Thus the estimated root is $u = 1.92288\ 41527$, which differs by six units in the last place from the correct 10D value, of which the last two digits are 33. The complete inverse interpolation process takes about 12 to 15 minutes on the desk calculator referred to above. Further values of x_i, say for $u_i = 1.95$ first, followed by $u_i = 1.89, 1.96, 1.88, 1.97$, etc., could be added if desired, and the existing results incorporated completely into the extended schedule up to the point at which the number of decimal places carried from the start ceases to justify it. In this example not more than two values can advantageously be added.

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[1] A. C. AITKEN, "On interpolation by iteration of proportional parts, without the use of differences." Edinburgh Math. Soc., *Proc.*, ser. 2, v. 3, 1932, p. 56–76.

[2] E. H. NEVILLE, "Iterative interpolation." Indian Math. Soc., *Jn.*, v. 20, 1933, p. 87–120.

[3] W. E. MILNE, *Numerical Calculus*. Princeton University Press, 1949, chap. III.

[4] J. R. WOMERSLEY, "Scientific computing in Great Britain." *MTAC*, v. 2, 1946, p. 110–117.

[5] W. M. KINCAID, "Solution of equations by interpolation." *Annals of Math. Statistics*, v. 19, 1948, p. 207–219.

[6] NBSMTP, *Tables of Lagrangian Interpolation Coefficients*. New York, Columbia University Press, 1944.

[7] E. T. WHITTAKER & G. ROBINSON, *The Calculus of Observations*. Fourth ed., London, 1948, chap. IV.

RECENT MATHEMATICAL TABLES

1144[F].—J. TOUCHARD, "On prime numbers and perfect numbers," *Scripta Math.*, v. 19, 1953, p. 35–39.

This note contains a small table of the function

$$C_p(n) = \sum_{k=1}^{n-1} k^p \sigma(k)\sigma(n-k)$$

for $p = 0(1)3$ and $n = 2(1)11$. Here $\sigma(k)$ denotes the sum of the divisors of k. These coefficients occur in the expansion of certain elliptic functions. More specifically,

$$\sum_{n=1}^{\infty} C_p(n)x^n = \Phi_{p-1,\,p}\Phi_{0,\,1}$$

where

$$\Phi_{r,\,s} = \sum_{n,m=1}^{\infty} m^r n^s x^{mn}.$$

D. H. L.

1145[I].—E. L. KAPLAN, "Numerical integration near a singularity," *Jn. Math. Phys.*, v. 31, 1952, p. 1–28.

The types of singularities considered in this paper are of the forms

$$x^{\frac{1}{2}}A(x), \quad x^{-\frac{1}{2}}A(x), \quad x^n\{A(x)\log x + B(x)\}, \quad n = 0, 1,$$

where A and B are regular near the origin. There are 14 principal tables of coefficients ranging from 3 to 6-term formulas. The accuracy is from 10 to 14D. Certain auxiliary tables of elementary symmetric functions of sets of 2, 3, or 4 small integers are included. The paper suffers from lack of illustrative material. The same problem for $x^{\frac{1}{2}}A(x)$ has been considered by Y. L. LUKE [*MTAC*, v. 6, p. 215–219], who discusses also the paper under review.

D. H. L.

1146[I].—TAKAHIKO YAMANOUCHI, *Tables of Coefficients of Everett's Interpolation Formula*. Report no. 1, Computation Institute, Tokyo, 1946, 19 p. 15 × 21 cm.

These tables give coefficients A and B for Everett's interpolation formula

$$y(x_0 + \theta h) = \theta y_1 + \phi y_0 - (A\delta^2 y_1 + B\delta^2 y_0)$$

where

$$A = \tfrac{1}{6}\theta(1 - \theta^2) \quad \text{and} \quad B = \tfrac{1}{6}\phi(1 - \phi^2), \quad \phi = 1 - \theta,$$

and for the inverse interpolation formula

$$\theta_r = \eta + \alpha A(\theta_{r-1}) + \beta B(\theta_{r-1}),$$

where

$$\eta = \theta_0 = (y - y_0)/(y_1 - y_0), \quad \alpha = \delta^2 y_1/(y_1 - y_0),$$
$$\beta = \delta^2 y_0/(y_1 - y_0) \quad \text{and} \quad x = x_0 + \theta_s h.$$

The coefficients A and B are tabulated for $\theta = 0(.001)1$ [also $\phi = 1(-.001)0$] to 5D, with differences and proportional parts.

