RECENT MATHEMATICAL TABLES

1018[D].—J. PETERS, *Sechsstellige Werte der Kreis- und Evolventen-Funktionen von Hundertstel zu Hundertstel des Grades nebst einigen Hilfstafeln für die Zahnradtechnik.* Zweite verbesserte Auflage, Bonn, Ferd. Dummlers Verlag, 1951, viii, 222 p., 14 × 20.8 cm.

The first edition of this work (viii, 182 p.) appeared in 1937 and the size of its page was 19.1 × 26 cm. For the second edition a photographic copy of these pages was made in reduced size, the only changes being in the addition on page viii of a preface for the second edition, and of nine new words on the title page. It will be observed, however, that an Appendix of 40 new pages has been added. These pages are mainly filled with tables of results useful in studying problems dealing with teeth of gears.

In the main body of the volume, with α° as argument at interval .01, one may read off on opposite pages 6D values of the 6 trigonometric functions; of the arc α; of the polar coordinates θ° and sec α of the evolute of a unit circle; of the radius of curvature, tan α of the evolute; and of arc $\theta = \text{ev}\,\alpha = \tan\alpha - \text{arc}\,\alpha$.

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1019[F, L].—L. A. DRAGONETTE, "Some asymptotic formulae for the mock theta series of Ramanujan," Amer. Math. Soc., *Trans.*, v. 72, 1952, p. 474–500.

The author is concerned chiefly with the function

$$f(q) = \sum_{n=0}^{\infty} q^{n^2}\{(1+q)(1+q^2)\cdots(1+q^n)\}^{-2}$$

introduced by RAMANUJAN. This function generates the function $A(n)$ defined by

$$f(q) = \sum_{n=0}^{\infty} A(n)q^n,$$

which is similar to the partition function $p(n)$ in that $A(n)$ possesses an asymptotic expansion similar to that of HARDY & RAMANUJAN.[1] Table 1 (p. 495) gives the exact values of $A(n)$ for $n = 0(1)100$. The asymptotic formula is

$$A(n) = \sum_{k<n^{\frac{1}{2}}} \lambda_n(k)(k(n-1/24))^{-\frac{1}{2}} \exp\{(\pi/k)(1/6)(n-1/24)^{\frac{1}{2}}\} + O(n^{\frac{1}{2}}\log n).$$

The coefficients $\lambda_n(k)$ are real and given in Table II (p. 497) as trigonometric polynomials in n for $k = 1(1)14$.

D. H. L.

[1] G. H. HARDY & S. RAMANUJAN, "Asymptotic formulae in combinatory analysis," London Math. Soc., *Proc.*, s. 2, v. 17, 1918, p. 75–115.

1020[G, K].—F. N. DAVID & M. G. KENDALL, "Tables of symmetric functions. Parts II and III," *Biometrika*, v. 38, 1951, p. 435–462.

Part I of this paper appeared in 1949 and is reviewed in *MTAC*, v 4, p. 146, where also will be found references to other symmetric function tables. The tables of Part I gave expressions for the monomial symmetric functions

$$(\alpha_1, \alpha_2, \cdots) = \sum x_1^{\alpha_1} x_2^{\alpha_2} \cdots$$

in terms of the power sums

$$s_r = (r) = \sum x_1^r$$

and inverse tables for all weights ≤ 12.

Part II gives the monomial symmetric functions in terms of the elementary symmetric function

$$a_r = (1^r) = \sum x_1 x_2 \cdots x_r$$

with inverse tables.

Part III gives the monomial symmetric functions in terms of the homogeneous product sums h_r defined by the generator

$$1 + h_1 t + h_2 t^2 + \cdots = (1 - a_1 t + a_2 t^2 - \cdots)^{-1},$$

with inverse tables.

Parts II and III extend also to weights ≤ 12 and have the same format as Part I. The tables of Part II have been compared with earlier tables. The tables of Part III are new.

D. H. L.

1021[G].—M. Osima, "On the irreducible representations of the symmetric group," *Canadian Jn. Math.*, v. 4, 1952, p. 381–384.

A table is given of the number of irreducible representations of the symmetric and alternating groups of order n for $n = 2(1)40$. If a representation by a Young diagram has its rows and columns interchanged we obtain an associated representation. The number of self-associated representations is also given.

D. H. L.

1022[I].—Z. Kopal, P. Carrus, & K. E. Kavanagh, "A new formula for repeated mechanical quadratures," *Jn. Math. Phys.*, v. 30, 1951, p. 44–48.

