On Modified Divided Differences I

Although divided differences of a function are of basic importance in the theory of numerical analysis, they are not nearly as useful as ordinary differences in application. In the first place, they are much harder to generate; not only because they involve unequal intervals of the argument, but because divisions are involved, often by small numbers. Furthermore, they are harder to interpret than ordinary differences, and the problem of making judgments at each step about the number of meaningful significant figures in the successive differences is one of considerable difficulty for high-speed computing machines, or for I.B.M. machines. We shall show that part of this difficulty at least can be overcome by modifying the definition of divided differences. These *modified* divided differences take on many of the characteristics of ordinary differences, based on equal intervals between successive arguments, and the familiar theory relating to ordinary differences becomes a special case of the more general theory. Moreover, the decimal point can remain fixed, and all differences can be carried to the same number of *decimals* as the entries themselves. Such modified differences can therefore serve as useful tools in analyzing functions which are available only at unequal intervals of the argument.

Definitions. Let a function $u(x)$ be given for a sequence of arguments $x_0, x_1, \cdots, x_n$, all included in a closed region X. No assumptions need be made about the ordering in the sequence. For brevity, let $u_k = u(x_k)$. The accepted definition[1] for the first divided difference of u_k and u_j is

$$[x_k x_j] = \frac{u_k - u_j}{x_k - x_j}.$$

The divided difference of order m, involving $u_0, u_1, u_2, \cdots, u_m$ is obtained from the divided differences of order $m - 1$ by the formula

$$[x_0 x_1 x_2 \cdots x_m] = \frac{[x_0 x_1 \cdots x_{m-1}] - [x_1 x_2 \cdots x_m]}{x_0 - x_m}.$$

It is well known that $[x_0 x_1 \cdots x_m]$ is a symmetric function of the arguments. If $u(x)$ possesses n continuous derivatives in X, then

$$(1.1) \qquad [x_0 x_1 \cdots x_n] = \frac{1}{n!} \frac{d^n u(t)}{dt^n}, \quad t \text{ in } X.$$

In particular, if $u(x)$ is a polynomial of degree n, then the nth divided difference is a constant, given by the right-hand side of (1.1). In the following discussion it will always be assumed that the function under consideration has n continuous derivatives in X, whenever the nth difference (ordinary or divided) is referred to.

It is to be noted from (1.1) that the order of magnitude[2] of a divided difference does not depend strongly on the size of the intervals between successive arguments. In contrast to this, the nth *ordinary* difference $\Delta^n u_0$ is related to the nth derivative of the function as follows:

$$\Delta^n u_0 = h^n \frac{d^n u(z)}{dz^n}, \quad z \text{ in } X, \tag{1.2}$$

where h is the constant difference between successive arguments. Hence $|\Delta^n u_0|$ *decreases* with h; this is a desirable property. In practice, h can often be chosen small enough so that an approximating polynomial of a given degree shall represent $u(x)$ with desired accuracy. We can introduce a *modified* divided difference possessing similar characteristics as follows:

Define the modified first divided difference of u_0 and u_1 by

$$[w; x_0x_1] = w[x_0x_1] = \frac{w}{x_1 - x_0}(u_1 - u_0), \tag{1.3}$$

where w is an arbitrarily chosen positive constant. We shall show in Section II that if there are N arguments x_k in X, then a useful constant to choose is one which is close in magnitude to

$$w = \sum_{k=1}^{N-1} \frac{|x_k - x_{k-1}|}{N - 1}. \tag{1.4}$$

If the arguments x_k form an increasing or decreasing sequence in X, then (1.4) becomes

$$w = |x_{N-1} - x_0|/(N - 1). \tag{1.5}$$

In many cases the order of magnitude of $x_k - x_{k-1}$ remains the same over the region X, even though the intervals are not all equal. If w is chosen as the average length of the grid intervals in X, $w/(x_1 - x_0)$ will usually be of the order of magnitude of unity. In that case the number of decimals in the first modified divided difference which have meaning is the same as in the entries $u(x)$ themselves.

