# OTHER AIDS TO COMPUTATION

ANALOGUE CALCULATION OF POLYNOMIAL AND TRIGONOMETRIC EXPANSIONS

Introduction: The use of polynomials and trigonometric expansions in science and engineering is well known, and hence the importance of machine application for their rapid evaluation does not need strong defense. In fact, the multiplicity of special purpose polynomial computers that have been designed and built points to the great desire for machine treatment of these functions. Although the method described in this paper is applicable to digital as well as analogue machines, the original intent of the authors was slanted towards the use of the analogue machine, and hence the presentation has an analogue machine background. The significance and importance of the new method is its simplicity and its use of standard computing machine equipment.

The Vector Multiplier: The key process which leads to the new method is the automatic multiplication of an arbitrary pair of vectors. The present paper will be limited to the treatment of the complex vector $(a + ib)$. The product of two such vectors, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, may be written

$$(1) \qquad z_3 = z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1),$$

and the operation performed by machine as shown in Figure 1.
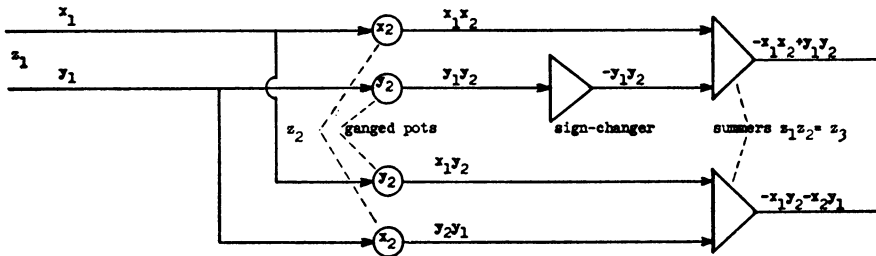
$$z_1 = x_1 + iy_1$$
$$z_2 = x_2 + iy_2$$
$$z_3 = z_1z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

FIG. 1. The Vector Multiplier.

In terms of operation on either the Reeves REAC or the comparable Goodyear Electronic Machine, x_1 and y_1 in the figure represent input voltages to a pair of ganged potentiometers or a set of four simple potentiometers (hereinafter referred to as pots) set respectively at x_2 and y_2. The outputs of these pots are, except for sign, the four terms on the right of equation (1). The term y_1y_2 is put through a sign changer, and then the terms are put into a pair of summers to obtain outputs which are the negatives of the real and imaginary parts of z_3. The amplifiers used on the machines referred to automatically change the sign of their summed inputs and hence

we find the complex vector multiplier described above giving an output vector which is the negative of the product z_3. This is in no way an impediment in the operation since many operations will call for sign changes, and hence, on the average the same sign change equipment would be needed even if the vector product operation produced the product without sign change. In what follows we shall call the input z_1 the multiplicand and the pot setting z_2 the multiplier.

Polynomials in the complex variable z with real coefficients: It is now fairly obvious how successive application of the vector multiplication will produce the successive powers z^2, z^3, z^4, $\cdots$, z^m. Figure 2 shows a part of the circuit diagram for a fifth degree polynomial. Unit voltages enter at the left and the successive output voltages representing respectively z, $- z^2$, z^3, $- z^4$ and z^5 are shown by the vertical output lines. If ganged pots are avail-
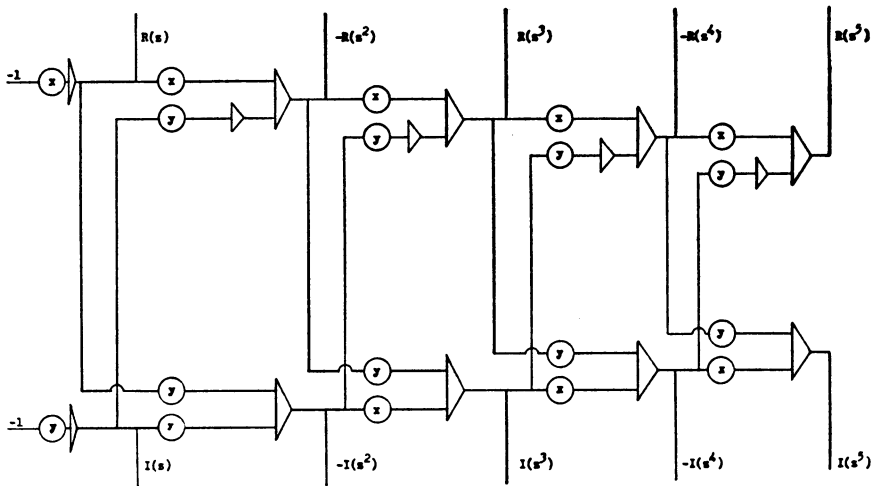

