The Zeros of the Partial Sums of e^z

The location of the zeros of certain entire functions has received considerable attention in the literature, and it was suggested by R. S. VARGA that a table of the zeros of certain truncated power series might be of interest. Accordingly, the zeros of the truncated exponential series $S_n(z) = \sum_0^n \frac{z^k}{k!}$ have been computed for values of n up to 23. They appear in the accompanying table with the complex zeros ordered as to modulus and with the real zero printed last.

Table of Zeros of $S_n(z)$

$n = 2$

−1.0000 0000 0000 ±	1.0000 0000 0000i

$n = 3$

−0.7019 6418 1008 ±	1.8073 3949 4452i
−1.5960 7163 7983	

$n = 4$

−1.7294 4423 1067 ±	0.8889 7437 6121i
−0.2705 5576 8932 ±	2.5047 7590 4362i

$n = 5$

−1.6495 0283 1735 ±	1.6939 3340 4349i
0.2398 0639 3753 ±	3.1283 3502 5970i
−2.1806 0712 4035	

$n = 6$

−2.3618 1018 0482 ±	0.8383 5027 7917i
−1.4418 0139 0549 ±	2.4345 2268 1808i
0.8036 1157 1031 ±	3.6977 0175 3629i

$n = 7$

−2.3798 8388 3168 ±	1.6289 9897 6372i
−1.1472 0068 9937 ±	3.1240 3923 8058i
1.4065 8592 8087 ±	4.2250 6684 4949i
−2.7590 0270 9962	

$n = 8$

−2.9645 9950 5160 ±.	0.8088 7832 7313i
−2.2864 2928 4171 ±	2.3777 1166 7793i
−0.7887 9362 0387 ±	3.7718 1078 3950i
2.0398 2240 9719 ±	4.7186 1488 3923i

$n = 9$

−3.0386 4807 2936 ±	1.5868 0119 5758i
−2.1108 3980 0302 ±	3.0899 1092 8725i
−0.3810 6984 5663 ±	4.3846 4453 3145i
2.6973 3346 1536 ±	5.1841 6206 2649i
−3.3335 5148 5269	

$n = 10$

−3.5538 7599 3928 ±	0.7894 2208 2895i
−3.0155 3577 0425 ±	2.3352 2385 7750i
−1.8716 6001 0419 ±	3.7701 9023 1409i
0.0662 0154 6301 ±	4.9676 7937 0404i
3.3748 7022 8472 ±	5.6260 2017 9698i

$n = 11$

−3.6641 5160 2429 ±	1.5570 4385 7798i
−2.9170 5082 0338 ±	3.0563 7182 9831i
−1.5814 4081 0406 ±	4.4223 5472 7709i
0.5460 7815 8920 ±	5.5249 0494 6051i
4.0692 9095 3063 ±	6.0474 9237 9889i
−3.9054 5175 7616	

$n = 12$

−4.1356 0823 9264 ±	0.7755 4204 9110i
−3.6888 9710 2446 ±	2.3027 5714 9247i
−2.7579 8923 1191 ±	3.7525 4838 2353i
−1.2491 2514 3358 ±	5.0495 5510 7284i
1.0534 2363 9656 ±	6.0594 9143 3864i
4.7781 9607 6918 ±	6.4511 7633 7446i

$n = 13$

−4.2712 4352 2040 ±	1.5348 5553 2416i
−3.6448 0690 0313 ±	3.0273 4405 4885i
−2.5489 2149 0908 ±	4.4259 0791 2207i
−0.8813 4146 1503 ±	5.6544 1733 8744i
1.5843 1494 9756 ±	6.5740 0727 9114i
5.4997 0440 2346 ±	6.8391 5923 4366i
−4.4754 1195 4676	

$n = 14$

−4.7125 8682 7652 ±	0.7651 0266 4661i
−4.3307 0823 3773 ±	2.2772 3593 3932i
−3.5438 3446 4412 ±	3.7319 2259 2644i
−2.2976 9841 1869 ±	5.0783 9242 5805i
−0.4831 5856 3940 ±	6.2391 4928 3405i
2.1356 7746 7731 ±	7.0705 6793 5765i
6.2323 0903 3917 ±	7.2131 4836 1604i

