derived. The main result of the paper is the probability distribution of the "waiting time" of a car arriving on the side road, from the moment of arrival to the earliest antiblock. The cumulative distribution function of the waiting time is

$$F(t) = e^{-NL} \sum_{r=0}^{h} (-e^{-NL})^r \{(r!)^{-1}[N(t - rL)]^r + (r + 1)^{-1}[N(t - rL)^{r+1}]\}$$

where h is the greatest integer contained in t/L. This distribution depends on two parameters NL and t/L. A table of $F(t)$ is presented for $NL = 0(.25)2(.5)6$, $t/L = 0(.5)3(1)7(2)11$ and selected larger values until $F(t)$ reaches .99. Properties of $F(t)$ are discussed. An empirical set of data is compared with a theoretical distribution and, on inspection, shows good agreement.

Z. W. Birnbaum

University of Washington & Stanford University
Seattle, Washington & Palo Alto, California

977[K].—P. R. Rider, "The distribution of the range in samples from a discrete rectangular population," Amer. Stat. Assn., *Jn.*, v. 46, 1951, p. 375–378.

The author considers the infinite statistical universe in which the integers 0, 1, 2, $\cdots$, $N - 1$, occur with equal frequency and derives the probability that in a random sample of n from it the range will be R. The values of this probability are tabulated to 4D for $N = 10, n = 2(1)10$, and $R = 0(1)9$. For comparison the like table is given for samples of n from the continuous rectangular distribution on the interval (0, 10).

C. C. C.

978[K].—S. S. SHRIKHANDE, "Designs for two-way elimination of heterogeneity," *Annals Math. Stat.*, v. 22, 1951, p. 235–247.

Designs are presented using ten or less replications for the following numbers of treatments and treatments per block.

Treatments per block	*Number of treatments*
3	6, 10, 13, 15*, 25
4	8, 10, 15
5	10, 24
6	12, 16
7	29, 48
8	17, 63
9	25, 26, 80
10	31

Methods are also presented to construct designs for s^2 treatments in blocks of size s.

R. L. ANDERSON

North Carolina State College
Raleigh, North Carolina

* Using either 5, 6 or 7 replications.

979[K].—C. H. SPRINGER, "A method for determining the most economic position of a process mean," *Industrial Quality Control*, v. 8, 1951, p. 36–39.

The discussion and tables given here deal with the problem of proper location of the process mean for the case in which the loss due to producing a piece above the upper specification is not equal to the loss of producing a piece below the lower specification.

Let X be a random variable (quality measure) with (process) mean $\bar{X}'$ and (process) standard deviation σ'. Let t be the transformed random variable with mean zero and standard deviation one. Let U be the upper and L the lower specification, C_U the cost of rejecting a piece $X > U$ and C_L the cost of rejecting a piece $X < L$. We have for $C(\bar{X}')$ the total expected unit cost of rejected product,

$$C(\bar{X}') = C_L \int_{-\infty}^{\frac{L-\bar{X}'}{\sigma'}} f(t)\,dt + C_U \int_{\frac{U-\bar{X}'}{\sigma'}}^{\infty} f(t)\,dt.$$

The first condition for $C(\bar{X}')$ a minimum yields

(1) $$C_L/C_U = f\left(\frac{U-\bar{X}'}{\sigma'}\right) \Big/ f\left(\frac{L-\bar{X}'}{\sigma'}\right)$$

which, together with the second minimizing condition, gives conditions on the density function $f(t)$ which are satisfied by (among others) the Gaussian and Pearson Type III considered by the author. For the former (1) reduces to

$$\bar{X}' = L + W\sigma', \text{ where } W = \tfrac{1}{2}\left(\frac{U-L}{\sigma'}\right) + \left(\frac{\sigma'}{U-L}\right) \ln C_L/C_U,\ C_L \leq C_U,$$

$$\bar{X}' = U - W'\sigma', \text{ where } W' = \tfrac{1}{2}\left(\frac{U-L}{\sigma'}\right) - \left(\frac{\sigma'}{U-L}\right) \ln C_L/C_U,\ C_L \geq C_U,$$

W and W' being equal for reciprocal cost ratios. Springer tables W for fifteen selected values of C_L/C_U (eight values, including unity, and their reciprocals) in the range .005 to 200, and for $\dfrac{U-L}{\sigma'} = 2.5(.5)6.0$.