FIG. 2. Circuit which produces the real and imaginary parts of z, $- z^2$, z^3, $- z^4$ and z^5.
$$(z = x + iy)$$

able, then the x pots and the y pots are set simultaneously by two or more operations depending on the multiplicity of pots per control. Of course the servo ganged pots usually provided with analogue calculators may be used for the ganged pot operation. Figure 3 shows the servo-pot diagram equivalent of Figure 2, with additional sign changers to provide the powers of z with both signs.

The outputs $R(z)$, $- R(z^2)$, $R(z^3)$, $- R(z^4)$ and $R(z^5)$, i.e., the real parts of the successive powers indicated, are respectively put through a set of pots which multiply them by the polynomial coefficients a_1, a_2, a_3, $\cdots$, a_5. The outputs of the coefficient pots, including a pot representing the polynomial constant, with appropriate signs, then go into a summing amplifier whose output defines the real part of the polynomial or its negative depending upon the computing requirements. A comparable process is used to calculate the imaginary part of the polynomial.

Polynomials in z with complex coefficients: In this instance it is only a matter of making vector multiplications of the coefficients a_m into the

corresponding vector terms $R(z^m) + iI(z^m)$ in place of using simple pot multiplication. The summing amplifiers used in the latter multiplications may of course be used for all or part of such multiplications depending upon the amplifiers input facilities. The real and imaginary parts of the polynomial may be projected on the screen of the cathode ray tube so that a visual display of the calculated points becomes available. Through this display one discovers rather quickly characteristics of the polynomial under study such as dominant terms and zeros.
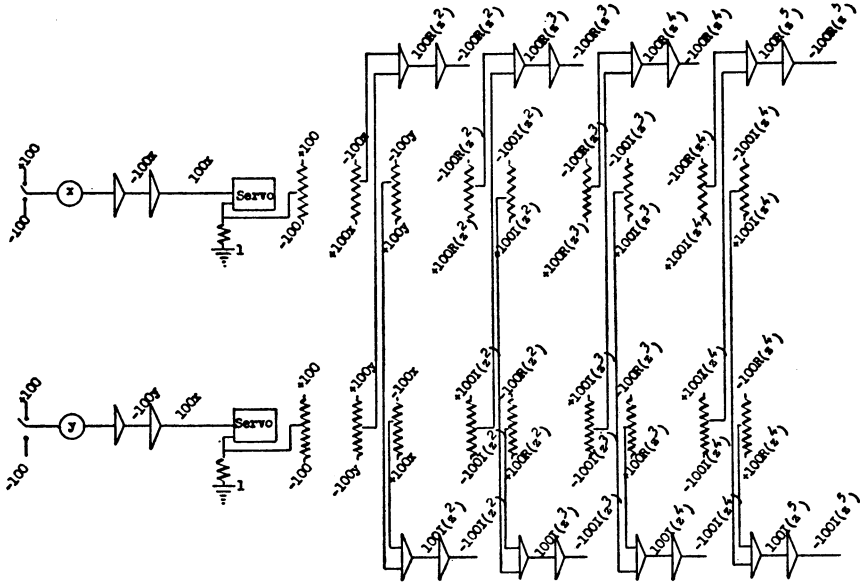

FIG. 3. Circuit in which the potentiometers of Fig. 2 have been replaced by servo multipliers.

