nated at each end at some distance from the computing portion to represent zero and infinity. To obtain the equivalent of phase the fixed contacts along the imaginary axis are doubled so that potential slope can be indicated. The report contains considerable information on both the construction and use of this device and an analysis of the effect of individual errors. A photograph shows the entire machine mounted on a laboratory cart.

F. J. M.

[1] Cf. A. R. BOOTHROYD, E. C. CHERRY & R. MAKAR, Inst. Elect. Eng., *Proc.*, v. 96, 1949, p. 163–177; *MTAC*, v. 5, p. 49–50.

18. J. R. SHAH & L. JACOBS, "Investigation of field distributions in symmetrical electron lens," *Jn. Appl. Physics*, v. 22, 1951, p. 1236–1241.

The potential field for the electron lens is obtained by the use of an electrolytic tank. It was determined experimentally that an impedance of 30 megohms for the probe was necessary. With these methods, the crossing of the nodes at the central symmetry point was accurate to within about a quarter of a degree. The field obtained from the electrolytic tank was compared with that obtained by a relaxation method. The maximum difference is about 1.5%.

F. J. M.

19. B. A. SOKOLOFF, "Principe et réalisation d'une machine mathématique dite 'Operateur Mathématique Electronique' OME," *Annales des Télécommunications*, v. 5, no. 4, April 1950, p. 143–159.

The mathematical machine described is an electronic differential analyzer with the customary feedback amplifier integration and addition. The output indicator is an oscilloscope and camera combination. Multiplication is by servo driven potentiometers. A discussion of stability is given for the constant coefficient case of linear differential equations, including the use of the determinantal conditions for positive definiteness.

F. J. M.

20. G. J. TAUXE & R. L. STOKER, "Analytical studies in the suppression of wood fires," A. S. M. E., *Trans.*, v. 73, 1951, p. 1005–1020.

Different methods of suppression of wood fires are analyzed thermally using electrical analogy methods. The latter effect the solution of the heat equation by means of an R-C network. The method has been previously described.[1]

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[1] V. PASCHKIS & H. D. BAKER, "A method for determining unsteady-state heat transfer by means of an electrical analogy," A. S. M. E., *Trans.*, v. 64, 1942, p. 105–110.

NOTES

139. THE RAND COLLECTION OF ILLUSTRATIVE APPROXIMATIONS.—In recent years, the Numerical Analysis Department at RAND has been preparing loose leaf sheets that contain interesting and useful approximations to a number of the higher transcendental functions. The data from a sample sheet are displayed at the end of this note together with a photographic reproduction of the error curve appearing on this sheet. Some sixty odd sheets in this series have now been prepared, and up to date collections of these sheets have been distributed to some three hundred people working in the field of numerical analysis throughout the United States and in a few foreign countries.

The approximations given in this series of loose leaf sheets are of both a practical and of an illustrative nature. In a high speed digital machine, these approximations may take the place of bulky tables or the place of awkward series developments. For the hand computer, an easily evaluated expression will often be of use when a required table is unavailable for consultation. Beyond this, however, the sheets will be useful for the insight they give the practical computer in the approximation of functions. Starting with approximations concerning the common logarithm, sheets for an ever increasing number of the useful transcendental functions have been prepared and distributed. Sheets of approximations in this series are already available concerning the logarithmic function, the exponential function, the inverse tangent, the sine function, the inverse sine, the Gamma function, the Gaussian error integral, the inverse Gaussian error integral, the complete elliptic integrals of first and second kind, the exponential integral and a number of other special functions.

As the approximations in this series are intended to be illustrative as well as practical, great care has been taken in the accurate leveling of the error curves. The sheets of approximations thus give interesting information concerning the location of roots—points at which function and approximation agree—and the location of extremals—points at which the deviation between function and approximation is locally a maximum in an absolute or relative sense. The accurate and carefully drawn error curves that appear on each sheet will do much to give the reader a feel for the nature of analytical approximation in a wide variety of typical cases of practical importance. In order to provide this important information, the coefficients in the approximations given have been recorded to perhaps two, three or four more decimals than would be required for practical considerations of utility. Should the coefficients given be of awkward size for use in a given computing machine, it is permissible to round the numbers quite severely. Of course the beauty and character of the error curve will be lost as a result of such mistreatment, but this is not a matter of practical concern in the use of the approximation.