Similar tables are given for the Pearson Type III function. Letting

$$K = \frac{1}{n}\frac{\sum_{i=1}^{n}(X_i - \bar{X}')^3}{(\sigma')^3}$$

where n is the sample size, Springer gives tables of W for $K = -0.3$, -0.7, -1.1, 0.3, 0.7, 1.1, for eleven selected values of C_L/C_U (six values, including unity, and their reciprocals) in the range .01 to 100, and for $\dfrac{U-L}{\sigma'} = 2.5(.5)5.5$.

HAROLD A. FREEMAN

Massachusetts Institute of Technology
Cambridge, Mass.

980[K].—D. R. WHITNEY, "A bivariate extension of the U statistic," *Annals Math. Stat.*, v. 22, 1951, p. 274–282.

Let $x_1, \cdots, x_l$; $y_1, \cdots, y_m$; $z_1, \cdots, z_n$ be independent variables having (by hypothesis) the same continuous cumulative distribution. Let U be the number of times a y precedes an x and V be the number of times a z precedes an x when the observations are arranged in order of size. Table I gives the number of sequences of the x, y and z's having specified U and $V(U = 0(1)18,\ V = 0(1)9)$ for the special case $l = 6$, $m = n = 3$. Table 2 gives 1000 times the cumulative distribution of U and V (and also a bivariate normal approximation for this) for the same special case and range of values for U and V.

The author states that tables of the joint distribution of U and V have been prepared for all cases where $l + m + n \leqslant 15$.

FRANK J. MASSEY

University of Oregon
Eugene, Oregon

981[L].—MILTON ABRAMOWITZ, "Table of the integral, $\int_0^x e^{-u^3}\,du$," *Jn. Math. Physics*, v. 30, 1951, p, 162–163.

The integral is tabulated for $x = 0(.1)2.5$ to 8D, with second central differences. For $x \leq 1.5$ the computation was carried out from the power series, for $x > 1.5$, by numerical integration. In both cases 10D were obtained and independent checks were made. Linear interpolation is said to be generally good to 4D.

A. E.

982[L].—I. BLOCH, M. H. HULL, JR., A. A. BROYLES, W. G. BOURICIUS, B. E. FREEMAN, & G. BREIT, "Coulomb functions for reactions of protons and alpha-particles with lighter nuclei," *Rev. of Modern Phys.*, v. 23, 1951, p. 147–182.

In many calculations in theoretical physics, the differential equation

$$\frac{d^2y}{d\rho^2} + \left[1 - \frac{2\eta}{\rho} - \frac{L(L+1)}{\rho^2}\right]y = 0$$

occurs. Its two most important solutions are the "regular Coulomb function," F_L, and the "irregular Coulomb function," G_L, which are determined by their asymptotic behavior for large (positive) ρ,

$$F_L \sim \sin\,(\rho - \tfrac{1}{2}L\pi - \eta \ln 2\rho + \sigma_L),$$
$$G_L \sim \cos\,(\rho - \tfrac{1}{2}L\pi - \eta \ln 2\rho + \sigma_L)$$

where

$$\sigma_L = \arg \Gamma(L + 1 + i\eta).$$

There are also several auxiliary functions. Constants C_L, D_L are defined by $C_0 = [2\pi\eta/(e^{2\pi\eta} - 1)]^{\frac{1}{2}}$, $L(2L+1)C_L = (L^2 + \eta^2)^{\frac{1}{2}}C_{L-1}$, $(2L+1)C_LD_L = 1$; and functions A_L, φ_L, Φ_L, Θ_L, Φ_L^*, Θ_L^* by

$$F_L = A_L \sin \varphi_L = C_L\rho^{L+1}\Phi_L,\ \ F_L' = C_L\rho^L\Phi_L^*,$$
$$G_L = A_L \cos \varphi_L = D_L\rho^{-L}\Theta_L,\ \ G_L' = D_L\rho^{-L-1}\Theta_L^*.$$