We define

$$[w; x_0x_1 \cdots x_n] = \frac{nw}{x_0 - x_n}\{[w; x_0x_1 \cdots x_{n-1}] - [w; x_1x_2 \cdots x_n]\}. \tag{1.6}$$

Again $nw/(x_n - x_0)$ will usually be of the order of magnitude of unity, and it is reasonable to retain the same number of decimals in all the divided differences as there are in $u(x)$.

Since w was introduced as a device to permit the retention of a fixed decimal point, it is enough to use a value of w which is close to, but not necessarily equal to, that defined in (1.4).

It follows from (1.1) and (1.6) that

$$[w; x_0x_1 \cdots x_n] = \frac{w^n d^n u(t)}{dt^n}, \quad t \text{ in } X. \tag{1.7}$$

The character of the modified divided differences parallels that of ordinary differences, and the rate at which the successive modified divided differences fall off again permits an estimation of the accuracy of a given approximation formula for $u(x)$ over X. Let

$$x - x_0 = pw; \quad x_k - x_0 = p_k w.$$

The well known identity for divided differences now assumes the following form:

$$\begin{aligned} u(x) &= u_0 + p[w; x_0x_1] + \frac{p(p-p_1)}{2!}[w; x_0x_1x_2] \\ &\quad + \frac{p(p-p_1)(p-p_2)}{3!}[w; x_0x_1x_2x_3] + \cdots \\ &\quad + \frac{p(p-p_1)(p-p_2)\cdots(p-p_{n-1})}{n!}[w; x_0x_1x_2\cdots x_n] \\ &\quad + \frac{p(p-p_1)(p-p_2)\cdots(p-p_n)}{(n+1)!}[w; xx_0x_1\cdots x_n]. \end{aligned} \tag{1.8}$$

If the right-hand side of (1.8) is truncated after the divided difference of order n, then the remainder R can be expressed by

$$R = M[w; xx_0x_1 \cdots x_n] = Mw^{n+1}\frac{d^{n+1}u(t)}{dt^{n+1}}, \quad t \text{ in } X, \tag{1.9}$$

where

$$M = \frac{p(p-p_1)(p-p_2)\cdots(p-p_n)}{(n+1)!}.$$

Whenever M is numerically less than unity, the magnitude of the divided differences of order $(n+1)$ provides an estimate for an upper bound of the truncation error R.

We can write

$$[w; x_0x_1x_2\cdots x_m] = \sum_{k=0}^{m} \frac{w^m m! u_k}{\prod_{j=0}^{m}{}'_k (x_k - x_j)}, \tag{1.10}$$

where the prime indicates that the factor corresponding to $j = k$ is to be omitted. In the special case of a divided difference of *even* order $2n$, chosen centrally around x_0, (1.10) becomes

$$[w; x_{-n}x_{-n+1}\cdots x_0x_1\cdots x_n] = \sum_{k=-n}^{n} \frac{(-1)^{n-k}w^{2n}\binom{2n}{n+k}u_k}{\prod_{j=-n}^{n}{}'_k\left(\frac{x_k - x_j}{k-j}\right)}, \tag{1.11}$$

$$[w; x_{-n}x_{-n+1}\cdots x_0\cdots x_n] = \sum_{k=-n}^{n} \frac{\binom{2n}{n+k}(-1)^{n-k}u_k}{\prod_{j=-n}^{n}{}'_k\left(\frac{p_k - p_j}{k-j}\right)}. \tag{1.12}$$

In the case where all intervals are equal to w, $p_k = k$, and (1.12) reduces to the ordinary central difference.

Since the only difference between the modified and ordinary divided difference of order m is the factor $w^m m!$ for all entries in a column of differences, it would appear that just as much information could be obtained from the entries without modification, provided the proper number of significant figures were retained in each difference. This is true. The chief advantages of modified divided differences over ordinary divided differences can be summarized as follows:

1) If w is suitably chosen the modified differences can be retained to the same number of places as the entries u_n themselves.

2) The numbers $nw/(x_k - x_j)$ by which one multiplies are generally of the order of magnitude of unity; hence it is easy to program their computation for high speed machines.