Trigonometric forms in one variable: Consider forms reducible to

$$(2) \quad T(\theta) = a_0 + a_1 \cos \theta + a_2 \cos 2\theta + \cdots + a_n \cos n\theta$$
$$+ b_1 \sin \theta + b_2 \sin 2\theta + \cdots + b_m \sin m\theta$$

in which $a_1, a_2, \cdots, a_n, b, b_2, \cdots, b_m$ are constants and m, n are integers with $m \geq n$. The form (2) is the real part of the following polynomial:

$$(3) \quad a_0 + (a_1 - ib_1)z + (a_2 - ib_2)z^2 + \cdots + (a_r - ib_r)z^r$$
$$+ \cdots + (a_m - ib_m)z^m,$$

in which $a_{n+1}, a_{n+2}, \cdots, a_m$ are all zero. To evaluate the trigonometric form for any specific range of θ one has but to evaluate the polynomial (3) over an x, y range corresponding to $\cos \theta, \sin \theta$.

The resolver servos available on the machines referred to would facilitate such treatment of the independent variables. In any case, the use of the sine and cosine columns in the trig table would make it relatively simple to range the variable appropriately. The treatment of mixed expansions of terms of the type $r^n \cos n\theta$, $r^n \sin n\theta$ is obvious.

Polynomials in Several Variables: We consider first a polynomial $P(z_1, z_2)$ in two variables (z_1, z_2). In this case the successive powers in z_1 and z_2 are respectively formed by the method described above and the required cross products such as $z_1{}^n$, $z_2{}^n$ are formed by the required vector multiplication. The vector cross product terms are then multiplied by the appropriate scalar or vector coefficient and the results fed into summing amplifiers to form the real and imaginary parts of $P(z_1, z_2)$. Treatment of the more general case is obvious. In the case of polymonials in several variables this machine calculation becomes very useful for obtaining exploratory information about the characteristics of such functions.

Polynomial transformations, approximations, zeros: When a polynomial has been coded on a machine, it becomes possible to obtain information by either manipulating the coefficient pots or the z pots or both. By using cathode ray screens and function tables one manipulates the coefficients to determine a transformation to take a contour C in the z plane into a contour C' in a P plane or one finds a polynomial approximation to a given contour C' in a P plane. Again one manipulates the x, y pots and determines a set of points in the z plane corresponding to a point p in a P plane, in other words one solves the equation $p = P(z)$ for a given value of p.

A systematic process for treatment of the latter problem is known in the literature[1,2] and will here be outlined briefly for the sake of completeness. The brief statement will enable most readers to apply the process without referring to the literature. We have the theorem that a closed contour C in the z plane is mapped by a polynomial into a closed contour C' enclosing a given point p if and only if C contains at least one of the points z_r satisfying the equation $p = P(z_r)$. One finds a rectangle C containing a point z_r and then gradually contracts it by systematic trial and error until z_r is located with the required accuracy or within the accuracy of the machine. Further accuracy may be obtained by the complementary use of a hand digital machine by such method as found in MILNE.[3]

It must be pointed out again that although the machine coding has been described in terms of ganged pots the coding may also be done with simple pots when ganged pots are not available. In fact the example cited below was worked out with and without ganged pots. With simple (x, y) pots all of which must be reset for each change in z, the treatment of the problem is somewhat longer. It was observed that some of the simple x, y pots dominate the changes in the polynomial and with this experience it did not take long to drive the polynomial values in the required direction.