The aim of the numerical tables presented here is to provide values, with an accuracy of one percent, required in the problems mentioned in the title.

There are 39 tables, pages 155–182. Tables I to V: F_0 to F_4. Tables VI to X: Φ_0 to Φ_4. Tables XI to XV: auxiliary functions for the computation of Φ_0 to Φ_4 in ranges in which Tables VI to X cannot be interpolated to 1 percent accuracy. Tables XVI to XIX: Φ_0^* to Φ_3^*. Tables XX to XXIV: Φ_0^*/Φ_0 to Φ_4^*/Φ_4. Tables XXV to XXIX: $\Phi_0\Theta_0$ to $\Phi_4\Theta_4$. Tables XXX to XXXIV: φ_0 to φ_4. Tables XXXV to XXXIX: A_0 to A_4.

$\log_{10}\eta = -.8(.1).6$ in the tables, and $0 < \rho \leq 6.2$ at varying intervals. Not all values of $\log_{10}\eta$ and of ρ occur in all tables. The values are given to 4D, and each table is accompanied by a note on accuracy and interpolation.

References to previously available tables are given on page 152. Methods for computation of the functions tabulated here are described in another paper[1] by the same authors, and on pages 153–154 of the present paper three numerical examples illustrate the use of the tables.

A. E.

[1] I. BLOCH, M. H. HULL, JR., A. A. BROYLES, W. G. BOURICIUS, B. E. FREEMAN, & G. BREIT, "Methods of calculation of radial wave functions and new tables of Coulomb functions," *Phys. Rev.*, v. 80, 1950, p. 553–560.

983[L].—R. J. BUEHLER & J. O. HIRSCHFELDER, "Bipolar expansion of coulombic potentials," *Physical Rev.*, v. 83, 1951, p. 628–633.

P_1 and P_2 being two points of three-dimensional space at distance r_{12} from one another, let P_1 be referred to a system (r_1, θ_1, ϕ_1) of spherical polar coordinates with pole O_1, and P_2 to a system (r_2, θ_2, ϕ_2) with pole O_2, it being understood that the polar axis is common to the two systems (and passes through O_1 and O_2): also R is the distance O_1O_2.

In the expansion

$$\frac{1}{r_{12}} = \sum_{n_1, n_2, m} B^m_{n_1, n_2}(r_1, r_2; R) P^m_{n_1}(\cos\theta_1) P^m_{n_2}(\cos\theta_2) \cos m(\phi_2 - \phi_1)$$

the coefficients B can be represented in simple closed forms if either $r_1 + r_2 < R$ or $|r_1 - r_2| > R$. In the remaining case, $|r_1 - r_2| \leq R \leq r_1 + r_2$, the B assume a more complicated form,

$$B^m_{n_1, n_2}(r_1, r_2; R) = (D^m_{n_1, n_2})^{-1} \sum_{i,j=0}^{2(n_1+n_2+1)} A^m_{n_1, n_2}(i, j) r_1^{i-n_1-1} r_2^{j-n_2-1} R^{n_1+n_2-i-j+1}.$$

In table I, the coefficients D and A are given exactly for $n_1, n_2 = 0(1)3$ and for all appropriate values of m, i, j. The results should be useful in the computation of the interaction of two overlapping Coulomb fields of potentials.