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1147[K].—R. A. BRADLEY & M. E. TERRY, "Rank analysis of incomplete block designs. I. The method of paired comparisons," *Biometrika*, v. 39, 1952, p. 324–345.

Given t-treatments with true non-negative preference ratings, π_i and estimates, p_i; ($\sum\pi_i = \sum p_i = 1$). The treatments are compared in blocks of two, $\binom{t}{2}$ blocks constituting a basic design; n repetitions of the basic design are considered. A likelihood test statistic (B) is given to test the null hypothesis, $H_0: \pi_i = 1/t$, against alternative hypotheses in which m groups of treatments can be formed so that the treatments differ from group to group but are alike within groups. Two special cases are considered: (i) $m = t$, (ii) $m = 2$.

Tables (Appendix A) are prepared to make significance tests for case (i), when $t = 3$ and 4. For $t = 3$, $n = 1(1)10$ and for $t = 4$, $n = 1(1)6$. These tables are in terms of all possible combinations of rank sums for each treatment, where r_{ijk} (= 1 or 2) is the rank of the i-th treatment in the k-th repetition of the block in which both treatments i and j appear. Values of the following are given for each combination of rank sums: p_i to 2D for $i = 1(1)t$; the test statistic, B_1 to 3D; the significance probability, P (probability of this or a smaller value of B_1) to 4D.

A combined analysis is also considered for g groups of rank sums in which the π's may change from group to group, so that it is not desirable to pool the rank sums. The test statistic B_1^c is found by adding values of B_1 for the individual groups. Appendix B gives values of B_1^c to 4D and P to 3D for $n = n'g, n', g = 2(1)5$.

The authors also set up a test of agreement of the rank sums from group to group, and large sample tests for this agreement and for B_1, B_2 and B_1^c.

An example is presented for $t = 3, g = 2, n' = 5$.

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1148[K].—H. E. Daniels, "The covering circle of a sample from a circular normal distribution," *Biometrika*, v. 39, 1952, p. 137–143.

If a sample of n observations is taken from the circular normal distribution

$$dF = (2\pi\sigma^2)^{-1} \exp[-(x^2 + y^2)(2\sigma^2)^{-1}]\, dxdy$$

the covering circle is defined as the least circle (radius r and centered at distance ρ from the mean) such that all the observations are on or within the circle. The joint distribution of r and ρ is given by

$$dF_n(r, \rho) = -n(n-1)\sigma^{-2}\{\exp[-(r^2 + \rho^2)\sigma^{-2}]\}r[P(r, \rho)]^{n-2}\, drd\rho$$

where

$$P(r, \rho) = \{\exp[-\rho^2(2\sigma^2)^{-1}]\} \int_0^r s\sigma^{-2}\{\exp[-s^2(2\sigma^2)^{-1}]\}I_0(s\rho\sigma^{-2})\, ds$$

and

$$I_\nu(z) = (2\pi)^{-1} \int_0^{2\pi} [\exp(z\cos\vartheta)] \cos\nu\vartheta\, d\vartheta.$$

The distribution for r taken singly is found to be

$$F_n(r) = n\{1 - \exp[-r^2(2\sigma^2)^{-1}]\}^{n-1} - (n-1)\{1 - \exp[-r^2(2\sigma^2)^{-1}]\}^n.$$

A table is exhibited showing the mean to 3D, the variance to 4D, and the 1%, 5%, 95% and 99% points to 3D of the distribution of r/σ for samples of $n = 2(1)15, 20(10)50, 100$.

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1149[K].—H. A. David, "Upper 5 and 1% points of the maximum F-ratio," *Biometrika*, v. 39, 1952, p. 422–424.

The ratio of the largest to the smallest of k sample variances, all with the same number of degrees of freedom ν, was introduced by Hartley[1] as a test statistic for the hypothesis that the population variances are equal, under the assumption of normality. Hartley gave a table of the upper 5% points for this ratio, containing values which were exact for $k = 2$ but were obtained from an approximate expression for $k > 2$. In the present paper numerical quadrature is used to evaluate the exact upper 5% and 1% points for this ratio with an accuracy which may possibly leave the third digit in doubt. Tables thus computed are presented for $k = 2(1)12$, $\nu = 2(1)10, 12,$

15, 20, 30, 60, ∞. It is found that Hartley's approximate table tended to underestimate the 5% points.

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[1] H. O. HARTLEY, "The maximum F-ratio as a short-cut test for heterogeneity of variance," *Biometrika*, v. 37, 1950, p. 308–312 [*MTAC*, v. 5, p. 145].