The authors start with the Hermite interpolation formula

$$(1) \qquad f(x) = \sum_{j=1}^{n} h_j(x) f(a_j) + \sum_{j=1}^{n} \bar{h}_j(x) f'(a_j) + [p_n(x)]^2 f^{(2n)}(u)/(2n)!$$

where

$$\begin{aligned} h_j(x) &= \{1 - (x - a_j)[p_n''(a_j)/p_n'(a_j)]\}[l_j(x)]^2, \\ \bar{h}_j(x) &= (x - a_j)[l_j(x)]^2, \\ l_j(x) &= p_n(x)/[(x - a_j)p_n'(a_j)], \\ p_n(x) &= (x - a_1)(x - a_2) \cdots (x - a_n), \end{aligned}$$

and u is an inner point of the range including all a_j and x. If the coefficients a_j are determined so as to satisfy

$$\int_{-1}^{1} \bar{h}_j(x) dx = 0, \quad j = 1, 2, \cdots, n,$$

then the well-known Gauss quadrature formula results. The authors derived new formulas by setting

$$\int_{-1}^{1} \bar{h}_1(x)dx = \int_{-1}^{1} h_j(x)dx = 0, \quad j = 2, 3, \cdots, n. \tag{2}$$

Let

$$\phi(x) = f'(x); \quad \int_{-1}^{1} f(x)dx = 2f(a_1) + U,$$

where

$$U = \int_{-1}^{1} dx \int_{a_1}^{x} \phi(y)dy.$$

When equations (2) are satisfied, there results the quadrature formula

$$U = 2f(a_1) + \sum_{j=2}^{n} H_j\phi(a_j) + R \tag{3}$$

where the remainder, R, involves the factor $\phi^{(2n-1)}(y)$; hence $R = 0$ if $\phi(y)$ is a polynomial of degree no higher than $2n - 2$. The authors tabulate the weight factors $H_j^{(n)}$ to 6, 7, or 8D, and the points a_j to 7D for $n = 2, 3, 4$, and to 6D for $n = 5$ and 7. They discovered the interesting fact that the system (2) yields more than one set of real solutions a_j for the values of n considered in sufficient detail. The following examples for $n = 3$ illustrate the character of the coefficients:

Type A ($a_1 = 0$)		Type B ($a_1 \neq 0$)	
$a_1 = 0$		$a_1 = \pm.5889711$	
$a_2 = .5477225$	$H_2 = .3042903$	$a_2 = \pm.2250452$	$H_2 = \mp.8445196$
$a_3 = -.5477225$	$H_3 = -.3042903$	$a_3 = \mp.5293628$	$H_3 = \mp.3334225$

No attempt was made by the authors to prove the existence of solutions of (2) for general n; but for odd values of n, solutions of Type A can be shown to exist.

The new formulas have advantages and disadvantages similar to those of the Gauss quadrature formula. It is of course true, as the authors infer, that by a simple linear transformation U can be made to represent the more general double integral

$$W = \int_{c}^{d} dx \int_{b}^{x} \phi(y)dy.$$

Indeed

$$W = U_1 + V_1; \quad U_1 = \tfrac{1}{4}(d - c)^2 \int_{-1}^{1} dw \int_{a_1}^{v} \phi_1(u)du;$$

$$V_1 = \tfrac{1}{4}(d - c)^2 \int_{s}^{a_1} \phi_1(u)du,$$

where

$$\phi_1(u) = \phi[\tfrac{1}{2}(d + c) + \tfrac{1}{2}(d - c)u]; \quad s = (2b - c - d)/(d - c).$$

However, it must be remembered that both V_1 and U_1 need to be evaluated, and that the argument for which ϕ must be computed may be cumbersome in practice. Where the calculations are not prohibitive, a further simplification is in fact possible. For U vanishes whenever $\phi(u)$ is an even function of u. Hence if $\phi(-y)$ is defined, write in (2)

$$\phi(y) = \tfrac{1}{2}(\phi_3 + \phi_4); \quad \phi_3 = \phi(y) + \phi(-y); \quad \phi_4 = \phi(y) - \phi(-y).$$

It is necessary to apply the quadrature formula only to ϕ_4, if a formula of Type A is used, and thus only half the number of multiplications have to be performed. In many cases arising in practice ϕ_4 may be just as simple to evaluate as ϕ itself.

There is a misprint in the authors' formula (21); in the coefficient of the cosine term replace the term $2x^2$ of the numerator by $2x^3$.

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1023[K].—H. A. DAVID, "Further applications of range to the analysis of variance," *Biometrika*, v. 38, 1951, pp. 393–409.

Consider the mean of k uncorrelated ranges for samples of size n from normal populations with common variance σ^2. Let c^2 be the expected value of $[(\text{mean range})/\sigma]^2$. Let ν equal the number of degrees of freedom for the χ^2-distribution which is approximately equivalent to the distribution of (mean range)$/c\sigma$. Table I contains values of c and ν for $n = 2(1)10$ and $k = 1(1)5, 10$. Table II contains values of c and ν for use in a double classification analysis. Here $n = 2(1)9$ and $k = 2(1)10, 20$. In Table III a split-plot design with l main treatments, m blocks, and N subtreatments (N is reviewer's notation) is considered. Table III contains values of c and $\nu' = \nu/l$ for $m = 2(1)10$ and $N = 2(1)9$. Values of c are given to 2D, those of ν and ν' to 1D. Let d_n be the expected value of (range)$/\sigma$ for a sample of size n while V_n is the variance of this quantity. Table IV contains values of d_n, d_n/V_n, d_n^2/V_n to 2D for use in analyses of single classification with unequal cell frequencies. Here $n = 2(1)20$.