A. E.

984[L].—F. R. ERSKINE CROSSLEY, "A hyperelliptic function as a non-linear oscillation," *Jn. Math. Phys.*, v. 30, 1951, p. 214–225.

Tables I–V give, for $\varphi_0 = 10°, 45°, 90°, 135°, 170°$, $\zeta = 10°(10°)90°$, and $e^2 = .10, .25, .5, .75, .90, 1$, values to 4D of ζ in radians, $\sin\zeta$, and of

$$\int_0^\zeta \left[\frac{1 - e^2(1 - 2k^2\sin^2 t)^2}{1 - k^2\sin^2 t}\right]^{\frac{1}{2}} dt, \qquad k = \sin\frac{\varphi_0}{2}.$$

Appended are values, in degrees, minutes, and seconds, of φ defined by

$$\sin\frac{\varphi}{2} = \sin\frac{\varphi_0}{2}\sin\zeta.$$

There are also some graphs connected with the integral.

A. E.

985[L].—W. R. DEAN, "Slow motion of viscous liquid in a semi-infinite channel," Cambridge Phil. Soc. *Proc.*, v. 47, 1951, p. 127–141.

Among the several short tables in this paper one finds 4D tables for $u = .9(-.1)-.9$ of

$$\frac{u(1+u^2)}{1-u^2} Im\left[\frac{1+u-i(1-u)}{1+u+i(1-u)}\right]^n + \frac{\pi}{32} nu^{n-1}(1+u)^4, \qquad n = 1(1)8,$$

$$\frac{1-u^2}{2u(1+u^2)}, \quad \frac{2u(1+u^2)}{1-u^2}\alpha\beta - \frac{\pi(1+u)^4}{16(1-u)} l, \quad \frac{u(1+u^2)}{1-u^2}\beta - \frac{\pi(1+u)^4}{32(1-u)},$$

$$\frac{(1+u)^3}{4(1-u)} - \frac{3(1+u)^4}{2\pi^2(1-u)} l^2 + \frac{16u(1+u^2)}{\pi^3(1-u^2)}(3\alpha^2\beta - \beta^3)$$

where $l = \log(1-u)$, $e^{\alpha+i\beta} = (2i)^{\frac{1}{2}} \dfrac{1-u}{1-iu}$.

A. E.

986[L].—Otto Emersleben, "Numerische Werte des Fehlerintegrals für $\sqrt{n\pi}$," *Zeit. angew. Math. Mech.*, v. 31, 1951, p. 393–394.

There is a table on p. 393 of 15D values of

$$2\pi^{-\frac{1}{2}} \int_0^x \exp(-t^2)\, dt$$

for $x = (\pi n)^{\frac{1}{2}}$, $n = 1(1)11$.

R. C. Archibald

Brown University
Providence, R. I.

987[L].—A. N. Gordon, "The field induced by an oscillating magnetic dipole outside a semi-infinite conductor," *Quart. Jn. Mech. Appl. Math.*, v. 4, 1951, 106–115.

Table I (p. 110) gives exact values of r^{-3} and 4D values of the real and imaginary parts of

$$-2b^3 \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)^2(n+3)}{(n+4)!}(br)^{n-1} \quad \text{and} \quad -2\frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}[I_1(\tfrac{1}{2}br)K_1(\tfrac{1}{2}br)]\right\}$$

for $r = .2(.2)1(.5)2$. Here $b = i^{\frac{1}{2}}$.

A. E.

988[L].—T. S. Kuhn, "A convenient general solution of the confluent hypergeometric equation, analytic and numerical development," *Quart. Appl. Math.*, v. 9, 1951, p. 1–16.

