vice for solving an algebraic equation. The third is an electrical network analogue for flow problems.

1130. D. M. Swingle, "Nomograms for the computation of tropospheric refractive index," I. R. E., *Proc.*, v. 41, 1953, p. 385–391.

1131. P. R. Vance & D. L. Haas, "An input-output unit for analog computers," I. R. E., *Proc.*, v. 41, 1953, p. 1483–1486.

The device described can be used as a recorder in plane cartesian coordinates or as a curve tracer type of function generator. The unit consists of a drum on which is wrapped a piece of graph paper. The drum's rotation corresponds to one variable and the axial movement of a carriage carrying either a pen or a potentiometer type transducer corresponds to the other. Movements in both coordinates are controlled by servos and static accuracies within 0.2 per cent of full scale are claimed.

For recording purposes the unit functions in the same way as a conventional plotting board. When used as a function generator the desired function is plotted with conducting ink or a soft lead pencil and attached to the drum. The transducer, which consists of a printed circuit potentiometer card, makes electrical contact with the curve and provides the carriage servo with the error voltage required to follow the curve as the drum turns. The output is furnished by a potentiometer driven by the carriage motion. Errors in following the curve, caused by dynamic limitations of the carriage servo, are compensated by adding the servo error voltage to the output potentiometer voltage. The over-all accuracy seems to be better than one per cent within the range for which the unit is intended. The device was designed for use with the GEDA computer.

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1132. G. B. Walker, "On the electric field in a multi-grid radio valve," Inst. Elect. Eng., *Proc.*, v. 98, part III, 1951, p. 64–67.

The use of electrolytic tanks for this purpose is described.

NOTES

160.—Inverse Interpolation for the Derivative in the Complex Plane. In a recent note[1] formulas were given for finding the argument for which a function $f(x)$ has a given derivative $f'(x)$, when that function is tabulated for x at equal intervals h. Those formulas are still applicable when dealing with an analytic function $f(z)$ which is tabulated in the complex plane, so long as the arguments lie equally spaced upon any straight line in the complex plane. But for $f(z)$ tabulated over a Cartesian grid $x + iy$ of length h, greater accuracy may be had by locating the arguments $z_k = z_0 + kh$ closer together by choosing k to be small (generally complex) integers. Thus the problem is to find P, or $z = z_0 + Ph$, when given the values of $f'_z \equiv f'(z) \equiv f'(z_0 + Ph)$ and $f_k \equiv f(z_k)$ at any conveniently located points z_k. We choose the following configurations of points z_k for the

n-point cases, $n = 3(1)7$, where the value of n depends upon the number of points z_k required for direct interpolation:

3—Point

$$\begin{array}{ll} \dot z_i & \\ \dot z_0 & \dot z_1 \end{array}$$

4—Point

$$\begin{array}{ll} \dot z_i & \dot z_{1+i} \\ \dot z_0 & \dot z_1 \end{array}$$

5—Point

$$\begin{array}{lll} \dot z_i & \dot z_{1+i} & \\ \dot z_0 & \dot z_1 & \dot z_2 \end{array}$$

6—Point

$$\begin{array}{lll} \dot z_{2i} & & \\ \dot z_i & \dot z_{1+i} & \\ \dot z_0 & \dot z_1 & \dot z_2 \end{array}$$

7—Point

$$\begin{array}{lll} \dot z_{2i} & & \\ \dot z_i & \dot z_{1+i} & \dot z_{2+i} \\ \dot z_0 & \dot z_1 & \dot z_2 \end{array}$$

The formula for P in terms of r, s, t, u and v is identical with that for p given previously,[1] namely,

$$\begin{aligned} P = r &- r^2s + r^3(2s^2 - t) + r^4(-5s^3 + 5st - u) \\ &+ r^5(14s^4 - 21s^2t + 3t^2 + 6su - v) \\ &+ r^6(-42s^5 + 84s^3t - 28st^2 - 28s^2u + 7tu + 7sv) + \cdots. \end{aligned}$$

But the r, s, t, u and v are now defined as follows:

3—Point

$$r = \{2hf'_z + 2(1-i)f_0 - (1-i)f_1 - (1-i)f_i\}/(2D), \qquad s = t = u = v = 0$$

where $D = -2if_0 + (1+i)f_1 - (1-i)f_i$.