3) In computing $u(x)$ from (1.8) the magnitudes of the first neglected divided differences in the region furnish an upper bound of the error, if $|M|$ in (1.9) is no greater than unity (a case of frequent occurrence).

4) The character of the differences is close to that of the familiar ordinary differences based on equal intervals between successive arguments.

In employing modified divided differences over a large region X, it is often desirable to consider several subregions, in each of which w is chosen to be close to the average length of the interval over the subregion.

Error Magnification. Consider the ordinary central difference of *even* order $2n$, based on the entries $u_{-n}, u_{-n+1}, \cdots, u_n$—usually denoted by $\delta^{2n}u_0$. From (1.12) it is seen that the entry u_0 appears in this difference with the coefficient $(-1)^n\binom{2n}{n}$. In the neighboring differences $\delta^{2n}u_k$, $|k| < n$, u_0 appears with the coefficient $(-1)^{n-k}\binom{2n}{n+k}$. It follows that if u_0 has an error ϵ_0 which is substantially larger numerically than $|\delta^{2n}u_k|/\binom{2n}{n}$, and if the errors in neighboring entries are of a lower order of magnitude, then the differences $\delta^{2n}u_k$ will alternate[3] in sign, and their numerical ratios to $\delta^{2n}u_0$ will be approximately $\binom{2n}{n-k}/\binom{2n}{n}$. Since the error ϵ_0 is magnified most in $\delta^{2n}u_0$, an incorrect entry can be picked out immediately, and even the order of magnitude of the error can be estimated from the difference pattern. (A divergence of the pattern from that of the binomial coefficient ratios indicates the presence of more than one error.) If a regular function $u(x)$ is tabulated at equally spaced arguments, systematic differencing of the entries constitutes the most powerful tool for checking the function.

The corresponding modified divided differences, based on unequal intervals between successive arguments, are inherently more difficult to interpret from the viewpoint of error indication. However, certain useful inequalities can be set down. In order to derive them, we shall limit the region X, and specify that the arguments x_k form either an increasing or a decreasing sequence. In application this is actually the most important case; and in a region where $x_k - x_{k-1}$ changes sign for some k, we may consider the subregions in which the sign of $x_k - x_{k-1}$ is constant. Furthermore, no essential restriction will be added if for convenience we consider only the case where $x_k > x_{k-1}$; this will now be assumed.

Just as in the case of ordinary differences, u_k enters into $2n + 1$ consecutive differences of order $2n$. It will be enough to examine a fixed u_0 and to study the coefficients with which it enters into the various differences of order $2n$. Let us write

$$\delta_w^{2n} u_0 = [w; x_n x_{n-1} \cdots x_0 \cdots x_{-n}] = \sum_{k=-n}^{n} M_{0,k}^{(2n)} u_k, \tag{2.1}$$

and more generally

$$\delta_w^{2n} u_m = [w; x_{m-n} x_{m-n+1} \cdots x_{m+n}] = \sum_{k=-n}^{n} M_{m,k+m}^{(2n)} u_{m+k}. \tag{2.2}$$

When no ambiguity is likely to arise, the superscripts will be dropped and we shall write $M_{m,k}$ for $M_{m,k}^{(2n)}$.

Let

$$w c_k = x_k - x_{k-1}.$$

Clearly c_k is positive, since $x_k - x_{k-1}$ is assumed positive.

From (1.11) we have

$$M_{0,0} = (-1)^n \tbinom{2n}{n} / V_{0,0}, \tag{2.3}$$

where

$$V_{0,0} = c_0 c_1 \left(\frac{c_0 + c_{-1}}{2}\right)\left(\frac{c_1 + c_2}{2}\right)\left(\frac{c_0 + c_{-1} + c_{-2}}{3}\right)\left(\frac{c_1 + c_2 + c_3}{3}\right) \cdots \times \left(\frac{c_0 + c_{-1} + \cdots + c_{-n+1}}{n}\right)\left(\frac{c_1 + \cdots + c_n}{n}\right),$$

$$\frac{M_{k,0}}{M_{0,0}} = \frac{(-1)^k \prod_{j=-nc}^{(-n-1+ck)c} (x_0 - x_j)}{\prod_{j=(n+1)c}^{(n+kc)c} (x_j - x_0)}, \begin{cases} 1 \leqq k^2 \leqq n^2 \\ c = 1, \text{ if } k \geqq 1. \\ c = -1, \text{ if } k < 0 \end{cases} \tag{2.4}$$