1150[K].—M. H. GORDON, E. H. LOVELAND, & E. E. CURETON, "An extended table of chi-square for two degrees of freedom, for use in combining probabilities from independent samples," *Psychometrika*, v. 17, 1952, p. 311–316.

These tables give the value of chi-square for two degrees of freedom corresponding to a given probability value. The values of chi-square are given to 4D for the argument, $p = 0(.001).999$. The major function of these tables, according to the authors, is to convert probabilities obtained from independent samples into corresponding values of chi-square so that they may be combined by the technique proposed by Fisher if one is interested in making a new test of significance of the combined data. A numerical illustration showing the application of the Fisher process and the use of the tables accompanies the article.

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1151[K].—E. J. GUMBEL, J. ARTHUR GREENWOOD, & D. DURAND, "The circular normal distribution: theory and tables," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 131–152.

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ denote the angles determined by n points on the circumference of a unit circle, with respect to a fixed radius, and let α_0 denote the angle determined by their centroid, whose radius vector is $\bar{a}$. Then α_0 is a species of mean given by $\sum_{i=1}^{n} \sin(\alpha_i - \alpha_0) = 0$. In this paper the probability density function $f(\alpha)$, for which the maximum likelihood estimate of its location parameter is a mean of the form α_0, is found to be $f(\alpha) = \{\exp[k\cos(\alpha - \alpha_0)]\}/[2\pi I_0(k)]$, where I_0 is the Bessel function of the first kind of purely imaginary argument. This is analogous to the ordinary normal distribution in the case where the n points are situated along a straight line instead of a circle. Table 2 gives the maximum likelihood estimate of the scale parameter k determined from the relation $I_0'(k) - \bar{a}I_0(k) = 0$, for the radius vector $\bar{a}$ for the given observations. Values of k are given to 5D for $\bar{a} = .00(.01).87$ with second central differences that are modified when $\bar{a} \geq .69$.

Table 3 gives 3D values for the function $\psi(\alpha) = \{[\exp(k\cos\alpha)]/I_0(k)\}^{\frac{1}{2}}$ for $k = 0(.1)4$ and $\alpha = 0°(10°)180°$. This function is used in equi-areal (i.e., area-preserving) plotting of the circular normal distribution $f(\alpha)$.

Table 4 gives 5D values of $\Phi(\alpha) = \int_{-\alpha}^{\alpha} f(x)dx$ for $\alpha = 5°(5°)180°$, $k = 0(.2)4$, together with the second central differences in each direction,

δ_α^2, δ_k^2. This table may be used for finding areas or significance levels. It is an abridgment of a 7D table, with intervals half as large, that was computed from the Fourier cosine series for $\exp(k \cos x)$, and from tables of $I_n(k)$ to $n = 14$ obtained (by arrangement) from the proof sheets of Part II of the BAASMTC's *Table of Bessel Functions*.[1]

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[1] BAASMTC, *Bessel Functions*, Part II. Cambridge, 1952 [*MTAC*, v. 7, p. 97].

1152[K].—A. K. GUPTA, "Estimation of the mean and standard deviation of a normal population from a censored sample," *Biometrika*, v. 39, 1952, p. 260–273.

The author recognizes two ways of censoring a sample: (1) observations above or below a given (truncation) point may be censored (type I); and (2) the $n - k$ smallest or greatest observations may be censored (type II). He derives maximum-likelihood equations for estimating the mean and standard deviation of a normal population from a type II censored sample. Although differing in algebraic form, the author's results when applied to any given sample with the largest (or smallest) measured sample observation, x_k, taken as the truncation point, lead to identical estimates as corresponding estimating equations derived earlier by STEVENS,[1] HALD,[2] and the reviewer.[3] The author indicates that samples considered in the above references are of type I since there, the truncation point is assumed known prior to collecting sample data. It must, however, be pointed out that estimating equations in each of these papers were derived with the total number of sample observations, n, the number of measured observations, k, and hence the number of censored observations, $n - k$, considered as known constants.