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1024[K].—P. M. GRUNDY, "The expected frequencies in a sample of an animal population in which the abundances of species are log-normally distributed. Part I," *Biometrika*, v. 38, 1951, p. 427–434.

Given N species in a population with the expected number of individuals per species (abundance), m, following a log normal distribution

$$f(m)dm = \frac{1}{\sqrt{2\pi}m\sigma} e^{-[\ln (m/a)]^2/2\sigma^2} dm,$$

where σ^2 is the variance and $\ln a$ the mean of $\ln m$ (hence a is the median abundance). The probability of obtaining r members of a given species in a sample is $e^{-m}m^r/r!$. Hence the expected proportion of species having r in the sample, ϕ_r, is

$$\phi_r = \int_0^\infty \frac{e^{-m}m^r}{r!} f(m)\, dm.$$

Table 1 presents values of $1 - \phi_0$ (expected proportion of species in the sample) and Table 2 presents values of ϕ_1 (expected proportion of species with singletons) to 4D for $\sigma^2 = 2(1)16$ and $\log_{10} a = -2.0(.25)3.0$.

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1025[K].—P. G. GUEST, "The fitting of polynomials by the method of weighted grouping," *Annals Math. Stat.*, v. 22, 1951, p. 537–548.

Given n equally spaced values of x and the corresponding n observed values of y, it is desired to fit a polynomial $u_p(x) = \sum_{j=0}^{p} b_{pj}x^j$ $(p < n)$ by a method of weighted grouping. To obtain an unbiased estimate of $b_{pp} = a_p$, the n points are divided into not more than $2p + 1$ groups of successive points, the number of points in the ith and in the $(n - i + 1)$th groups being equal. The sum of the y's in each of $p + 1$ of these groups is to be assigned a non-zero weight. Because of symmetry, not more than $(p + 2)/2$ different weights are involved if p is even. If p is odd, $p + 1$ different weights are involved but half of these are the negatives of the other half. The groupings are determined for each n and p in such a way that the variance of a_p, assuming equal variances of the y's, is a minimum. To estimate b_{pj}, $j = 0, 1, \cdots, p - 1$, the a_j, $j = 0, 1, \cdots, p - 1$ are calculated and the estimates of b_{pj} are obtained from the relation, estimate of $b_{pj} = a_j + \beta_{j+1,j}a_{j+1} + \cdots + \beta_{pj}a_p$. (Alternate β's are zero.)

A table (p. 541–545) gives the groupings, the weights (exact), the β's (β_{20} and β_{31}, exact; β_{42} and β_{53}, 10S; β_{40} and β_{51}, 9S) for $n = 7(1)55$, $j = 0(1)5$.

The relative (compared with least square procedure) efficiencies of the estimates of b_{pj} are discussed and a table of some limiting relative efficiencies is given. The lowest limiting relative efficiency appearing in the table is .889 for estimates of a_1. Computational schedules and an example are given.

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1026[K].—H. O. HARTLEY & E. S. PEARSON, "Moment constants for the distribution of range in normal samples," *Biometrika*, v. 38, 1951, p. 463–464.

The moments indicated by the title are the mean μ_1'; the central moments μ_r, $r = 2(1)6$; $\sigma = \sqrt{\mu_2}$; $\beta_1 = \mu_3^2/\mu_2^3$; $\beta_2 = \mu_4/\mu_2^2$; $\kappa_4 = \mu_4 - 3\mu_2^2$; $\kappa_5 = \mu_5 - 10\mu_3\mu_2$; $\kappa_6 = \mu_6 - 15\mu_4\mu_2 - 10\mu_3^2 + 30\mu_2^3$. These moments are tabulated for $n = 2(1)20$ except for μ_5, μ_6, κ_4, κ_5, κ_6, which are given for $n = 2(1)12$. The tabled values are given to the number of decimals followed in parentheses by the number of significant digits as follows: μ_1', 5(6); μ_2, 5(5); μ_3, 4(4); μ_4, 3(4); μ_5, 3(3); μ_6, 1(2–3); σ, 4(4); β_1, 4(4); β_2, 3(4); κ_4, 3(2–3); κ_5, 2(0–2); κ_6, 2(0–3).

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1027[K].—L. Kaarsemaker & A. van Wijngaarden, *Tables for Use in Rank Correlation*, Report R 73, Computation Department of the Mathematical Centre, Amsterdam, 1952, 17 p. mimeographed. 21.0 × 32.4 cm.