The confluent hypergeometric equation is written in the form

$$\frac{d^2U^{(l,n)}}{dr^2} + \left[-\frac{1}{n^2} + \frac{2}{r} - \frac{l(l+1)}{r^2}\right] U^{(l,n)} = 0,$$

and $z = (8r)^{\frac{1}{2}}$. The author puts

$$U^{(l,n)} = \sum_{k=0}^{\infty} n^{-2k} U_k^{(l)}(z)$$

and gives explicit expressions of $U_k^{(l)}(z)$ and $\frac{1}{2}z \frac{d}{dz} U_k^{(l)}(z)$ for $l = 0, 1$ and $k = 0(1)7$ in terms of Bessel functions of order 0 and 1: the coefficients of the Bessel functions are polynomials in z. Two particular solutions arise if the Bessel function is J or Y, and for these particular solutions the explicit expressions were evaluated, and are tabulated to 4D, for $z = 3.5(.5)7.5$.

A. E.

989[L].—R. O. GUMPRECHT & C. M. SLIEPCEVICH, *Tables of light-scattering functions for spherical particles*. xvi + 574 p., $6.50.
Tables of Riccati Bessel functions for large arguments and orders. xvi + 260 p., $3.50.
Tables of functions of first and second partial derivatives of Legendre polynomials. xii + 310 p., $3.50. University of Michigan Engineering Research Institute, *Special Publications*. Ann Arbor, 1951. 15 × 23 cm.

The diffraction of electromagnetic waves on a sphere has been investigated by several authors, most completely by G. MIE,[1] and the theory is presented in standard text-books.[2] The infinite series occurring in Mie's theory converge the slower the larger the radius of the sphere (in comparison to the wave-length of the incident light-wave). The NBSCL tables[3] provide data for spheres whose radius does not exceed the wave-length of the incident light. From the practical point of view much larger droplets are also of interest, and the present tables supply data for spheres whose circumference is up to 400 times the wave-length. On account of the slow convergence of the series, it is estimated by the authors that the computation of the tables with a standard desk-computer would have taken 25 man-years. Using IBM equipment this period could have been reduced to 3 man-years. Actually, the computation was carried out on the ENIAC, and less than two weeks' time was required once the details of programming had been worked out.

The notation adopted is that of the NBSCL tables, and is explained in greater detail in *MTAC*, v. 3, 1949, p. 484. The relevant parameters are α, m, γ and the following notations are used:

$$\beta = m\alpha, \quad x = \cos\gamma, \quad \pi_n = \frac{dP_n(x)}{dx}, \quad \pi_n' = \frac{d^2P_n(x)}{dx^2},$$

$$S_n(z) = \left(\frac{\pi z}{2}\right)^{\frac{1}{2}} J_{n+\frac{1}{2}}(z), \quad C_n(z) = (-1)^n\left(\frac{\pi z}{2}\right)^{\frac{1}{2}} J_{-n-\frac{1}{2}}(z)$$

where $P_n(x)$ is the Legendre polynomial, J the Bessel function of the first kind, and S_n and C_n are called Riccati-Bessel functions. There are also two complex quantities A_n and P_n which are compounded of Riccati functions and their derivatives, the variables being α and β. [This P_n should not be confused with the Legendre polynomial in the definition of π_n and π_n'.] The quantities of physical interest are the intensity functions

$$i_1 = \left|\sum_{n=1}^{\infty} \{A_n\pi_n + P_n[x\pi_n - (1 - x^2)\pi_n']\}\right|^2,$$

$$i_2 = \left|\sum_{n=1}^{\infty} \{A_n[x\pi_n - (1 - x^2)\pi_n'] + P_n\pi_n\}\right|^2$$

and the scattering function

$$K(m, \alpha) = \frac{2}{\alpha^2} \sum_{n=1}^{\infty} \frac{|A_n|^2 + |P_n|^2}{(2n + 1)/n^2(n + 1)^2}.$$

[In the definition of the intensity functions the square brackets in place of | | are presumably a misprint.]