In particular

$$\frac{M_{1,0}}{M_{0,0}} = \frac{-(x_0 - x_{-n})}{(x_{n+1} - x_0)}; \quad \frac{M_{-1,0}}{M_{0,0}} = \frac{-(x_n - x_0)}{(x_0 - x_{-n-1})}. \tag{2.5}$$

From (2.4) we have

THEOREM 1. *The coefficients $M_{k,0}$ of u_0 alternate in sign.*

THEOREM 2. *If $p \leq c_k \leq P$, for $-n \leq k \leq n$, then*

$$\binom{2n}{n} \frac{1}{P^{2n}} \leq (-1)^n M_{0,0} \leq \binom{2n}{n} \frac{1}{p^{2n}}, \tag{2.6}$$

$$\frac{n}{n+1} \frac{p}{P} \leq \frac{-M_{\pm 1,0}}{M_{0,0}} \leq \frac{n}{n+1} \frac{P}{p}, \tag{2.7}$$

$$\frac{\binom{2n}{n+k}}{\binom{2n}{n}} \left(\frac{p}{P}\right)^k \leq \frac{(-1)^k M_{\pm k,0}}{M_{0,0}} \leq \frac{\binom{2n}{n+k}}{\binom{2n}{n}} \left(\frac{P}{p}\right)^k. \tag{2.8}$$

The above inequalities follow from (2.3), (2.4), and (2.5).

If $P = p = 1$ the inequalities give the well known relations among the coefficients of the ordinary central differences. From (2.6) and (2.8) we have

$$\frac{p^k}{P^{2n+k}}\binom{2n}{n+k} \le |M_{\pm k,0}| \le \binom{2n}{n+k}\frac{P^k}{p^{2n+k}}. \tag{2.9}$$

THEOREM 3. *If $P \leqq \{2(2m+1)/(m+1)\}^{\frac{1}{2}}$, the coefficients $|M_{0,0}^{2n}|$ increase with n, for $n \geqq m$.*

Proof.

$$|M_{0,0}^{(2n+2)}| = \frac{|M_{0,0}^{2n}|w^2(2n+2)(2n+1)}{(x_0 - x_{-n-1})(x_{n+1} - x_0)} \ge \frac{|M_{0,0}^{2n}|2(2n+1)}{P^2(n+1)}$$

$$\ge |M_{0,0}^{(2n)}|\frac{2n+1}{2m+1}\frac{m+1}{n+1} \ge |M_{0,0}^{(2n)}|,$$

since $2n + 1/(n+1)$ is an increasing function of n.

COROLLARY 3.1. *If $p \ge \{2(2n+1)/(n+1)\}^{\frac{1}{2}}$, $|M_{0,0}^{(2n)}|$ decreases with n.*

Before applying the previous results, it will be desirable to distinguish between three types of errors in the entries, just as in the case when ordinary differences are under consideration:

Errors of Type (a). Systematic errors, prevalent in all entries over a region. Such errors cannot be discovered by differencing methods; hence they will be eliminated from the discussion.

Errors of Type (b). Under this category we shall include

1. An isolated error: An error in an entry—say in u_0—which is considerably larger numerically than the errors of most entries in the region.

2. Multiple errors: This term will be used when two or more entries in a region have errors which are considerably larger numerically than the errors of most entries in the region.

The phrase *considerably larger* is necessarily qualitative, and depends on what degree of magnitudes the tests applied can distinguish.

Errors of Type (c).

1. Errors due to rounding approximate entries to a fixed number of decimals.

2. Errors too small to be considered in Type (b).

3. Errors of a non-systematic type which occur in too many entries of a region to be picked up by differencing tests. Usually all errors of Type (c) will occur in the last two decimal places of the entries. A discussion of errors of Type (c) will be given in Part II of this paper.