For a given number n of subjects ranked according to two criteria, the size of Kendall's coefficient of rank correlation is proportional to the score S.[1] If S is the sample value, on the null hypothesis of no association $E(S) = 0$; to test this null hypothesis $P_n(S \geqq \mathrm{S})$, the probability that in n subjects $S \geqq \mathrm{S}$, on the assumption of no association, is used. It is assumed for this paper that no ties in ranks occur. Kendall[2] tabulated $P_n(S \geqq \mathrm{S})$ for $n = 1(1)10$ for all possible values of S and the authors extend this table giving $P_n(S \geqq \mathrm{S})$ to 3D for $n = 1(1)40$. They divide the table into two parts, Tables I and II, for S even and odd, respectively. Table III gives for $n = 4(1)40$ and for $\alpha = .005, .01, .025, .05$ and .1, the smallest value of S for which $P_n(S \geqq \mathrm{S}) \leq \alpha$. Since for $n > 40$ the normal distribution is a good approximation to that of S, values of the standard deviation of S, σ_S are listed in Table IV for $n = 40(1)100$ to 5D. There is a typographical error in the formula for σ_S where it is first written on p. 2; it appears correctly on p. 3 and in the table. A recurrence relation used in the calculation of Tables I and II is developed.

C. C. C.

[1] M. G. Kendall, *Rank Correlation Methods*, London, Griffin, 1948.
[2] Loc. cit., p. 141.

1028[K].—K. C. S. Pillai, "Some notes on ordered samples from a normal population," *Sankhyā*, v. 11, 1951, p. 23–28.

Consider a sample of size n from a normal population with zero mean. Table I contains 1% and 5% points of $T = 2$ (midrange)/(range) to 2 and 3D for $n = 3(1)10$. Table II presents values of coefficients for approximate determination of the distribution of the median (average of two central order statistics) to 6D for $n = 4, 6, 8$. Next consider two independent samples from normal populations. Form the range for each sample and let the sample size be n_1 for the larger range and n_2 for the smaller range. Table III contains 1% and 5% points of $F' =$ (larger range)/(smaller range) to 2D for $n_1, n_2 = 2(1)8$. Let σ_1 be the standard deviation for the population yielding the larger range and σ_2 the standard deviation of the other population. Table IV contains a power function comparison of the F'-test and the Snedecor F-test to 3D for $(n_1, n_2) = (3, 7), (4, 4), (8, 2), (5, 2), (8, 3)$, and $\sigma_1/\sigma_2 = 1(.5)2.5$.

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1029[K].—J. E. Walsh, "Some nonparametric tests of whether the largest observations of a set are too large or too small," *Annals Math. Stat.*, v. 21, 1950, p. 583–592.

This paper is apparently the first published work covering nonparametric tests for the detection of outlying observations. For the nonparametric tests, the usual restriction of normality can, of course, be removed and the sample considered to be drawn randomly from one or more continuous symmetrical populations. Assuming that the sample values are arranged in increasing order of magnitude: $x(1) \leq x(2) \leq \cdots \leq x(n)$, the tests considered are (a) the detection of whether the r largest observations are too large to be consistent with the hypothesis that the populations from which the sample values came have a common median, (b) whether the r largest observations are too small and (c) whether the populations are symmetrical in the tails. With regard to (a) and (b), the proposed tests also cover similar hypotheses for the r smallest observations because of symmetry conditions. The tests, based on the order statistics, covered in the paper have the somewhat fascinating property in that the Type I error or significance level, α, for the sample statistics used is independent of the sample size n for values of n permitted. (Generally speaking, n should be large, in which case the significance level tends toward the value α; however, for no admissible value of n does the significance level exceed 2α.)

The details of the actual tests are rather complicated for description here, but Table 1 gives the necessary specifications for 21 tests and the corresponding significance levels, α, to 4D, for use with $r \geq 4$.

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1030[L].—S. Chandrasekhar, Donna Elbert, & Ann Franklin, "The X- and Y-functions for isotropic scattering. I," *Astrophys. Jn.*, v. **115**, 1952, p. 244–268.

S. Chandrasekhar & Donna Elbert, "The X- and Y-functions for isotropic scattering. II," *Astrophys. Jn.*, v. 115, 1952, p. 269–278.

The functions in question are solutions of the integral equation

$$X(\mu) = 1 + \mu\omega_0 \int_0^1 (\mu + \mu')^{-1}[X(\mu)X(\mu') - Y(\mu)Y(\mu')]d\mu',$$

$$Y(\mu) = e^{-\tau/\mu} + \mu\omega_0 \int_0^1 (\mu - \mu')^{-1}[Y(\mu)X(\mu') - X(\mu)Y(\mu')]d\mu'.$$

Approximations were described in Chandrasekhar's book *Radiative Transfer*, Chapter VIII.