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Approximations in Numerical Analysis

Function:

$$-\mathrm{Ei}\,(-X) = \int_X^\infty \frac{e^{-t}\,dt}{t}$$

Range:

$$1 \leq X < \infty$$

Approximation:

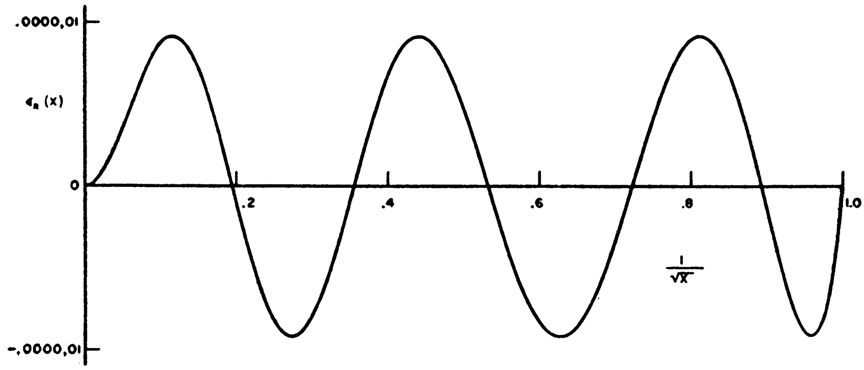
$$-\mathrm{Ei}^*(-X) = \frac{e^{-X}}{X}\left\{\frac{a_0 + a_1X + a_2X^2 + X^3}{b_0 + b_1X + b_2X^2 + X^3}\right\}$$

$a_0 = .2372, 9050 \qquad b_0 = 2.4766, 3307$

$a_1 = 4.5307, 9235 \qquad b_1 = 8.6660, 1262$

$a_2 = 5.1266, 9020 \qquad b_2 = 6.1265, 2717$

Error Curve (Approximation − Function)/(Function):



140. An Alternative "Predictor-Corrector" Process.—The purpose of this note is to report the successful use, on a system of 3 non-linear ordinary differential equations of order 2 (with initial conditions), of a pair of formulas analogous to Milne's.[1]

The system of differential equations is of the form

(1) $$\ddot{x}_i = f_i(\dot{x}_1, \dot{x}_2, \dot{x}_3, x_1, x_2, x_3, t), \quad (i = 1, 2, 3).$$

The formulas used for *prediction* are

(2) $$\dot{x}_i^{(n+1)} = 5\dot{x}_i^{(n-1)} - 4\dot{x}_i^{(n)} + 2h[\ddot{x}_i^{(n-1)} + 2\ddot{x}_i^{(n)}], \quad (i = 1, 2, 3).$$

The *correction* formulas are

(3) $$\dot{x}_i^{(n+1)} = \dot{x}_i^{(n)} + \frac{h}{12}[5\ddot{x}_i^{(n+1)} + 8\ddot{x}_i^{(n)} - \ddot{x}_i^{(n-1)}], \quad (i = 1, 2, 3)$$

and their companions

(4) $$x_i^{(n+1)} = x_i^{(n)} + \frac{h}{12}[5\dot{x}_i^{(n+1)} + 8\dot{x}_i^{(n)} - \dot{x}_i^{(n-1)}], \quad (i = 1, 2, 3).$$

In the above, h denotes the integration step.

The process is as follows: Given two sets of "starting" values, $x_i^{(n-1)}$, $\dot{x}_i^{(n-1)}$, $\ddot{x}_i^{(n-1)}$ and $x_i^{(n)}$, $\dot{x}_i^{(n)}$, $\ddot{x}_i^{(n)}$ [the values at $t = t_{n-1}$ and $t_n = t_{n-1} + h$, resp.], the first approximations ${}_1\dot{x}_i^{(n+1)}$ are predicted by (2), then (4) are used to obtain ${}_1x_i^{(n+1)}$. These first approximations are substituted in (1) to obtain ${}_1\ddot{x}_i^{(n+1)}$, which are then used in (3) to obtain ${}_2\dot{x}_i^{(n+1)}$. The above steps may be repeated, to give ${}_3\dot{x}_i^{(n+1)}$, etc. If the ${}_j\dot{x}_i^{(n+1)}$ prove acceptable (see below), then ${}_jx_i^{(n+1)}$ are computed and accepted.