Tables are provided in this paper for computing estimates when the right (larger) tail of the sample is censored. The function, $z(\psi, p)$, used in the estimation process, is tabulated to 4D for $\psi = .05(.05).95$ and for $p = .1(.1)1.0$, where

$$\psi = s^2/(s^2 + d^2) = (1 + \eta z - z^2)/(1 + \eta z)$$

and

$$z = \eta + \left(\frac{1}{p} - 1\right)\{\exp(-\eta^2/2)\}\left(\int_z^\infty \exp(-t^2/2)\, dt\right)^{-1}$$

in which s^2 is the sample variance, $d = x_k - \bar{x}$, x_k is the largest measured observation, $\bar{x}$ is the sample mean, $p = k/n$, and $\eta = (x_k - \mu)/\sigma$, where μ and σ are the population mean and standard deviation. For computing asymptotic variances and covariances of the estimates, tables of σ_{ij} $(i, j = 1, 2)$ are given to 5D for $p = .05(.05).95(.01).99$, where $[\sigma_{ij}] = [\nu_{ij}]^{-1}$ in which ν_{ij} are elements of the maximum likelihood variance-covariance matrix.

Best (minimum variance) linear estimates, μ^* and σ^* from type II samples for small n are also considered in this paper, and a table with entries to 5D of the coefficients β_i is provided, where

$$\mu^* = \sum_{i=1}^{k} \beta_i x_{i|n}, \quad k = 2(1)n - 1;\ n = 3(1)10$$

in which the $x_{i|n}$ are the x_i, $i = 1, \cdots, k$, rearranged so $x_{1|n} < x_{2|n} < \cdots < x_{k|n}$. A similar table with entries also to 5D is given of the coefficients γ_i, where

$$\sigma^* = \sum_{i=1}^{k} \gamma_i x_{i|n}, \quad k = 2(1)n; \; n = 2(1)10.$$

Two final tables with entries to 5D are given of the variances of estimates σ^* and μ^* for $n, k = 2(1)10$.

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[1] W. L. STEVENS, "The truncated normal distribution" (Appendix to paper by C. I. BLISS, "The calculation of the time mortality curve"), *Ann. Appl. Biol.*, v. 24, 1937, p. 815–852.

[2] A. HALD, "Maximum likelihood estimation of the parameters of a normal distribution which is truncated at a known point," *Skandinavisk Aktuarietidskrift*, v. 32, 1949, p. 119–134. *MTAC*, v. 5, p. 74.

[3] A. C. COHEN, JR., "Estimating the mean and variance of normal populations from singly truncated and doubly truncated samples," *Ann. Math. Stat.*, v. 21, 1950, p. 557–569.

1153[K].—M. R. SAMPFORD, "The estimation of response-time distributions. II. Multi-stimulus distributions," *Biometrics*, v. 8, 1952, p. 307–369.

"The types of response time distribution occurring when two or more stimuli act on a sample of individuals are discussed. Two situations are discussed in considerable detail; the 'accidental death' model, in which each response can be related to its appropriate stimulus, and the 'natural death' model in which the exact cause of any death cannot be determined, but the distribution of 'potential survival times' to the two stimuli can be assumed bivariate normal. Maximum likelihood methods of estimation are developed for these situations, and tables are given to simplify the calculations." (From the author's summary.) Tables of ν, λ, and ξ are given to 5D for $\eta = -5(.1) - 3(.01)3(.1)5$, where $Z(\eta) = (2\pi)^{-\frac{1}{2}} \exp(-\eta^2/2)$, $Q(\eta) = 1 - \int_{-\infty}^{\eta} Z(u)du$, $\nu = Z(\eta)/Q(\eta)$, $\lambda = \nu(\nu - \eta)$, and $\xi = \nu - \eta\lambda$. It should be noted that ξ is always positive. In the tables a curious misprint occurs for all $\eta \geqq -2.75$ in attributing to ξ a minus sign.

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1154[K].—M. E. TERRY, "Some rank order tests which are most powerful against specific parametric alternatives," *Annals Math. Stat.*, v. 23, 1952, p. 346–366.

A statistic $c_1(R)$ based on a method of HOEFFDING[1] yields a most powerful rank order test of H_0: two samples of m and n observations come from the same continuous population, against the alternative H_1: the two samples come from two normal populations with common variance σ^2, and with means θ and ϕ, respectively, where $(\theta - \phi)/\sigma$ is positive and small. An approximation to the null distribution of c_1 in terms of the t-distribution and a comparison of the exact and approximate critical values for $N = 6(1)10$ for certain m and n is given. Under certain conditions, the asymptotic distribution of c_1 under H_0 is shown to be normal. Table 1 gives the exact null distribution of $c_1(R)$ to 2D for $N = 2(1)10$ for $m \leq n$, $m + n = N$ and distinct permutations of R (i.e. permutations of m 0's and n 1's), together with the corresponding values of the MANN-WHITNEY[2] U test. Table 2 gives the probability p of exceeding the critical values c_1 under H_0 for all $p \leq .1$ to 3D for $N = 6(1)10$.