In the first of these papers the approximate solutions were improved by iteration and are tabulated for $0 \leq \mu \leq 1$ for the following values of the parameters: $\omega_0 = .95, .9, .8, .5$, $\tau = .05(.05).25, .5, 1$. The tables are to 4 or 5D, and the difference between the tabulated solution and the corrected second approximation (of Chandrasekhar's book) is also given.

In the second paper, "standard solutions" are defined in the case $\omega_0 = 1$. They satisfy the relations

$$\int_0^1 X_s(\mu)d\mu = 2, \quad \int_0^1 Y_s(\mu)d\mu = 0.$$

The moments of X, Y, X_s, Y_s are called α_n, β_n, α_n^s, β_n^s. The laws of diffuse reflection are expressed in terms of the solutions

$$X^*(\mu) = X_s(\mu) + Q\mu[X_s(\mu) + Y_s(\mu)],$$
$$Y^*(\mu) = Y_s(\mu) - Q\mu[X_s(\mu) + Y_s(\mu)],$$

where

$$Q = - \frac{\alpha_1{}^s - \beta_1{}^s}{(\alpha_1 + \beta_1)\tau + 2(\alpha_2 + \beta_2)}.$$

Table 1a gives α_0, β_0, $\alpha_0{}^*$, α_1, β_1, and

$$\bar{s} = 1 - \{(2 - \omega_0\alpha_0)\alpha_1 + \omega_0\beta_0\beta_1\}$$

for $\tau = .05(.05).25, .5, 1$ and $\omega_0 = .95, .90, .80, .5$. Table 1b gives $\alpha_0 + \beta_0$, $\alpha_1 + \beta_1$, $\alpha_2 + \beta_2$, $\alpha_1{}^s$, $\beta_1{}^s$, $(\alpha_1{}^s)^2 - (\beta_1{}^s)^2$, Q, and $\bar{s}$ for the same values of τ and $\omega_0 = 1$.

Table 2 gives X^*, Y^*, X_s, Y_s for $\omega_0 = 1$, $\tau = .05(.05).25, .5, 1$, $\mu = 0(.01)1$.

Table 3 contains certain combinations of the moments.

A. E.

1031[L].—W. Chester, "The reflection of a transient pulse by a parabolic cylinder and a paraboloid of revolution," *Quart. Jn. Mech. Appl. Math.*, v. 5, 1952, p. 196–205.

Table I (p. 204) gives 4D values of

$$4\pi^{-\frac{1}{2}} \int_0^\infty e^{-(\tau+1)r^2} \left\{ \frac{4}{\pi} \left[1 - re^{-r^2} \int_0^r e^{z^2} dz \right]^2 + r^2 e^{-2r^2} \right\}^{-1} dr$$

and of

$$\frac{1}{2} \int_0^\infty e^{-(\tau+1)r} \left\{ \left[1 + \tfrac{1}{2} re^{-r} \int_{-r}^\infty z^{-1} e^{-z} dz \right]^2 + \tfrac{1}{4}\pi^2 r^2 e^{-2r} \right\}^{-1} dr$$

for $\tau = 0, .1, .2(.2)2$. In the last expression the integral with respect to z is meant as a Cauchy principal value.

A. E.

1032[L].—G. Diemer & H. Dijkgraaf, "Langmuir's ξ, η tables for the exponential region of the $I_a - V_a$ characteristic," *Philips Res. Rep.*, v. 7, 1952, p. 45–53.

The integral $\int^\eta [f(x) + m^2]^{-\frac{1}{2}} dx$, where

$$f(x) = e^x(1 + \text{erf}\, x^{\frac{1}{2}}) - 1 - 2(x/\pi)^{\frac{1}{2}},$$

was tabulated by Freeman[1] in the range $0 \leq \eta \leq 20$, $0 \leq m^2 \leq 20$. Modern microwave diodes and triodes make it desirable to extend these tables considerably. The present paper contains 3S values for $10^{-2} \leq \eta \leq 60$, $0 \leq m^2 \leq 10^6$. The intervals vary. The two tables overlap and there are discrepancies in the overlapping regions.

A. E.

[1] J. J. Freeman, "Noise spectrum of a diode with a retarding field," NBS, *Jn. of Research*, v. 42, 1949, p. 75–88.

1033[L].—S. Goldstein, "On diffusion by discontinuous movements, and on the telegraph equation," *Quart. Jn. Mech. Appl. Math.*, v. 4, 1951, p. 129–156.

The $\gamma(n, \nu)$ may be defined by the recurrence relation $(r + 1)\gamma(n + 1, \nu) = r[\gamma(n, \nu - 1) + \gamma(n, \nu + 1)] - (r - 1)\gamma(n - 1, \nu)$, together with the initial conditions

$$\gamma(1, 1) = \gamma(1, -1) = \tfrac{1}{2}, \quad \gamma(1, \nu) = 0 \text{ if } \nu \neq \pm 1.$$

The author obtains the generating function of the $\gamma(n, \nu)$, also integral representations and an asymptotic expansion for n large and $\nu^2 = O(n)$. To test the asymptotic expansion, he tabulates (p. 137) 7D values of $\gamma(15, \nu)$ for $r = \frac{1}{2}, 1, 2, 5$ and $\nu = 1(2)15$.