Errors of Type (b). Consider an isolated error ϵ_0 in u_0. Let $\bar{u}_0 = u_0 + \epsilon_0$ be the tabulated entry, and suppose that $M_{0,0}^{(2n)}\epsilon_0$ is of a higher order of magnitude numerically than $\delta_w^{2n} u_k$.

Let $\bar{\delta}_w^{2n}u_k$ be the approximate value of the $(2n)th$ divided difference in the region, and let

$$S\varphi_0^{(2n)} = \delta_w^{2n}u_0 - \delta_w^{2n}u_k, \quad S = \text{signum}\,[\delta_w^{2n}u_0 - \bar{\delta}_w^{2n}u_k]. \tag{2.10}$$

On the assumption that $S\varphi_0^{2n}$ is due primarily to the error, we may write

$$(-1)^n\rho\epsilon_0 M_{0,0}^{(2n)} = S\varphi_0^{(2n)},$$

where ρ is a positive number close to unity.

$$\left|\frac{\varphi_0^{(2n)}}{M_{0,0}}\right| = |\rho\epsilon_0|. \tag{2.11}$$

From (2.6)

$$\frac{\varphi_0^{(2n)}p^{2n}}{\binom{2n}{n}} \le |\rho\epsilon_0| \le \frac{\varphi_0^{(2n)}P^{2n}}{\binom{2n}{n}}. \tag{2.12}$$

The estimation of $\delta_w^{2n}u_k$ for the region can be avoided. For we may study

$$\delta_w^{2n}u_0 - \delta_w^{2n}u_1 = \rho S\varphi_{0,1}^{(2n)}, \text{ say.} \tag{2.13}$$

If ϵ_0 is an isolated error, then

$$(M_{0,0} - M_{1,0})\epsilon_0\rho \cong S\varphi_{0,1}^{(2n)}, \tag{2.13a}$$

$$\rho S(-1)^n\epsilon_0 \cong \frac{\varphi_{0,1}^{(2n)}}{(-1)^n M_{0,0}\left(1 - \dfrac{M_{1,0}}{M_{0,0}}\right)}.$$

From (2.7)

$$\frac{(-1)^n\varphi_{0,1}^{(2n)}}{M_{0,0}\left(1 + \dfrac{n}{n+1}\dfrac{P}{p}\right)} \le \rho S(-1)^n\epsilon_0 \le \frac{(-1)^n\varphi_{0,1}^{(2n)}}{M_{0,0}\left(1 + \dfrac{n}{n+1}\dfrac{p}{P}\right)},$$

$$\frac{\varphi_{0,1}^{(2n)}p^{2n}}{\binom{2n}{n}\left(1 + \dfrac{n}{n+1}\dfrac{P}{p}\right)} \le \rho S(-1)^n\epsilon_0 \le \frac{\varphi_{0,1}^{(2n)}P^{2n}}{\binom{2n}{n}\left(1 + \dfrac{n}{n+1}\dfrac{p}{P}\right)} \tag{2.14}$$

THEOREM 4. *If ϵ_0 is an isolated error, then*

$$\frac{\delta_w^{2n}u_0 - \delta_w^{2n}u_1}{\delta_w^{2n}u_0 - \delta_w^{2n}u_{-1}} \cong \frac{1 + \dfrac{x_0 - x_{-n}}{x_{n+1} - x_0}}{1 + \dfrac{x_n - x_0}{x_0 - x_{-n-1}}}. \tag{2.15}$$

Proof.

$$\delta_w^{2n}u_0 - \delta_w^{2n}u_1 \cong \epsilon_0(M_{0,0} - M_{1,0}) \cong \epsilon_0 M_{0,0}\left(1 - \frac{M_{1,0}}{M_{0,0}}\right),$$

$$\delta_w^{2n}u_0 - \delta_w^{2n}u_{-1} \cong \epsilon_0 M_{0,0}\left(1 - \frac{M_{-1,0}}{M_{0,0}}\right).$$

Then (2.15) follows from (2.4) and the above. Since the right-hand side (2.15) is easy to evaluate, we have a criterion as to whether a possible error is an isolated one. The estimate of the error is furnished by (2.12) or (2.14).