An estimate of the error in the second approximation to $\dot{x}_i^{(n+1)}$ is available from the error terms of (2) and (3), viz. $h^4x_i^v(\xi)/6$ and $-h^4x_i^v(\eta)/24$, respectively, where both ξ and η are on the interval $t_{n-1} < t < t_{n+1}$ and where x_i^v denotes the fifth derivative of x_i. If we let $D_{ij} = {}_{j+1}\dot{x}_i^{(n+1)} - {}_j\dot{x}_i^{(n+1)}$, then, as in Milne,[1] we find that the error in ${}_2\dot{x}_i^{(n+1)}$ is approximately $D_{i1}/5$. Thus, if all the $|D_{i1}/5|$ are insignificant, one iteration is probably sufficient, whereas if any $|D_{i1}/5|$ is intolerably large, a smaller h and/or more iterations are called for.

In the problem to which the process was applied (it was carried out on our IBM CPC), $h = 0.1$ was chosen initially. This proved to be a good choice, with 2 iterations, from $t = 0$ to $t = 4.0$, as shown by D_{i1} and D_{i2}, which were all listed for monitoring. In most instances, some $|D_{i1}/5|$ was too large ($> \frac{1}{2} \cdot 10^{-4}$), whereas no $|D_{i2}|$ exceeded 10^{-3}, thus indicating that further iterations would probably not affect the 3rd decimal place. The combination $h = 0.1$, and two iterations, was actually used to $t = 4.4$, at which place some $|D_{i2}|$ increased significantly; the h was then decreased to 0.05 and the process continued from $t = 4.0$ with two iterations (although the previously computed values at $t = 4.3$ were probably satisfactory, it was felt best, for the sake of smoothness and safety, to back up to $t = 4.0$ before changing to $h = 0.05$). The new combination gave good results as far as $t = 5.15$, when some $|D_{i2}|$ again became significant. Here it was decided to use $h = 0.05$ again, but increase to 4 iterations; this worked successfully as far as $t = 4.90$.

It may be of interest to note that the need for smaller h and/or more iterations arose because the f_i in (1) each would (apparently) have become infinite in the neighborhood of $t = 4.7$. The results were checked by differencing the (accepted) values of the x_i. In addition, there was an internal check available in the problem, viz. that $\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 = K$. Interestingly enough, this check failed approximately at the same time that the last computed $|D_{ij}|$ became intolerably large.

One of the advantages of this process is that it requires only two sets of "starting" values rather than four. This means that in case it becomes necessary to decrease h at some point of the process, only one additional "point" is needed. This added point can be calculated, e.g., by means of the modified Euler method,[2] as can the additional point needed initially. Another "advantage" (rather a nebulous one, actually, since it is based on esthetics rather than mathematics) is that one has a feeling of greater confidence in values based on others in their more immediate neighborhood (two starting values as opposed to four). Obviously, examples exist which would vitiate this belief.

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[1] W. E. Milne, *Numerical Calculus*. Princeton, 1949, p. 135.
[2] J. B. Scarborough, *Numerical Mathematical Analysis*. Baltimore, 1950, p. 235.

141. A New Calculation of Euler's Constant.—In 1878 J. C. Adams published his notable paper[1] on the evaluation of Euler's constant. By two separate calculations he determined approximations which agreed to 263D.

Since that time no attempt seems to have been made to extend the approximation beyond that point, until a recent study of Adams' work induced me to undertake the task.

Following the procedure adopted by Adams, I used the Euler-Maclaurin summation formula applied to the harmonic series, which yields the equation

$$\gamma = \sum_{k=1}^{n} k^{-1} - \ln n - \frac{1}{2n} + \sum_{r=1}^{m} \frac{(-1)^{r+1}B_r}{2rn^{2r}} + R_m.$$

For fixed n, the minimum absolute value of R_m is nearly equal to the numerical value of the term of the asymptotic series given by taking $m = [n\pi - \frac{1}{4}]$.

For this calculation of γ it was decided to take $n = 1000$, principally because of the resultant ease of evaluating the sum of the asymptotic series to about 350D.