4—Point

$$r = \{2hf'_z + 3(1-i)f_0 + 2if_1 - 2f_i - (1-i)f_{1+i}\}/(2D)$$

$$s = \{3(1+i)f_0 - 3(1-i)f_1 + 3(1-i)f_i - 3(1+i)f_{1+i}\}/(2D), \qquad t = u = v = 0,$$

where $D = -4if_0 + (3+i)f_1 - (3-i)f_i + 2if_{1+i}$.

5—Point

$$\begin{aligned} r = \{&10hf'_z + 5(4-3i)f_0 + 20if_1 - (2+i)f_2 - 4(2+i)f_i \\ &- 10f_{1+i}\}/(10D), \end{aligned}$$

$$\begin{aligned} s = \{&15(1+3i)f_0 - 15(5-i)f_1 + 6(1-2i)f_2 + 3(13-i)f_i \\ &+ 15(1-3i)f_{1+i}\}/(10D) \end{aligned}$$

$$\begin{aligned} t = \{&-5(1+i)f_0 + 10(1-i)f_1 + (1+3i)f_2 - 2(3-i)f_i \\ &+ 10if_{1+i}\}/(5D), \quad u = v = 0 \end{aligned}$$

where $10D = 5(3-11i)f_0 + 20(3+2i)f_1 - 3(3-i)f_2 - 4(9+2i)f_i - 10(3-2i)f_{1+i}$.

6—Point

$$r = \{20hf'_s + 40(1 - i)f_0 + 16(1 + 2i)f_1 - (3 - i)f_2 - 16(2 + i)f_i - 20(1 - i)f_{1+i} - (1 - 3i)f_{2i}\}/(20D)$$

$$s = \{45(1 + i)f_0 - 24(2 - 3i)f_1 - 3(2 + 3i)f_2 + 24(3 - 2i)f_i - 54(1 + i)f_{1+i} - 3(3 + 2i)f_{2i}\}/(8D)$$

$$t = \{- 4f_0 - 2(1 + 3i)f_1 + f_2 - 2(1 - 3i)f_i + 6f_{1+i} + f_{2i}\}/(D)$$

$$u = \{5(1 - i)f_0 + 4(3 + i)f_1 - (1 - 2i)f_2 - 4(1 + 3i)f_i - 10(1 - i)f_{1+i} - (2 - i)f_{2i}\}/(8D)$$

$$v = 0$$

where $4D = - 30if_0 + 32f_1 - (1 - 3i)f_2 - 32f_i + 24if_{1+i} + (1 + 3i)f_{2i}$.

7—Point

$$r = \{20hf'_s + 4(12 - 11i)f_0 + 40(1 + i)f_1 - (1 - 7i)f_2 - 8(3 + 4i)f_i - 20(3 - i)f_{1+i} + 8if_{2+i} - (3 - i)f_{2i}\}/(20D)$$

$$s = \{9(7 + 9i)f_0 - 12(7 - 15i)f_1 - 33f_2 + 12(12 - i)f_i - 18(3 + 13i)f_{1+i} - 36f_{2+i} - 15if_{2i}\}/(8D)$$

$$t = \{- (17 + 3i)f_0 - 4(4 + 9i)f_1 + (5 - 6i)f_2 - 2(11 - 9i)f_i + 6(7 + 5i)f_{1+i} + 6(1 - i)f_{2+i} + (2 + 3i)f_{2i}\}/(2D)$$

$$u = \{(21 - 13i)f_0 + 20(3 + i)f_1 + (3 + 14i)f_2 + 4(3 - 11i)f_i - 10(9 - i)f_{1+i} + 16if_{2+i} - 3(2 + i)f_{2i}\}/(8D)$$

$$v = \{- 3(1 - 3i)f_0 - 12(2 - i)f_1 + 3(2 + i)f_2 + 6(1 + 3i)f_i + 30(1 - i)f_{1+i} - 6(1 + i)f_{2+i} + 3f_{2i}\}/(20D)$$

where $20D = 2(8 - 99i)f_0 + 32(9 - 2i)f_1 + (27 + 31i)f_2 - 96(2 + i)f_i - 40(4 - 7i)f_{1+i} + 32(1 + i)f_{2+i} - (11 - 15i)f_{2i}$.