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[1] W. HOEFFDING, " 'Optimum' nonparametric tests," *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, Univ. of California Press, Berkeley, Calif., 1951, p. 83–92.

[2] H. B. MANN & D. R. WHITNEY, "On a test of whether one of two random variables is stochastically larger than the other," *Annals of Math. Stat.*, v. 18, 1947, p. 50–60.

1155[K].—J. E. WALSH, "Operating characteristics for tests of the stability of a normal population," Am. Stat. Assn., *Jn.*, v. 47, 1952, p. 191–202.

Let n independent observations be drawn from a normal population with unknown mean m' and unknown standard deviation σ'. The problem is to test whether a small sample came from a specified normal population with known mean m and known standard deviation σ. In this situation, the null hypothesis tested is that $m' = m$ and $\sigma' = \sigma$.

The most useful result of the paper is a set of tables which give the operating characteristic (OC) function values for the following three types of tests. The first test, and the most common, is based on the "limits" for the sample mean, $\bar{x}$, and the standard deviation, s (using $n - 1$). In the second test, let $\bar{x}$ be replaced by the t-statistic

$$t = n^{\frac{1}{2}}(\bar{x} - m)/s,$$

so that the limits for t and s are used. In the third test, the statistic s is replaced by

$$s(m) = \left[\sum_{i=1}^{n} (x_i - m)^2/n\right]^{\frac{1}{2}}$$

to give a test based on the limits of $\bar{x}$ and $s(m)$.

Table 1 provides "limits," given to 3D, which define the critical region for t and for s at $\alpha = .005$, $.01(.01).05$ and $n = 3, 5$. Table 2 provides "limits," given to 3D, which define the critical region for $\bar{x}$ and for $s(m)$ at the same values of α and n as given in Table 1. If one or both of the statistics fall outside these limits, the null hypothesis is rejected.

The OC function for a test is defined to be the probability that both statistics have values within their limits given the true values of m' and σ'. However, in this paper, the OC function values are given for alternative hypotheses which are expressed in terms of two equivalent parameters a and b, where

$$a = n^{\frac{1}{2}}(m - m')/\sigma', \quad b = \sigma/\sigma'.$$

Table 3 contains an OC function comparison, given to 3D, for the three types of tests at values of $\alpha = .01$; $n = 3, 5$; $a = 0(1)4$; $b = 1/8, 1/4, 1/2, 1, 2, 4, 8$. Table 4 contains the OC function values, given to 3D, for the test based on $\bar{x}$ and s at values of $\alpha = .001, .005, .01, .02, .05$; $n = 3, 5, 7$; $a = 0(1)4$; $b = 1/8, 1/4, 1/2, 1, 2, 4, 8$.

The results of this paper are directly applicable to those quality control situations for which the assumptions underlying the test are satisfied. This reviewer would enjoy the extension of the OC function values to the $\bar{x}$ and R test together with the three types of tests included in this paper.

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1156[L].—S. Chandrasekhar & Donna Elbert, "The roots of $J_{-(l+\frac{1}{2})}(\lambda\eta)J_{l+\frac{1}{2}}(\lambda) - J_{l+\frac{1}{2}}(\lambda\eta)J_{-(l+\frac{1}{2})}(\lambda) = 0$," Cambridge Phil. Soc., *Proc.*, v. 49, 1953, p. 446–448.

The authors tabulate to 6S the first root, λ_1, of the equation mentioned in the title, together with 6S or 7D values of

$$J_{\pm(l+\frac{1}{2})}(\lambda_1), \quad J_{\pm(l+\frac{1}{2})}(\lambda_1\eta), \quad \frac{2}{(\lambda_1\pi)^2}\left(\frac{J^2_{l+\frac{1}{2}}(\lambda_1\eta)}{J^2_{l+\frac{1}{2}}(\lambda_1)} - 1\right)$$

for $\eta = .2$, $l = 1(1)5$; $\eta = .3$, $l = 1(1)6$; and $\eta = .4, .6, .8$, $l = 1(1)15$.

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1157[L].—W. J. Duncan, *Normalised Orthogonal Deflexion Functions for Beams*. Aeronautical Research Council *Reports and Memoranda*, No. 2281. His Majesty's Stationery Office, London, 1950. 23 p.