A. E.

1034[L].—HARVARD UNIVERSITY, COMPUTATION LABORATORY, *Annals*, v. 23: *Tables of the error function and of its first twenty derivatives*. Cambridge, Mass., Harvard University Press, 1952, xxviii, 276 p., 19.5 × 26.7 cm. $8.00.

The functions tabulated in this volume are

$$\phi^{(-1)}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt, \quad \phi^{(n)}(x) = \frac{d^n}{dx^n}\left[\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right], \quad n = 0, 1, 2, \cdots.$$

The $\phi^{(n)}$ are connected with Hermite polynomials.

Tables of these functions are listed in section 15 of the FMR *Index*. Additional tables are mentioned in *MTAC*, v. 3, p. 521, and tables on punched cards are listed in *MTAC*, v. 5, p. 203, 204.

Modern developments in statistics, probability theory, mathematical physics, and noise and communications theory made new tables desirable over a more extensive range with a finer interval, and, most of all, including derivatives of higher orders. The authors of the present work expect that their tables will satisfy all present needs.

Tables I and II give 6D values for $\phi^{(-1)}$ to $\phi^{(10)}$, where x runs from 0 at intervals of .004 up to the point where ϕ ceases to change. The limiting value is .5 for $\phi^{(-1)}$, and zero for all other $\phi^{(n)}$. The last x increases with n, being 4.892 for $n = -1$, and 8.236 for $n = 10$.

Tables III and IV give values to 7S or 6D for $\phi^{(11)}$ to $\phi^{(20)}$, and x runs from 0 at intervals of .002 up to the point where ϕ vanishes to 6D identically. This point is 8.518 for $n = 11$, and 10.902 for $n = 20$.

The introduction contains a chapter (by WARREN L. SEMON) giving a collection of formulas and integral representations for $\phi^{(n)}$ and their connection with Hermite polynomials; a brief section (by the same author) explaining the computation of the tables and the steps taken to eliminate errors; a chapter (by DAVID MIDDLETON) on applications of the functions tabulated in this volume; and a 10D table of all the zeros of all the functions tabulated in this volume. Each section is accompanied by a list of references.

The tables maintain the high standards set by previous volumes of this series.

A. E.

1035[L].—TH. LAIBLE, "Höhenkarte des Fehlerintegrals," *Zeit. angew. Math. Physik*, v. 2, 1951, p. 484–486.

Relief diagrams of erf $(x + iy)$ for (i) $0 \leq x \leq 5$, $0 \leq y \leq 6$, and (ii) $0 \leq x \leq 1.7$, $0 \leq y \leq 2.2$. 4S values of the real and imaginary parts of the first 5 zeros (in the first quadrant) of erf (z).

A. E.

1036[L].—C.-B. LING, "Tables of values of the integrals $\int_0^\infty x^m dx/\sinh^p x$ and $\int_0^\infty x^m dx/\cosh^p x$," *Jn. Math. Phys.*, v. 31, 1952, p. 58–62.

This paper gives values of the above integrals multiplied by the auxiliary factor $p^{m+1}/2^p(m)!$ mostly to 5D for all admissible pairs of p and m, $p = 1(1)8$, $m = 0(1)15$. The reciprocal of the above factor is given exactly or to 6S for the same range. C. W. NELSON[1] has tabulated the first integral to 12D for $m = p = 1(1)40$. Entries common to both tables are in perfect agreement.

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[1] C. W. NELSON, "A Fourier integral solution for the plane-stress problem of a circular ring with concentrated radial loads," *Jn. Appl. Mech.*, v. 18, 1951, p. 173–182.

1037[L].—R. C. LOCK, "The velocity distribution in the laminar boundary layer between parallel streams," *Quart. Jn. Mech. Appl. Math.*, v. 4, 1951, p. 42–63.

The "standard solution," $g(\xi)$, of the Blasius equation $ff'' + 2f''' = 0$ is that solution which has the expansion

$$g(\xi) = -1 + e^{\frac{1}{2}\xi} - \tfrac{1}{4}e^{\xi} + \cdots$$

for $\xi \leq 0$.

Table II (p. 58) gives 4D values of g, g', g'' for $\xi = -\infty, -16(2) - 10(1) - 5(.4).2$, and $.5537 = \xi_0$, where $g(\xi_0) = 0$.