In the first volume Part I gives values (mostly to 4S) of the intensity functions for $\gamma = 90°$, and of the scattering function, for $m = 1.2, 1.33, 1.4, 1.44, 1.5, 1.6$ and $\alpha = 1(1)6(2)10(5)100(10)200(50)400$. Values of the intensity functions for other angles γ are not tabulated. They may be computed, however, from the values of A_n, P_n, π_n, etc. Part II of the first volume gives 6D values of the real and imaginary parts of A_n and P_n for the same α and m as above, and for a suitable range of n. The practical difficulties in computing the series for larger α (and β) are illustrated by the following examples indicating various ranges of n appropriate to the different values of α, and $m = 1.2$:

α	n
1	1(1)4
10	1(1)17
100	1(1)114
400	1(1)421.

In the second volume under review those Riccati-Bessel functions are tabulated which were required for the computation of the A_n and P_n. Values (mostly to 6S) are given of $S_n(z)$, $S_n'(z)$, $C_n(z)$, $C_n'(z)$, in Table I for $z = \alpha$ with α and n ranging as above, and in Table II for $z = m\alpha$, with m, α, and n ranging as above, except that those values which have already appeared in Table I are omitted. Certain other omissions, especially for $m = 1.33$ and $m = 1.6$, also occur. Judging from the manner in which the infinite series converge, the authors believe that the tabulated values are accurate to within one unit of the last decimal place given.

The last of the three volumes under review contains material for the computation of intensity functions. Specifically, it gives tables of π_n and of $x\pi_n - (1 - x^2)\pi_n'$ to 9 or less S for $n = 1(1)420$ and $\gamma = 10°(10°)170°(1°)180°$. The tabulated functions are stated to be accurate to within one unit of the fifth significant figure, except when fewer than 5S are given in which case one unit of the last place is the maximum error.

The first of the three volumes has a foreword by P. DEBYE. Each volume has an introduction giving definitions of the tabulated quantities, a brief description of the computation, notes on accuracy and interpolation, and a request that users communicate to the authors any errors or omissions noticed. In the reviewer's copy there is one correction in ink: i_1 for $\alpha = 35$, $m = 1.33$, $\gamma = 90°$, should read 9.926 instead of 14.07.

The tables should be very useful in practical computations of scattering from droplets. On the whole they are well arranged and well produced, and perhaps it is ungracious to complain about details. There are a few blemishes, though, most of which could have been avoided. In the second volume the table headings were stencilled, and it is difficult to distinguish between C and c. Also, the tables in this volume have been reduced considerably, perhaps excessively, in the photographic process. In the third volume the table headings were typed, and the same symbol was used for the variable x and the multiplication cross $\times$.

A. E.

[1] GUSTAV MIE, "Beiträge zur Optik trüber Medien," *Annalen der Physik*, s. 4, v. 25, 1908, p. 377–445.
[2] J. A. STRATTON, *Electromagnetic Theory*. New York, 1941.
[3] NBSCL, *Tables of Scattering Functions for Spherical Particles*. Washington, D. C., 1948. See *MTAC*, v. 3, 1949, p. 483–484.

990[L].—F. W. J. OLVER, "A further method for the evaluation of zeros of Bessel functions and some new asymptotic expansions for zeros of functions of large order," Cambridge Phil. Soc., *Proc.*, v. 47, 1951, p. 699–712.

In a previous paper,[1] the author described a method for determining zeros of Bessel functions $J_n(x)$ and $Y_n(x)$. The method is independent of tabular values and zeros are obtained by numerical integration of a third order non-linear differential equation. The order of the functions is fixed and integration is performed with respect to a variable t. The numerical procedure was not satisfactory for the early zeros and an alternative approach was developed. Here the technique amounts to the computation and inverse interpolation of the phase of the Hankel function.