Example: The polynomial equation

$$P(z) = z^5 - z^4 - 3z^3 + z^2 + 12z + 5 = 0$$

was solved on both Reeves and Goodyear Analogue Machines. One root was located in the rectangle whose vertices were $(0, 0)$, $(2, 0)$, $(2, 2i)$ and $(0, 2i)$. The rectangle was systematically contracted until the root was located with an optimum of accuracy. The polynomial was then reflected in the origin and a pair of roots located in the same rectangle. Since one of these was clearly a real root it was located with an optimum of accuracy by an obvious systematic search on the real segment of the rectangle. The first contraction of this rectangle which left out the axis of reals gave a con-

tour in the P plane which simply inclosed the origin. It was then a matter of further contractions to obtain the best estimate of the root. The table below gives a comparison of the roots as calculated by both analogue and digital methods.

| Analogue Method | − .454 | 1.94 ± 960i | − 1.21 ± .954i |
| Digital Method | − .4517 | 1.935 ± .9466i | − 1.216 ± .9595i |

Max G. Scherberg
John F. Riordan

Wright-Patterson Air Force Base
Ohio

This paper was written at the Flight Research Laboratory, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio.

[1] L. Bauer & S. Fifer, "The solution of polynomial equations on the REAC," *Symposium I on REAC Techniques*. U. S. Navy Special Devices Center and Reeves Instrument Corporation, New York, 1951, p. 31–36.

[2] E. C. Titchmarsh, *Theory of Functions*. Oxford University Press, 1939, Second Edition, p. 116.

[3] W. E. Milne, *Numerical Calculus*. Princeton University Press, 1949, p. 53–57.

BIBLIOGRAPHY Z

1010. J. Euler & R. Ludwig, "Zwei Nomogramme zum Gebrauch bei Messungen mit optischen Pyrometern," *Zeitschrift f. angew Physik*, v. 2, 1950, p. 362–373.

The nomograms involve two-dimensional plotting for the inputs.

F. J. M.

1011. H. Glubrecht, "Elektrisches Rechengerät für Gleichungen höheren Grades," *Zeitschrift f. angew. Physik*, v. 2, 1950, p. 1–8.

The apparatus is set up to represent both the z and w plane and the roots of the polynomial are located by exploration. The null point of the polynomial is obtained by means of a special cathode ray tube arranged so that when both the real and imaginary parts of w are zero an electron beam goes through a narrow hole and signals the presence of a zero. As far as practical, quantities are represented by voltages having complex significance, thus multiplication is by means of a mixing tube and the resolution into real and imaginary parts is accomplished by a passive network associated with the ordinary type of cathode ray tube upon which z and w are represented. Normally a circle is mapped in the z plane and its w image also is given. The amplitude of this circle can be either varied by hand or swept automatically. The article discusses also two other apparatuses of this type due to Tischner and Rasch.

F. J. M.

1012. E. H. Gamble & B. W. Hatten, "Design and analysis of a conservative dynamic load simulator," *Jn. Appl. Phys.*, v. 22, 1951, p. 1250–1257.

The design of a load simulator for the dynamic testing of actual control-system components such as aircraft autopilots is described. The use of an electrical computer for determining the load and a hydraulic servomecha-

nism for actually applying the load to the test unit results in greater flexibility than a load simulator in which the loading is determined mechanically. A feature of this simulator which should prove of great importance in the testing of large servomechanisms is the fact that virtually all the energy absorbed from the unit being tested is returned to the electrical lines and need not be dissipated as heat.

RICHARD C. BOOTON, JR.

Massachusetts Institute of Technology
Cambridge 39, Massachusetts

1013. G. A. KORN & T. M. KORN, *Electronic analog computers* (D-c *analog computers*). McGraw-Hill Book Co., New York, 1952, xv + 378 p., 15 × 23 cm. Price $7.00

This book is concerned with d.c. analog equipment of the type manufactured by Reeves, Goodyear and Boeing and to a certain extent with that of Philbrick. There is also a clear intention to indicate how less expensive versions of these machines may be set up for special purposes. Two special installations, the Curtiss Wright Analog Computer and the RAND analog facility, are discussed.