The orthogonal deflexion functions are

$$S_n(\xi) = \frac{(4n+1)^{\frac{1}{2}}}{(2n)!}\frac{d^{2n-2}}{d\xi^{2n-2}}[\xi^{2n}(1-\xi)^{2n}],$$

$$A_n(\xi) = -\frac{(4n+3)^{\frac{1}{2}}}{(2n+1)!}\frac{d^{2n-1}}{d\xi^{2n-1}}[\xi^{2n+1}(1-\xi)^{2n+1}],$$

and can be shown to be numerical multiples of $(1 - x^2)^2P_m''(x)$ where $x = 1 - 2\xi$, $m = 2n$ for S_n, $m = 2n + 1$ for A_n, and P_m is the Legendre polynomial.

The Green's functions for a doubly built-in uniform beam are

$$G_1(\xi, \tau) = \tfrac{1}{6}\xi^2(1-\tau)^2[3\tau - \xi(1+2\tau)], \quad 0 \le \xi \le \tau \le 1,$$

$$G_1(\xi, \tau) = G_1(\tau, \xi), \quad G_2(\xi, \tau) = \frac{\partial}{\partial\tau}G_1(\xi, \tau).$$

Tables 1 to 5 were computed under the direction of R. A. Fairthorne, and give 5D values of S_1, S_2, S_3, A_1, A_2 for $\xi = 0(.01)1$.

Tables 6 and 7 were computed by S. Kirkby and give 5D values of G_1 and G_2 for $\xi, \tau = 0(.05)1$.

A. E.

1158[L].—COSTANTINO FASSO, "Di un integrale intervenuto in una questione di idraulica," Ist. Lombardo Sci. Lett., *Rend.*, *Cl. Mat. Nat.*, s. 3, v. 15, 1951, p. 471–497.

The author denotes by $\Phi(x, z)$ a suitably defined indefinite integral of

$$\frac{x^5}{z + x^5}$$

with respect to x. He gives 5D values of $\Phi(x, z)$ for $x = .86, .9(.02)1.1, 1.14$ and $z = 10, 5, 2.5, 1.5, 1, .5, .25, .1, 0$ and for fewer x values and $-z = .25, .5, .7, 1, 1.2, 1.5, 2.5, 5, 10$. He also gives some numerical data and diagrams of the level curves of the surface,

$$y = \Phi(x, z) - \Phi(1, z)$$

in (x, y, z) space.

A. E.

1159[L].—MARION C. GRAY, "Legendre functions of fractional order," *Quart. Appl. Math.*, v. 11, 1953, p. 311–318.

The author gives a 6D table of $P_\nu(\cos \theta)$ for $\nu = .1(.1)2$, $\theta = 10°(10°)170°, 175°$. In comparison with the tables described in RMT **1110**, the present table is much shorter in the ν direction, much thinner in the θ direction, and there are no associated Legendre functions; but on the other hand there is one more decimal, and the tabulation extends to $\theta = 175°$ (instead of $\theta = 90°$).

There is a discussion of the zeros of $P_\nu(\cos \theta)$ for fixed θ and variable ν (see also RMT **1120** and the literature quoted there), and there are several useful expansions for numerical computation.

A. E.

1160[L, V].—J. BARKLEY ROSSER & R. J. WALKER, *Properties and Tables of Generalized Rocket Functions for Use in the Theory of Rockets with a Constant Slow Spin.* Cornell University, 1953, 21.5 × 28 cm.

This report treats certain indefinite integrals that describe the motion of fin-stabilized rockets having a constant slow spin about the axis of symmetry. The integrands are products of trigonometric functions and Fresnel integrals. In terms of functions defined in *MTAC*, vol. 2, p. 213 and *MTAC*, v. 3, p. 474, the integrals tabulated are

$$Rrc(\alpha, x) = \int_x^\infty Rr(w) \cos \alpha(w - x)\, dw$$

and the similar integrals obtained when $Rr(w)$ is replaced by $Ri(w)$ and when cos is replaced by sin. About half the space is devoted to a tabulation of these four integrals to 5D for $x = 0.0(0.1)5.0$ and $\alpha/\pi = 0.1(0.1)2.0$. For a lesser range of x, the tables extend over $\alpha/\pi = 2.0(0.1)8.0$. Second differences with "throw-back" are given for x. Auxiliary formulas and tables enable the entire range of real α and x to be covered. Careful attention is given to the accuracy attainable in the various regions. About half the space is devoted to definitions of the various rocket functions, to their connection with differential equations, to a review of the properties of the simpler *Rr* and *Ri* functions referred to above, to a development of the properties of the functions tabulated, and to the methods of computation of the tables.