The first part of the present paper gives a new method for rapidly determining the early zeros. The idea is to treat t as fixed and regard the order of the Bessel function n as a continuous variable. If $\rho(n, t)$ is a zero of the real cylinder function $C_n(x)$, then the starting point is an equation of WATSON,[2] p. 508,

$$\frac{\partial \rho}{\partial n} = F(\rho, n) = 2\rho \int_0^\infty K_0(2\rho \sinh u) e^{-2nu}\, du. \tag{1}$$

The asymptotic expansion of (1) is determined for large ρ and n. With $\lambda = n/\rho$, the result is

$$F(\rho, n) = \sum_{m=0}^{\infty} (-1)^m f_m(\lambda) n^{-2m},$$

$$f_m(\lambda) = (-1)^m (2m)!(\tfrac{1}{2}\lambda)^{2m} \int_0^\infty \operatorname{sech}^{2m+1} u\, a_m(\lambda \operatorname{sech} u)\, du. \tag{2}$$

Values of a_m are given by Watson,[2] p. 316, for $m = 0, 1, 2, 3$. In the applications $0 < \lambda < 1$, and it is advantageous to tabulate $f_m(\lambda)$ over this range so that interpolation is easily accomplished. For this purpose, the author defines

$$I_m(\lambda) = \lambda^{m-1} \int_0^\infty (\cosh u + \lambda)^{-m}\, du. \tag{3}$$

Values of I_1 and I_2 are easily obtained and I_m is determined recursively. The f_m are then expressed in terms of (3). The author derives a formula so that values of the derivative of a function at its own zero can be found by quadrature. The formula involves an asymptotic expansion whose coefficients depend on $g_m(\lambda) = (-1)^{m-1} f_m'(\lambda)$ which are written in terms of (3). In a similar fashion expansions are developed so that zeros of $C_n'(x)$ and stationary values of $C_n(x)$ can be obtained by numerical integration. Here again the necessary coefficients depend on (3).

To facilitate the calculations, 10D values of $I_m(\lambda)$ have been computed for $m = 1(1)13$, $\lambda = 0(.05)1$. These values were then used to compute $f_m(\lambda)$, $g_m(\lambda)$ for $m = 0(1)4$ and $\lambda = 0(.01)1$. These tables are not given in the present paper. The procedures of the paper are being used to evaluate the zeros of J, Y and their derivatives as part of the program of the Royal Society Mathematical Tables Committee (see *MTAC*, v. 3, p. 340). The reviewer anticipates that tables of $f_m(\lambda)$, etc., will be published simultaneously with zeros of J, Y, etc. To illustrate techniques of paper, extracts of tables (1 and 2) are given showing the numerical integration process for computing the first zero of $J_n(x)$ to 10D over the range $n = 6(.5)20$.

In the second part of the paper, asymptotic expansions are given for the direct evaluation of the following:

(a) positive zeros ρ of the real cylinder function $C_n(x)$
(b) values of $C_n'(\rho)$
(c) positive zeros σ of $C_n'(x)$
(d) values of $C_n(\sigma)$

The first few coefficients needed in (a) to (d) are tabulated in Tables 3 to 5, respectively. Values are given corresponding to the first five zeros of $J_n(x)$, $Y_n(x)$, etc. Asymptotic expansions are given for item (b) and also its reciprocal. It is for the latter that the coefficients are given in Table 4. Coefficients for the former can be easily evaluated in terms of Table 3 and a portion of Table 4. Accuracy obtained using tabulated coefficients is judged by examples. If $s = 1$, $n = 20$, seven-figure accuracy can be obtained from items (a) and (c); six-figure accuracy, in associated values of items (b) and (d). If $s = 5$, $n = 20$, the corresponding results are five and four-figure accuracy. Accuracy increases with n and decreases with s.

The basic coefficients needed for the above tabulations depend on tables of the Airy integral.[3] The present tabulations are virtually all new and supersede the earlier work of MEISSEL[4] and AIREY[5] whose results for the most part are in error. Airey's results are quoted in JAHNKE & EMDE.[6]

YUDELL L. LUKE

Midwest Research Institute
Kansas City, Missouri

[1] F. W. J. OLVER, "A new method for the evaluation of zeros of Bessel Functions and of other solutions of second-order differential equations," Cambridge Phil. Soc., *Proc.*, v. 46, 1950, p. 570–580.