$n = 15$

−4.8669 6227 5703 ±	1.5176 2913 7005i
−4.3271 6518 2166 ±	3.0027 7099 6726i
−3.3948 7438 4748 ±	4.4177 0410 7602i
−2.0103 3299 7335 ±	5.7117 2749 1737i
−0.0585 5212 5116 ±	6.8056 2430 6791i
2.7050 4956 8386 ±	7.5509 4048 4655i
6.9747 8093 2988 ±	7.5745 6159 4666i
−5.0438 8707 2612	

Table of Zeros of $S_n(z)$—*Continued*

$n = 16$

−5.2863 1780 3783	±	0.7569 4362 0482i
−4.9527 8138 2083	±	2.2566 6238 4481i
−4.2704 2428 9874	±	3.7119 3355 5596i
−3.2047 4966 4526	±	5.0858 8483 8575i
−1.6915 4717 7729	±	6.3274 3910 8059i
0.3892 9335 7090	±	7.3554 4605 9723i
3.2904 2476 4254	±	8.0166 1911 0269i
7.7261 0219 6653	±	7.9245 9187 5073i

$n = 17$

−5.4551 0328 9602	±	1.5038 3976 7334i
−4.9806 7273 2968	±	2.9819 4653 1042i
−4.1680 2106 9422	±	4.4053 7601 5639i
−2.9788 2514 8596	±	5.7375 9875 5455i
−1.3451 2663 7403	±	6.9268 7649 6075i
0.8577 8157 7290	±	7.8899 9871 2830i
3.8901 4238 0621	±	8.4688 8076 9632i
8.4854 1859 0153	±	8.2642 5429 2176i
−5.6111 8734 0141		

$n = 18$

−5.8576 9828 4668	±	0.7503 7782 8924i
−5.5616 1575 6798	±	2.2397 2181 6511i
−4.9588 1020 0174	±	3.6935 9122 5275i
−4.0258 8765 2371	±	5.0838 2295 5316i
−2.7214 0644 5610	±	6.3738 9942 0060i
−0.9741 5991 2999	±	7.5112 3498 4979i
1.3447 6355 9230	±	8.4104 8615 4221i
4.5028 0973 4285	±	8.9088 2705 4271i
9.2520 0495 9109	±	8.5944 2118 7876i

$n = 19$

−6.0379 0688 9211	±	1.4925 3401 9264i
−5.6146 0932 9746	±	2.9641 6550 4574i
−4.8936 3517 6394	±	4.3919 0787 8943i
−3.8487 8978 9420	±	5.7480 1406 1058i
−2.4360 0924 7000	±	6.9957 5579 1547i
−0.5812 0465 9773	±	8.0815 7686 4150i
1.8484 3722 1795	±	8.9179 6283 4128i
5.1272 4541 2251	±	9.3374 1621 1924i
10.0252 3949 6630	±	8.9158 4881 4122i
−6.1775 3407 8278	±	

$n = 20$

−6.4273 0264 0861	±	0.7449 7139 4210i
−6.1611 0097 0908	±	2.2255 2308 7417i
−5.6209 8880 2007	±	3.6770 7242 0762i
−4.7903 2735 9920	±	5.0773 1588 8906i
−3.6406 2228 5764	±	6.3986 7391 1969i
−2.1255 4693 1251	±	7.6040 5463 9680i
−0.1684 0640 7437	±	8.6388 4961 0732i
2.3672 7408 2252	±	9.4133 5826 0670i
5.7624 3706 9983	±	9.7554 8801 5501i
10.8045 8424 5908	±	9.2291 9790 4677i

$n = 21$

−6.6167 8934 1171	±	1.4830 8180 5631i
−6.2346 1169 3725	±	2.9488 4393 5502i
−5.5863 1103 9618	±	4.3785 4166 8723i
−4.6531 2333 7897	±	5.7502 8769 4696i
−3.4046 1507 1622	±	7.0364 9649 6532i
−1.7924 6476 9782	±	8.1996 0590 2952i
0.2624 1464 0116	±	9.1839 0151 8883i
2.8999 6466 8596	±	9.8974 9663 3020i
6.4075 0945 9944	±	10.1637 8334 0590i
11.5895 7008 7260	±	9.5350 4977 9337i
−6.7430 9089 3669		