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1161[L].—TATUDIRO SASAKI, *Tables of* $\mu(x) = R\psi(ix)$ *and* $\sigma_0(x) = \arg \Gamma(1 - ix)$. Numerical Computation Bureau, Tokyo, 1950. 9 pages, mimeographed.

These tables contain values of $\mu(x) = \mathrm{Re}\,\psi(ix)$ and $\sigma_0(x) = \arg \Gamma(1 - ix)$ where x is a positive variable, $\Gamma(z)$ is the gamma function and $\psi(z)$ is its logarithmic derivative. The values of $\mu(x)$ are tabulated to 6D with second central differences for $x = 0(.01).5(.02)2(.05)2.5$. The table of $\sigma_0(x)$ is also to six decimal places with second central differences for $x = 0(.01)3$. The auxiliary function $S_0(y)$ defined by the relation $-\sigma_0(x) = x \ln x - x + S_0(y)$, $y = 1/x$ is given to 6D for $y = 0(.01).35$ with first differences.

Comparison with the tables of the same functions published by NBSCL[1] showed a transposition in the value for $x = .30$ in the table of $\mu(x)$ at the top of page 1. There is also a gradually increasing discrepancy of one to four units in the values of $\sigma_0(x)$ from $x = 2.40$ to $x = 3.00$.

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NBSCL

[1] NBSCL, *Tables of Coulomb Wave Functions*, v. 1. AMS no. 17, Washington, 1952 [*MTAC*, v. 7, p. 101–102].

1162[L].—L. J. SLATER, "On the evaluation of the confluent hypergeometric function," Cambridge Phil. Soc., *Proc.*, v. 49, 1953, p. 612–622.

This paper contains an 8S table of

$$M(a, b, x) = \sum_{n=0}^{\infty} \frac{\Gamma(b)\Gamma(a + n)x^n}{\Gamma(a)\Gamma(b + n)n!}$$

for $a = -1(.1)1$, $b = .1(.1)1$, $x = 1(1)10$, and the asymptotic expansion of this function for large x with converging factors which improve the accuracy quite considerably.

The tables were computed from the power-series expansions, about thirty terms of the series being required. The computations were carried out on the EDSAC at the Cambridge Mathematical Laboratory.

A. E.

1163[L].—UNIVERSITY OF CALIFORNIA, Department of Meteorology, *Tables relating to Rayleigh scattering of light in the atmosphere.* Computed for the project *Investigation of polarization of sky light* by direction of ZDENEK SEKERA, Project Director. Computations performed under supervision of GERTRUDE BLANCH, by the U. S. Department of Commerce, NBS, Los Angeles, INA. November, 1952, xxvi + 85 p., 1 p. Errata. 21 × 27 cm.

This is a collection of nine tables useful in the study of the polarization of the light in the earth's atmosphere. The numerical work presented here is based on the theory developed by S. CHANDRASEKHAR,[1] who in 1947 found the exact solution to this classical problem. A helpful feature of the publication is the careful definition of all tabulated quantities as well as a description of their physical significance. The fundamental functions on which the solution of the whole problem depends, are the four pairs of functions $X^{(k)}(\mu, \tau)$ and $Y^{(k)}(\mu, \tau)$ $(k = 1, 2, 3, 4)$ defined by the simultaneous integral equations

$$(1)\quad \begin{aligned} X^{(k)}(\mu) &= 1 + \mu \int_0^1 [X^{(k)}(\mu)X^{(k)}(t) - Y^{(k)}(\mu)Y^{(k)}(t)] \frac{\psi_k(t)}{t+\mu}\, dt, \\ Y^{(k)}(\mu) &= e^{-\tau/\mu} + \mu \int_0^1 [Y^{(k)}(\mu)X^{(k)}(t) - X^{(k)}(\mu)Y^{(k)}(t)] \frac{\psi_k(t)}{\mu - t}\, dt \end{aligned}$$

where

$$\psi_1(t) = \tfrac{3}{8}(1 + t^2 - 2t^4), \quad \psi_2(t) = (1 + 2t^2 + t^4),$$
$$\psi_3(t) = \tfrac{3}{4}(1 - t^2), \quad \text{and} \quad \psi_4(t) = \tfrac{3}{8}(1 - t^2).$$

Successive approximations $X_n^{(k)}(\mu, \tau)$ and $Y_n^{(k)}(\mu, \tau)$ to the solutions of (1) were obtained by an iteration procedure, starting with the values

$$X_0^{(k)}(\mu, \tau) = 1, \quad Y_0^{(k)}(\mu, \tau) = e^{-\tau/\mu}.$$

All calculations were made with IBM machines. Depending on the value of τ, six or seven iterations were carried out. Table I contains the values of $X_n^{(k)}(\mu, \tau)$ and $Y_n^{(k)}(\mu, \tau)$ to 5D,

for $\tau = 0.15$ and 0.25, $n = 6$, $\mu = 0(.02).32(.04).96(.02)1$
and for $\tau = 1$, $n = 7$, $\mu = 0(.02).2(.04).96(.02)1$.