Let us now examine the effect of w on the location of errors. From the right-hand side of (2.10) it is clear that if w is chosen excessively small, then significant figures of u_k will be moved off quickly to the right and it will be impossible to pick up significant errors of the last few places of the entries. Moreover, both P and p will tend to be large, since $x_k - x_{k-1} = c_k w$

is fixed by the given grid for all k. In that case (2.14) will furnish a poor estimate of ϵ_0, due to the sharp variations in the inequalities. On the other hand, the choice of a very large w will simply have the effect of magnifying both small and large errors. A desirable w is one which tests as well as possible all the significant figures of the entries u_k. For that purpose, w should be chosen as the *average* $|x_k - x_{k-1}|$ in the region. For then both P and p will tend to be of the order of magnitude of unity, provided successive intervals vary with reasonable continuity.

If w is chosen as the average of the values $x_k - x_{k-1}$, then there may be some regions where P is relatively large with respect to w; and other regions where P is quite small. The question arises: If there is an error of Type (b), will (2.13) show it? From (2.13a) and (2.3) it is clear that $\varphi_{1,0}^{(2n)}$ will tend to be small in regions where the intervals $x_k - x_{k-1}$ are *large* with respect to the average interval of the space. Hence a small error will not show up. If the spread between P^{2n} and p^{2n} is small, in such a region then (2.14) can be used, to examine critically a fairly small $\varphi_{0,1}^{(2n)}$. However, if an isolated group of $k < n$ intervals is excessively large, then p and P will vary considerably, and the work of testing every doubtful entry may become too laborious. By the same considerations, a region where the values of $x_k - x_{k-1}$ are excessively small with respect to the average will show a big error magnification, and such errors will be readily traceable. Example 2 illustrates some features of error patterns.

If P varies much from p, the inequalities given here will not offer a good estimate of the magnitude of an isolated error. However, from Theorem 1 it is assured that a sufficiently large error will show up, because of sign variation in the vicinity of an error, if the magnitude of the modified divided difference is of lower order than the error. Small errors, on the other hand, can be masked by the rounding errors in the vicinity, often to a greater degree than in the case of equally spaced entries.

Let us now examine briefly the consequences of Theorem 3. In ordinary differences, where all c_k are unity, the numerical value of $M_{0,0}$ increases with n. Thus the magnification of an error increases in successive differences. If differences of high enough order are taken and the interval is sufficiently small for the correct differences to fall off numerically, an isolated error will be noticed more and more in successive differences. To have this property carry over to regions where all c_k are not equal to unity, Theorem 3 requires that no c_k be numerically greater than $2[(n + \frac{1}{2})/(n + 1)]^{\frac{1}{2}}$. Thus in regions where the intervals between successive entries are numerically larger than twice the *average* interval in the region, successive differencing will not tend to magnify the error, and so detection of the error may become difficult. It would also be desirable to have the ratio $|M_{\pm 1,0}/M_{0,0}|$ as small as possible. If the c_k are all equal to unity, this ratio is $n/(n + 1)$ for the difference of order $2n$. Hence with increasing n, this ratio is so close to unity as not to offer much help in the *variation of magnification* even if all c_k are unity. However, the fact that the *sign of* $M_{\pm 1,0}/M_{0,0}$ *is always negative* is of the utmost importance. For even if this ratio is numerically close to unity, the fact that successive $M_{k,0}$ differ in sign will help in detecting an error—especially so in regions where $|M_{0,0}|$ increases with n.

Although attention was fixed on even central differences, the main characteristics apply also to differences of odd order.

The following examples are instructive.