[2] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*. Cambridge, 1945.

[3] BAASMTC, Pt. v. B. *The Airy Integral*. Cambridge, 1946.

[4] E. MEISSEL, "Beitrag zur Theorie der Bessel'schen Functionen," *Astr. Nach.*, v. 128, 1891, cols. 435–438.

[5] J. R. AIREY, "The numerical calculation of the roots of the Bessel Function $J_n(x)$ and its first derivative $J_n'(x)$," *Phil. Mag.*, s. 6, v. 34, 1917, p. 189–195.

[6] E. JAHNKE & F. EMDE, *Tables of Functions*. New York, Dover, 1945, p. 143.

991[L].—K. M. SIEGEL, D. M. BROWN, H. E. HUNTER, H. A. ALPERIN, & C. W. QUILLEN, *The Zeros of the Associated Legendre Functions* $P_n^m(\mu')$ *of Non-Integral Degree*, University of Michigan, Engineering Research Institute, *Report* No. UMM-82, Ann Arbor, Mich., 1951, vii + 20 p.

In a previous discussion of a paper by CARRUS and TREUENFELS (CT), a difference test indicated that some of the early zeros of the associated Legendre function $P_n^1(\cos\theta) = 0$ as a function of n were incorrect (*MTAC*, v. 5, p. 152–153). The investigation of the present article also reveals some errors. The authors give an alternative proof of an equation due to MACDONALD[2] for determining the early zeros of $P_n^m(\cos\theta) = 0$ where θ is near π. For $m = 1$, $\theta = 165°$, this formula gives 1.035 as an approximation to the first zero. Employing power series, it is shown that the first zero must be between 1.0316 and 1.0321. The value reported by CT is 1.053 and so is in error. Application of the Macdonald formula shows that for $165° \leq \theta \leq 180°$, the corresponding values of n decrease with increasing θ. For $m = 1$, $\theta = 170°$, the first zero given by CT is 1.05 and thus is also incorrect. Numerical analysis of early zeros for values of θ other than those cited above is not given, but sufficient evidence now exists to show that the CT tables should be used with caution.

YUDELL L. LUKE

Midwest Research Institute
Kansas City, Missouri

[1] P. A. CARRUS & C. G. TREUENFELS, "Tables of roots of incomplete integrals of associated Legendre functions of fractional orders," *Jn. Math. Phys.*, v. 29, 1951, p. 282–299 [*MTAC*, v. 5, p. 152–153].
[2] H. M. MACDONALD, "Zeros of the spherical harmonic $P_n^m(\mu)$ considered as a function of n," London Math. Soc., *Proc.*, s. 1, v. 31, 1900, p. 264–278.

MATHEMATICAL TABLES—ERRATA

In this issue references to Errata have been made in RMT's **989, 990,** and **991.**

203.—AKADEMIÎA NAUK, SSSR. *Tablitsy znachenii Funktsii Besselîa ot mnimogo Argumenta.* [*MTAC*, v. 5, p. 151–152.]

p.	x	Function	*For*	*Read*
10	.444	ΔiH_0	2367	2267
19	.899	ΔiH_0	667	657
42	2.031	H_1	593738	493738
42	2.032	H_1	481922	381922
106	5.237	iH_0	153939	132939
114	5.650	ΔH_1	788	782
115	5.700	x	5.605	5.700
118	5.815	ΔH_1	476	469
118	5.816	ΔH_1	449	456
163	8.061	ΔH_1	949	849
166	8.235	ΔH_1	001	81991
195	9.654	iH_0	276029	276022
205	.074	ΔK_1	82290	81290
206	.139	ΔK_0	76	876
220	.815	ΔK_0	693	683
220	.848	ΔK_0	645	685
230	1.312	K_0	380745	381745