A second solution, $g_1(\xi)$, defined for $\xi > \xi_0$, depends on a parameter $\rho\mu$. It is a solution of the Blasius equation, subject to the initial conditions

$$g_1(\xi_0) = 0, \quad g_1'(\xi_0) = g'(\xi_0), \quad g_1''(\xi_0) = (\rho\mu)^{\frac{1}{2}} g''(\xi_0).$$

With $a = [g_1'(\infty)]^{-\frac{1}{2}}$, $b = \xi_0$, two further functions f_1 and f_2 are defined by

$$f_1(\eta_1) = a g_1(a\eta_1 + b), \quad f_2(\eta_2) = a g(a\eta_2 + b).$$

These are also solutions of the Blasius equation, $\eta_1 > 0$, $\eta_2 < 0$.

Tables III–VI (p. 59–62) give 4D values of $f_1, f_1', f_1'', f_2, f_2', f_2''$ for, respectively, $\rho\mu = 5.965 \times 10^4$, 100, 10, 1. η_1 and η_2 range over their respective ranges at intervals suited to the physical problem.

Table VII (p. 63) gives, for the case $\rho\mu = 1$, a similar tabulation of that solution for which $f_2'(-\infty) = .5014$ (instead of 0 as in the previous cases).

A. E.

1038[L].—B. MISHRA, "Wave functions for excited states of mercury and potassium," Cambridge Phil. Soc., *Proc.*, v. 48, 1952, p. 511–515.

The radial wave function $P(nl|r)$ for excited state (n, l) of Hg satisfies the differential equation

$$\left\{\frac{d^2}{dr^2} + \epsilon(nl) + \frac{2Z_p(r)}{r} - \frac{l(l+1)}{r^2}\right\} P(nl|r) = 0,$$

where r is the radial distance, $\epsilon(nl)$ is an energy parameter, and $Z_p(r)$ is the total effective nuclear charge. The wave function is normalized so that

$$\int_0^\infty [P(nl|r)]^2 dr = 1.$$

For the numerical integration the variables $\rho = \ln(1000\, r)$ and $S = r^{-\frac{1}{2}}P$ were used.

Table 1, p. 512, gives 3 to 5S values of Z_p and 4D values of S for the states $(6s)$, $(6p)$, $(6d)$, $(7s)$, $(7p)$, $(7d)$, for $\rho = 0(1/3)2(1/6)4(1/12)11\cdot25$. The values of Z_p were derived from existing tables for the neutral atom, and the wave functions found by numerical integration of the differential equation.

The differential equation for K contains an additional term $V_p(r) = -\frac{1}{2}\alpha(r^2 + r_0^2)^{-2}$. Table 3 gives 4D values of $P(4p|r)$ for $r = 0(.02).3(.05).6(.1)1.2(.2)3.2(.4)16(1)30$.

Tables 2 and 4 give the numerical values of certain integrals.

A. E.

1039[L].—A. Papoulis, "On the accumulation of errors in the numerical solution of differential equations," *Jn. Appl. Phys.*, v. 23, 1952, p. 173–176.

By solving a differential equation with various initial conditions the author exhibits a method for determining the error in the numerical solution of such equations. Consider the differential equation (†) $y' = f(x, y)$ and assume that the interval $[a, b]$ is divided into n subintervals by the points of subdivision x_k. Let β_k be the difference between the value of the correct solution at $x = x_k$ and the computed solution y_k at x_k. (β_k is the sum of truncation and round-off errors.) If we assume that $y(x)$ is correct up to x_{k-1} and from x_k to b satisfies (†), then at $x = b$ we shall have an error $E_k = y(b) - y_n$. Let $y(x)$ be a solution of (†) with the initial condition $y(x_0) = y_0$ and $Y(x)$ a solution of (†) with $Y(x_0) = y_0 + M$ (M a constant); then under the assumption (A): $r(x) = Y(x) - y(x)$ is small compared with $y(x)$, the author shows that the total error R is the sum of the E_k $(= \beta_k[r(b)/r(x_k)])$, and hence for n large,

$$R = [nr(b)/(b-a)] \int_a^b [\beta(x)/r(x)]dx.$$

Under assumption (A), $f(x, Y(x)) - f(x, y(x)) \doteq r(x)\varphi(x)$, where $\varphi(x) = \partial f(x, y(x))/\partial x$ and hence $r(x)$ is determined from the differential equation $r'(x) = \varphi(x)r(x)$. If $\beta(x)$ is the result of round-off errors alone and if they are random with variance d, then the dispersion of R is

$$[d^2nr^2(b)/(b-a)] \int_a^b [1/r^2(x)]dx.$$

Extensions to systems of two equations, $dx/dt = f(x, y, t)$, $dy/dt = g(x, y, t)$, are given in detail with some remarks on systems of higher order.

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1040[L].—A. van Wijngaarden, *Table of the integral*

$$\int_0^1 \exp(-v^{-2} - xv)v^{-p}dv.$$

Mathematical Centre, Amsterdam, *Report* R 176, 1952, 6 mimeographed leaves.