$n = 22$

−6.9955 1940 2365	±	0.7404 3625 9274i
−6.7537 1576 2515	±	2.2134 4348 1101i
−6.2643 4318 6033	±	3.6622 7895 1044i
−5.5151 2445 5329	±	5.0687 9151 8941i
−4.4855 3901 9504	±	6.4113 1616 4046i
−3.1434 8080 8797	±	7.6621 3795 8098i
−1.4388 3815 7314	±	8.7831 4914 1824i
0.7096 9887 5451	±	9.7174 9498 5677i
3.4453 7712 5688	±	10.3711 1273 0512i
7.0616 9981 9937	±	10.5629 5976 6282i
12.3797 8501 1070	±	9.8339 1911 3179i

$n = 23$

−7.1926 8590 7451	±	1.4750 5919 5082i
−6.8443 1411 8665	±	2.9355 2075 2798i
−6.2551 3130 4907	±	4.3657 9349 6476i
−5.4112 0644 7035	±	5.7481 1045 3530i
−4.2905 2911 0822	±	7.0608 9517 9930i
−2.8595 2302 8914	±	8.2762 1397 0162i
−1.0664 4595 7935	±	9.3553 6015 8909i
1.1720 9815 7227	±	10.2403 1777 0398i
4.0025 2534 8326	±	10.8348 6485 1671i
7.7243 3865 4741	±	10.9536 0414 4523i
13.1748 6483 8907	±	10.1262 6417 8727i
−7.3079 8221 4646		

The work was done on the Harvard Mark IV Calculator[1] using the quadratic factor method[2] together with a routine which supplied initial approximations chosen on a rectangular mesh in the complex plane. Mark IV operates with a fixed decimal point and carries sixteen decimal digits, but to reduce round-off errors a floating point routine was used. As a final check, the sum and product of the roots of each polynomial were compared with the appropriate coefficients of $S_n(z)$, and agreement to twelve significant digits was obtained for all $n < 21$.

The plot of the zeros in Fig. 1 exhibits the regularity of the family of curves joining the n zeros of a given $S_n(z)$. The broken curves which join zeros of the same rank (when ordered as to modulus for each n) have immediate application in that the zeros of the partial sum $S_n(z)$ may be located approximately from a knowledge of the zeros of the partial sums of lower order. This could be used to provide good first approximations in extending the table.

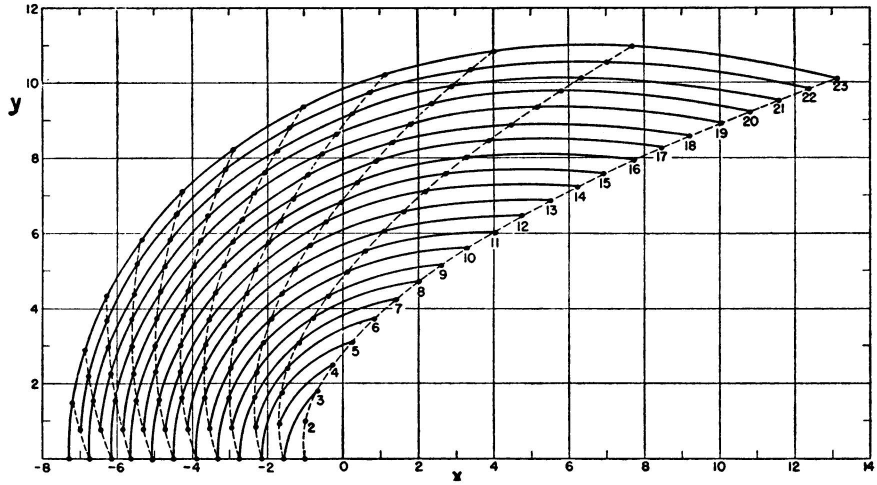

FIG. 1.