Example 1

(a) Unmodified Divided Differences of $Y_0(x)$

x	$Y_0(x)$						
1.00	.08825696	.7640470					
1.04	.11881884	.742922	−.42250	.124			
1.05	.12624806	.7264190	−.41258	.1038	−.10	.1	
1.08	.14804063	.6670239	−.395967	.086985	−.0800	.0600	—
1.20	.22808350	.6026670	−.378570	.071895	−.06561	.0540	−.021
1.25	.25821685	.5735327	−.364179	.05965	−.05102	.0390	−.047
1.28	.27542283	.5485412	−.357021	.05290	−.0397	.034	−.02
1.32	.29736448	.5169806	−.350673	.04769	−.0326	.03	—
1.37	.32321351	.4859780	−.344473	.04343	−.0266		
1.41	.34252663	.4622297	−.339261				
1.44	.35651952						

Comment. In obtaining the unmodified divided differences, the computer was instructed to retain in $[x_i x_{i+1} \cdots x_{i+n+1}]$ the same number of significant figures as there were in $[x_i x_{i+1} \cdots x_n] - [x_{i+1} \cdots x_{n+1}]$. Another reasonable method is to retain in a column only the number of significant figures that are common to most entries in the column. The difficulty here is that the computer does not know, until the column is finished, what this common number should be.

(b) Modified Divided Differences of $Y_0(x)$, in Units of the 8th Decimal Place

$w = 0.044$

x	$Y_0(x)$	δ_w	δ_w^2	δ_w^3	δ_w^4	δ_w^5	δ_w^6
1.00	.08825696	3361807					
1.04	.11881884	3268857	−163592	6338			
1.05	.12624806	3196244	−159751	5305	−900	198	
1.08	.14804063	2934905	−153318	4446	−720	119	−74
1.20	.22808350	2651735	−146582	3675	−590	107	−11
1.25	.25821685	2523544	−141010	3049	−459	077	−24
1.28	.27542283	2413581	−138239	2704	−357	067	− 8
1.32	.29736448	2274715	−135781	2437	−293	059	− 9
1.37	.32321351	2138303	−133380	2219	−239		
1.41	.34265263	2033811	−131362				
1.44	.35651952						

Example 2

Modified Divided Differences of x^3, in Units of the Fifth Decimal Place

$w = 0.35$

x	x^3	δ_w	δ_w^2	δ_w^3	δ_w^4
−2.4	−13.82400	509600			
−2.0	− 8.00000	330750	−139106		
−1.5	− 3.27500	170100	−124950		
−1.1	− 1.33100	78050	− 71595	11434	
− .6	− .21600	15050	− 44100	40016	22231
.1	− .00100	1050	− 12250	20621	−14291
.2	.00800	13650	14700	25725	4203
.5	.12500	66850	93100	25725	0
.6	.31600	9450	−200900		
.7	.34300	76650	117600		
1.0	1.00000	152600	75950		
1.4	2.74400				

Comments on Example 2. The entries corresponding to $x = -1.5$ and 0.6 contain errors of the same magnitude. Yet the error in $u(.6)$ is already evident from $\delta_w^2 u$, but not so the error in $u(-1.5)$. This illustrates the fact that errors are magnified most in regions where the intervals $c_k w$ are small with respect to w. Let

$$x = .6 = x_0, \quad n = 1; \quad \text{then} \quad \frac{x_0 - x_{-1}}{x_2 - x_0} = 1/4 = \frac{x_1 - x_0}{x_0 - x_{-2}}$$

using (2.15),

$$\left(1 + \frac{x_0 - x_{-n}}{x_{n+1} - x_0}\right) \Big/ \left(1 + \frac{x_n - x_0}{x_0 - x_{-n-1}}\right) = 1.$$

From the table of differences, the left-hand side of (2.15) for this example is equal to

$$\frac{-3.185}{-2.940} = 1.08$$

Hence the error appears to be an isolated one.