The first three chapters discuss the principles of these devices, practical set up procedure and a variety of specific applications. The fourth chapter is concerned with the theory of linear computing networks and specifically the necessity for using amplifiers and the effect of this use on the solution. The fifth chapter discusses the design of d.c. amplifiers for computer applications. Various simple types of such amplifiers are discussed as well as the modern drift stabilized amplifiers as developed by WILLIAMS, TARPLEY & CLARK, and the GOLDBERG amplifier. The titles of Chapters 6, 7 and 8 are, respectively, "Multipliction and Function Generation," "Auxiliary Circuits and Computer Operation," and "The Design of Complete D-c Analog-computer Installations."

The case for the use of these computers is skillfully presented and the authors' concern with the accuracy of individual components is admirable. However, the book discusses only briefly the possibilities for large network computers and presumably many other types of analogs were deliberately excluded from the scope, for instance, a.c. analogs, continuous analogs such as electrolytic tanks and conducting sheets, and the many types of stress and strain analogs.

F. J. M.

1014. M. TRIBUS, "Intermittent heating for aircraft ice protection with application to propellers and jet engines," ASME *Trans.*, v. 73, 1951, p. 1117–1130.

The author uses an electrical analog computer for the study of deicing of airplane parts. Thermally speaking, the problem is that of an idealized fin in transient heat flow. Heat is generated in the body which is to be deiced, for example the propeller, by means of electric heating wires; the heat is lost to the surrounding environment across a boundary conductance. The paper deals in considerable detail with the problems of deicing but mentions only

very briefly the computer used. No reference is made to the large amount of literature on the paper.

As far as computing technique goes the main point of interest is an electronic circuit used to represent the boundary conductance which is not constant but rather a function of temperature. This nonlinear condition has heretofore on other computers been represented in discrete finite steps. The author shows in figure 8 a circuit for representation of such boundary conductance. The check of the computations with actual experiments is rather unsatisfactory possibly because of poor assumption of physical constants of the system; the correction of assumptions determines in analog computers the validity of the result. The author, in designing the computing circuit, disregards a number of influences (for example the thermal resistance of the ice).

VICTOR PASCHKIS

Heat and Mass Flow Analyzer Laboratory
School of Engineering
Columbia University, New York, N. Y.

1015. F. WEINER, "Further remarks on intermittent heating for aircraft ice protection," ASME *Trans.*, v. 73, 1951, p. 1131–1137.

The paper deals with the deicing of a propeller of ovoid cross section. The butt end of this ovoid will be called A, the extension, B. Heat is generated by electric heaters only in the region A and is conducted to the region B. A cross section through the propeller in the region A shows (proceeding inwards) the steel blade (extending all the way to B), a layer of nylon, the heater, another layer of nylon, and finally sponge rubber. Disregarding the heat flow along the propeller length, the problem is two dimensional.

The greatest problem in solving two dimensional problems on analog computers, one on which little information is available, is that of how to section the body in which heat flow occurs. The authors disregard the thermal resistance across the thickness of the steel, and along the length of the nylon and heater layers. The sponge rubber is represented by a number of parallel sections, ending in a fictitious center with zero volume (Fig. 3 of the paper). This design is not discussed or analyzed; thus, the most crucial problem, from a computational view-point, is not dealt with in the paper. Regarding deicing, the paper shows the desirability of using high rates of energy production, which results in a lower total heat consumption than heating at a lower rate.

VICTOR PASCHKIS

Heat and Mass Flow Analyzer Laboratory
School of Engineering
Columbia University, New York, N. Y.

NOTES

143.—ANALYTICAL APPROXIMATIONS. [EDITORIAL NOTE: In Note **139,** *MTAC,* v. 6, p. 251–253, CECIL HASTINGS has described the RAND *Collection of Illustrative Approximations.* The interest that these approximations have aroused during the past year is considerable. It is hoped to publish as Notes