It is interesting to consider the present numerical results in the light of the following known properties of $S_n(z)$:

(1) $\lim\limits_{n\to\infty} \dfrac{r_n}{n} = \dfrac{1}{2} - \dfrac{1}{e\pi}$ where r_n is the number of zeros of $S_n(z)$ lying in the right half-plane.[3]

(2) $S_{2n}(z)$ and $S_{2n+1}(z)$ each have $2n$ complex zeros.[4]

(3) The semi-infinite strip[5] $|y| < \sqrt{6}$, $x > 0$, contains no zeros of any $S_n(z)$. Figure 1 suggests that much larger zero-free regions exist.

(4) Every zero of $S_n(z)$ satisfies the inequality[6]

$$n > |z| > \frac{n}{e^2}.$$

(5) The zero of smallest modulus of $S_{2n+1}(z)$ is real and negative. An unpublished proof has been given by D. J. NEWMAN.

(6) The convex polygon formed by the zeros of $S_n(z)$ encloses all the zeros of the partial sums of lower order. This is based on the fact that $S_n'(z) = S_{n-1}(z)$ and on the following theorem:[7] Any convex

polygon which contains all the zeros of a polynomial $p(z)$ also contains all the zeros of the derivative $p'(z)$.

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[1] *A Description of the Mark IV Calculator.* Computation Laboratory of Harvard University, *Annals*, v. 28 (In preparation).

[2] W. E. Milne, *Numerical Calculus.* Princeton 1949, p. 53; or D. R. Hartree, *Numerical Analysis.* Oxford 1952, p. 205.

[3] G. Szegö, "Über eine Eigenschaft der Exponentialreihe," Berliner Mathematischen Gesellschaft, *Sitzungsberichte*, 1924, p. 50–64.

[4] J. Berghuis, "Truncated power series," Mathematical Centre, Amsterdam, *Report* R173, 1952.

[5] R. S. Varga, "Semi-infinite and infinite strips free of zeros," Universita e Politecnico di Torino, Seminario Matematico, *Rendiconti*, v. 11, 1951–1952, p. 289.

[6] K. S. K. Iyengar, "A note on the zeros of $\sum_0^n \frac{x^r}{r!}$." *Mathematics Student*, v. 6, 1938, p. 77.

[7] M. Marden, *The Geometry of the Zeros of a Polynomial in a Complex Variable.* American Mathematical Society. New York, 1949.

RECENT MATHEMATICAL TABLES

1094[B].—Marchant Calculating Machine Co. *Table No. 80 of Factors for 8-Place Square Roots*, 1951, 4 p.; *Table No. 81 ··· for 6-Place Roots*, 1952, 4 p.; and *Table No. 82 ··· for 5-Place Roots*, 1952, 2 p. *Publications* of Marchant Calculators, Inc., Oakland, California. 21.5 × 28 cm.

The tables for 5- and 6-place roots resemble a former one for 5-place roots of the same publisher (*MTAC*, v. 1, p. 356). As in the former table the root results from adding a tabular number to the number N of which root is desired, and dividing this sum by an adjacent tabular number. The table for 8-place roots requires two divisions but without need of intermediate copying. Mathematically the table for 8-place roots is equivalent to a similar two-division table reported in *MTAC*, v. 5, p. 180.

Doubtless the most welcome of these tables will be the one for 6-place roots because this number of places is specified by the majority of work sheets for general surveying and military uses. There has not been available a means of obtaining 6-place roots by the use of tables of this type and a single division.

Significant improvement has been made in tables Nos. 81 and 82 as compared with the previous 5-place table (1) by eliminating need of determining which of two arguments is nearer to N, (2) by reducing the number of tabulated divisors—resulting from using a single column of divisors for the entire range of N from 1 to 100 and from altering the range of error from (0 to $+5$) to (-5 to $+5$) in units of last place, (3) by using the leeway afforded by integer arguments to reduce the number of digits of the divisors to as few as possible.

The method of selecting argument intervals and the error expression of tables of the single-division type are described in Willers, *Practical Analysis*, Dover edition, Art. 6–21, 1948 (*MTAC*, v. 3, p. 493). Obvious alteration of the error expression is necessary when applying them to these tables because of change of the range of error.