The integral mentioned in the title is given to 5D ("the last decimal is not absolutely reliable") for

$$x = 0(.25)3.5(.5)14.5,$$
$$p = 0(1)10.$$

For $x = 0$ the integral becomes

$$\tfrac{1}{2}\int_1^\infty t^{(p-3)/2}e^{-t}dt,$$

an incomplete gamma function. A special table for $x = 0$ to 6D is given for $p = -25(1)11$. Several methods were used to compute the tables including power series and recurrence relations resulting from integration by parts.

D. H. L.

1041[L].—E. M. Wilson, *Solutions of the Equations* $(y'')^2 = yy'$ *and Two Other Equations*, Admiralty Research Laboratory, Teddington, Middlesex, November, 1951.

The first part of this report is concerned with the solution of $y'' = -\sqrt{yy'}$ such that $y(0) = 0$, $y'(0) = 1$. Solution depends on the inversion of the integral $x = \int_0^y (1 - t^{\frac{3}{2}})^{-\frac{2}{3}}dt$, where x is the independent variable. A generalization of this integral has been studied by R. Grammel (*MTAC*, v. 5, p. 155). When x approaches $4\pi\sqrt{3}/9$, y approaches unity while y' and y'' approach zero. Starting at this point the function tabulated is the solution of $y'' = +\sqrt{yy'}$ which admits of an integral representation similar to the previous one. Series solutions are also given. A photostat table is available giving y to 6D for $x = 0(.002)6$ and $\log y$ to 6D for $x = 6(.01)7$. The report tabulates y to 6D for $x = 0(.05).5(.1)6$. Author believes maximum error is less than 0.7 unit in the last figure given. Second differences, mostly modified, are also tabulated. To facilitate interpolation near the origin, $y = 4x^{\frac{5}{2}}/15$ is tabulated to 6D for $x = 0(.05).25$.

In the second part, values of the integral

$$f(\beta, \rho) = (2e^{-\rho^2}/\beta^2\sqrt{\pi}) \int_0^\beta I_0(2\rho\eta)e^{-\eta^2}\eta d\eta$$

are tabulated to 4D for $\beta = 0(.25)4$, $\rho = 0(.25)5$. I_0 is the modified Bessel function. The author claims entries are correct to within one unit of the last figure. The table has been subtabulated so that entries are in intervals of .05 and is available in photostat form. A more extensive tabulation of a function simply related to the above has been made by S. R. Brinkley, Jr. & R. F. Brinkley (*MTAC*, v. 2, p. 221) and S. R. Brinkley, Jr., H. E. Edwards & R. W. Smith, Jr. (*MTAC*, v. 6, p. 40).

Part three tabulates to 3D that zero of $u \sin x - \cos x + e^{-ux}$ which lies between π and 2π for $u = .1(.01).3(.02)2$ and $\sqrt{u} = 0(.02).5$. The author states that the last figure should be correct to within 0.7 of a unit. Linear interpolation yields full accuracy and first differences are provided.

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MATHEMATICAL TABLES—ERRATA

In this issue references have been made to errata in RMT **1022**, **1032**.

214.—E. P. ADAMS & R. L. HIPPISLEY, *Tables of Elliptic Functions*, *Smithsonian Miscellaneous Collections*, v. 74, no. 1, Washington 1939, 1947.

The heading of page 294

for $E' = 1.5629622295$
read $E' = 1.5631622295$.

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215.—(1) A. A. GERSHUN, "Berechnung des Volumleuchtens," *Physikalische Z. d. Sowjetunion*, v. 2, 1932, p. 149–185 [*MTAC*, v. 2, p. 191].

(2) E. SCHMIDT, "Die Berechnung der Strahlung von Gasräumen," *Zs. Verein Deutscher Ingenieure*, v. 77, 1933, p. 1162–1164.

(3) S. GOLDSTEIN, "On the vortex theory of screw propellers," Roy. Soc., London, *Proc.*, v. 123A, 1929, p. 440–465.

Three tables are given in (1), on p. 172, 175, and 180, respectively. Table I, containing the function

$$F_1(x) = 1 - (1 - x)e^{-x} + x^2 \operatorname{Ei}(-x) = 1 - 2x^2 \int_x^\infty \frac{e^{-t}}{t^3}\, dt$$

to 4D for $x = 0(.01).02$, .05, .1, .2(.2)1, 1.6(.4)2.4, 3, 4, was read against the same function given in (2) in complementary form on p. 1163, also to 4D mainly, for $x = 0(.01).02$, .05, .1(.1)1.0(.2)2.0, 2.4, 2.5, 3(1)5. The discrepancies, and the extra values given in (2), were checked, revealing the following errors in (1) and (2):

	F_1	*for*	*read*
(1)	0.4	.4925	.4854
	3.0	.9822	.9821
(2)	0.3	.6000	.6001
	0.9	.2516	.2514
	1.2	.1680	.1679
	1.4	.1296	.1292
	1.6	.1011	.0998
	1.8	.0777	.0774
	2.5	.0328	.0326
	4.0	.00545	.00552
	5.0	.00175	.00176