The evaluation of the sum of the first thousand terms of the harmonic series to 350D was accomplished by means of an artifice used by Adams. He replaced that sum by an equivalent, namely $-43 + \sum_{i=1}^{168} a_i p_i^{-k_i}$, where $p_i^{k_i} < 1000 < p_i^{k_i+1}$, $k_i \geq 1$, and a_i is a positive integer $< p_i^{k_i}$ determined uniquely by decomposing the terms of the partial sum of the harmonic series into partial fractions with prime-power denominators, adding all the component fractions involving the same prime p_i, and finally reducing the numerator modulo $p_i^{k_i}$. This part of the calculation also was facilitated by the use of an accurate manuscript table of the complete decimal periods of the reciprocals of all primes less than 1000, which I had previously calculated in order to check Gauss' table of circulating decimals.[2]

The only remaining datum required was ln 1000. This was deduced from the 330D approximation to ln 10 calculated by H. S. Uhler.[3]

Inasmuch as Uhler guaranteed his value of ln 10 to within 2 units in the 329th decimal place, and all other parts of this calculation were carefully checked, it is believed that the following approximation to Euler's constant is correct to at least 328D. It confirms the accuracy of Adams' approximation to 262D.

γ = 0.57721 56649 01532 86060 65120 90082 40243 10421 59335 93992
35988 05767 23488 48677 26777 66467 09369 47063 29174 67495
14631 44724 98070 82480 96050 40144 86542 83622 41739 97644
92353 62535 00333 74293 73377 37673 94279 25952 58247 09491
60087 35203 94816 56708 53233 15177 66115 28621 19950 15079
84793 74508 57057 40029 92135 47861 46694 02960 43254 21519
05877 55352 67331 39925 40129 674(28)

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[1] Roy. Soc. London, *Proc.*, v. 27, 1878, p. 88–94.
References to earlier calculations are given in FMR, *Index*, p. 95.
[2] *MTAC*, v. 4, 1950, p. 222, 223.
[3] Nat. Acad. Sci., *Proc.*, v. 26, 1940, p. 205–212.

142. The Determination of a Large Prime. [Editorial Note: The primality of $(2^{148} + 1)/17$ as established by A. Ferrier was very briefly announced in *MTAC*, v. 6, p. 61. Since this is probably the last "largest" prime to be identified by hand computing methods (primes like $2^{1279} - 1$ would require more than a century of desk calculator work to establish by any known method), it may be of interest to give the following details quoted from Ferrier's letter of July 14, 1951.]

I have established that

$$N = (2^{148} + 1)/17 = \begin{array}{l} 2098\ 89366\ 57440\ 58648\ 61512 \\ 64256\ 61022\ 25938\ 63921 \end{array}$$

is a prime and is in fact the largest prime known. The method used is the following.

I first found that N divides $3^{17N} - 3^{17}$. In fact for $n = 2^{144}$ we have

$$\begin{aligned} 3^n &\equiv -566\ 68397\ 79443\ 57425\ 64352 \\ &\equiv U \pmod{N} \end{aligned}$$

and

$$U^2 \equiv 3^{2n} \equiv -9 \pmod{N}.$$

Hence

$$3 \cdot (3^n)^{16} = 3^{17N} \equiv 3 \cdot (-9)^8 \equiv 3^{17} \pmod{N}.$$

That is, N divides $3^{17N} - 3^{17}$. Next we see that

$$N - 1 = (2^{148} - 2^4)/17 = 2^4(2^{72} - 1)(2^{72} + 1)/17.$$

The largest prime factor of $N - 1$ is that dividing $2^{72} + 1$ and is

$$p = 48\ 78248\ 87233.$$

Writing $N - 1 = pm$ we find that $3^{17m} - 1$ is prime to N. Applying the theorem quoted and corrected in *MTAC*, v. 3, p. 497, and v. 5, p. 259, respectively, it follows that every prime factor of N is of the form $px + 1$ as well as $296x + 1$, that is of the combined form

$$14439\ 61666\ 20968x + 1 = qx + 1.$$

For $q = 1(1)11$, $qx + 1$ is divisible by small primes. Hence every prime factor of N exceeds $12q$.

Writing $N = A^2 - B^2$ we have

$$2A < 12q + N/(12q) < 1.3 \cdot 10^{28}.$$

However

$$\begin{aligned} 2A &\equiv N + 1 \pmod{q^2} \\ &\equiv 1885\ 97808\ 71263\ 54966\ 31614\ 94450 \pmod{q^2} \end{aligned}$$

so that $2A > 1.8 \cdot 10^{28}$.

Thus A does not exist and N is a prime.

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