The intervals used in the numerical integrations were such that the entries are guaranteed only to within 0.0001 for $\mu = 0.02$, and to within 0.00003 for $\mu \geq 0.04$. In order to exhibit the behavior of successive approximations to X and Y, in Table IX representative values are listed to 7D. Since for $k = 3$ the solutions of (1) are not unique, the functions tabulated (called "standard solutions") are those defined by the relations

$$X^{(s)} = X^{(3)} + q\mu[X^{(3)} + Y^{(3)}]; \quad Y^{(s)} = Y^{(3)} - q\mu[X^{(3)} + Y^{(3)}]$$

where

$$q = y_0^{(3)}/[x_1^{(3)} + y_1^{(3)}],$$

$x_j^{(k)}$ and $y_j^{(k)}$ being the "moments" of the X and Y functions:

$$x_j^{(k)} = \int_0^1 \psi_k(t)X^{(k)}(t, \tau)t^j\, dt,$$

$$y_j^{(k)} = \int_0^1 \psi_k(t)Y^{(k)}(t, \tau)t^j\, dt.$$

Using the various formulas given by the theory, the authors of this publication finally illustrate, in Table VIII, the predicted behavior of the sky radiation by considering the intensity and polarization on the principal meridian (containing the sun) for various elevations of the sun. The corrections to the intensity and polarization predicted by the theory for a ground surface reflecting according to Lambert's law with albedos $\lambda = 0.10, 0.25, 0.50$, and 0.80, have also been included in Table VIII.

The tables presented here not only provide the numerical results predicted by an exact theory of atmospheric scattering according to Rayleigh's laws, but also provide a basis for a quantitative evaluation of the nonmolecular component of the sky radiation.

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[1] S. CHANDRASEKHAR, *Radiative Transfer*. Oxford, 1950.

1164[L].—KLAUS ZWEILING, *Grundlagen einer Theorie der biharmonischen Polynome*. Verlag Technik, Berlin, 1952, viii + 130 p., 5 plates, 1 insert, 18 × 24 cm.

The polynomials tabulated in this volume are

$$P_{0,0} = 1, \quad P_{1,0} = x, \quad P_{1,1} = xy$$

$$P_{2n,0} = \tfrac{1}{2}x(w^{2n-1} + \bar{w}^{2n-1}) \qquad n = 1(1)6,\ w = x + iy,\ \bar{w} = x - iy$$

$$P_{2n,1} = \frac{x(w^{2n} - \bar{w}^{2n})}{4ni} \qquad n = 1(1)5$$

$$P_{2n+1,0} = \frac{1}{4n}\left[2nx(w^{2n} + \bar{w}^{2n}) - iy(w^{2n} - \bar{w}^{2n})\right] \qquad n = 1(1)5$$

$$P_{2n+1,1} = -\frac{y(w^{2n+1} + \bar{w}^{2n+1}) + i\,(2n+1)x(w^{2n+1} - \bar{w}^{2n+1})}{4n(2n+2)} \qquad n = 1(1)5$$

$$p_{11,1} = x^{-3}y^{-1}P_{11,1}, \quad p_{12,0} = x^{-2}P_{12,0}, \quad p_{12,1} = x^{-2}y^{-1}P_{12,1},$$

all for $x, y = 0(.1)1$.

The zero lines of these polynomials are also tabulated, graphs of the $P_{i,j}$ are given, as are explicit representations in Cartesian and polar coordinates, various formulas and properties, and the expression of $\int P_{m,n}P_{i,k}\,dx$ in terms of products of these polynomials, for $m + n,\ i + k \leq 6$.

The polynomials tabulated here are useful in problems concerning the bending of elastic plates. Similar quantities have also been tabulated by THORNE.[1]

A. E.

[1] C. J. THORNE, "A table of harmonic and biharmonic polynomials and their derivatives," Utah Engineering Experiment Station, *Bulletin*, no. 39, 1949, supplement. Salt Lake City.