Let us now consider what information can be obtained from (2.12) about the nature of the error. In the usual case, the true value of $\delta_w^{(2n)}$ is unknown, and $\varphi_0^{(2n)}$ is estimated from adjacent differences. In the present case, the second differences at $x = 0.1$ and 0.2 warrant a guess that the second modified difference at $x = 0.6$ is about 0.4. This is based on the assumption that the correct third modified difference in the region is about 0.25. Since, from (1.6),

$$[w; x_0x_1x_2x_3] \frac{(x_3 - x_0)}{3w} + [w; x_0x_1x_2] = [w; x_1x_2x_3]$$

and $(x_3 - x_0)/3w$ is approximately $\frac{1}{2}$ in this region, we have, applying the above formula twice,

$$\delta_w^2 u \cong 0.15 + 0.25 = 0.4, \quad \text{at } x = 0.6.$$

Using the above estimate in (2.12) we arrive at the following calculations:

$$P = .86; \quad p = .285; \quad P^2 = 0.74; \quad p^2 = 0.081,$$
$$|\varphi^{(2)}| \cong |-2.009 - 0.4| = 2.409; \quad |\varphi^{(2n)}/\tbinom{2n}{n}| \cong 1.205.$$

Thus (2.12) implies

$$1.205(.081) \leq |\epsilon| \leq 1.205(.74); \quad \text{or} \quad 0.097 \leq |\epsilon| \leq 0.90.$$

The true error does lie between the above limits, but the actual size of the error must be obtained from recomputation. The important fact to observe is that the pattern of the second differences correctly shows that $u(0.6)$ is in error.

The error in $u(-1.5)$ shows up in the fourth difference.

Example 3. The following four values of $f(y)$ are available.

y	$f(y)$	Δ	Δ^2
3.7416573868	+.00824 2550		
3.777	+.00003 5971	−230753	+79
3.778	−.00019 4782	−230674	
3.779	−.00042 5456		

The differences of the equally-spaced values have been entered. It is required to find the zero of $f(y)$. If it were certain that the three last values of $f(y)$ are correct, and that $\Delta^3 y$ is negligible, this zero could be easily obtained by quadratic inverse interpolation. However, it happens that the computation of $f(y)$ is quite laborious, and it is desirable to check the accuracy of $f(y)$. Let us obtain modified divided differences, using the interval $w = .001$, since in this problem we merely want to have an indication of the magnitude of $\Delta^2 f$ in this region, at an interval of .001 in y. The values follow.

$w = .001$

y	$f(y)$	δ_w	δ_w^2
3.7416573868	+.00824 2550		
		−232201	
3.777	+.00003 5971		80
		−230753	
3.778	−.00019 4782		79
		−230674	
3.779	−.00042 5456		

The fact that the two values of δ_w^2 differ by only one unit is assurance that the values of $f(y)$ are correct, and that quadratic inverse interpolation is adequate. The solution is $y_0 = 3.77715\ 586$, to eight decimal places. [It can be shown that $\lambda = \frac{1}{4}y_0^2$ satisfies the system $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \lambda\right) u = 0$, $u = 0$ on the boundary C, $u > 0$ in interior of region where C is the ellipse $\frac{1}{4}x^2 + y^2 = 1$.]

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[1] L. M. MILNE-THOMSON, *The Calculus of Finite Differences.* London and New York, 1933.

[2] The numbers u_1 and u_2 will be said to have the same order of magnitude if $1/10 < |u_1/u_2| < 10$.

[3] See J. C. P. MILLER, "Checking by Differences I," *MTAC*, v. 4, 1950, p. 3–11.

A Minimum Problem Solved by Mesh Methods

Introduction. In the following, the function $y(x)$, subject to $y(0) = y(1) = 1$, is sought which will minimize the integral

$$I = \int_0^1 y^{-1}(1 + y'^2)^{\frac{1}{2}}\, dx. \tag{1}$$

This problem can be solved by the usual methods of the calculus of variations[1] but the differential equation involved is rather complicated. It is proposed here to solve the problem by mesh or "assumed polynomial" methods.[2] Methods of this sort have been rather extensively applied to the solution of differential equations and boundary value problems, and, recently, also to the determination of characteristic numbers or eigenvalues.[3]

Basic Equations. To solve the above problem by mesh methods it is first assumed that the interval $0 \leq x \leq 1$ is divided into segments each of equal length $h = 1/(2n)$ by the points x_i, $i = 0(1)2n$. It should be noted