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[1] H. E. SALZER, "Formulas for finding the argument for which a function has a given derivative," *MTAC*, v. 5, 1951, p. 213–215.

161.—THE NUMERICAL INTEGRATION OF $x''(t) = G(x)$. The formula used by ECKERT, BROUWER, & CLEMENCE[1] in their monumental integration of the equations of motion of the five outer planets, is a modification of one used by COWELL in his investigation of the motion of Halley's comet from 1759 to 1910. In order to get the next position in the ephemeris, use is made not only of those already obtained but of first and second summations.

It seems desirable to have formulas for such work that do not involve summations and such are given here.

Let $x = f(t)$ be a solution of

$$x''(t) = G(x)$$

and put $F(t) = G(f(t))$. We then have

$$f(a + w) = f(a) + wf'(a) + \tfrac{1}{2}w^2f''(a) + \cdots,$$
$$f(a - w) = f(a) - wf'(a) + \tfrac{1}{2}w^2f''(a) + \cdots.$$

We add, express derivatives in terms of *forward* differences, and then replace differences by their values in terms of forward function values, all according to Newton's scheme. Changing w to $-w$, we see that we can write, assuming nth differences of F to be constant,

$$f(a + w) = 2f(a) - f(a - w) + w^2Q,$$

where

$$Q = A_0F(a) + A_1F(a - w) + a_2F(a - 2w) + \cdots + A_nF(a - nw).$$

Let $f(t) = t^2$. Then $F(t) = 2$, and

$$(a + w)^2 = 2a^2 - (a - w)^2 + w^2 2(A_0 + A_1 + A_2 + \cdots + A_n),$$

so that

$$A_0 + A_1 + A_2 + \cdots + A_n = 1.$$

Thus if C is the common denominator of the A's, the sum of their numerators is also C, giving a very simple check.

The following table gives the coefficients for $n = 5(1)9$.

n	5	6	7	8	9
C	240	1 20960	1 20960	36 28800 = 10!	36 28800 = 10!
CA_0	317	1 68398	1 76648	55 37111	57 66235
CA_1	−266	−1 85844	−2 43594	−92 09188	−112 71304
CA_2	374	3 17946	4 91196	213 90668	296 39132
CA_3	−276	−3 11704	−6 00454	−313 23196	−505 69612
CA_4	109	1 84386	4 73136	308 31050	597 00674
CA_5	−18	−60852	−2 34102	−203 32636	−492 02260
CA_6	0	8630	66380	86 46188	278 92604
CA_7	0	0	−8250	−21 48868	−103 97332
CA_8	0	0	0	2 37671	22 99787
CA_9	0	0	0	0	−2 29124
Sum	800	6 79360	12 07360	666 42688	1252 98432
	−560	−5 58400	−10 86400	−630 13888	−1216 69632
	240	1 20960	1 20960	36 28800	36 28800

The formulas were checked by using simple functions, for instance in the case $n = 9$, the functions $f(t) = t^{10}$ and $f(t) = t^{11}$, with $w = 1$.

The formula used by Eckert, Brouwer, and Clemence is for $n = 9$, and by proper transformations it was carried over into a form that avoided summation. The coefficients obtained were identical with those directly arrived at. Several of the coefficients in the original formula had ten figures in the numerators, while none of those above exceeds eight.

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[1] W. J. ECKERT, DIRK BROUWER & G. M. CLEMENCE, "Coordinates of the Five Outer Planets, 1653–2060," *Astronomical Papers Prepared for the Use of the American Ephemeris and Nautical Almanac*, v. 12, Washington, 1951.

162.—Historical Note on Root–Finding Machine. A machine which escaped the attention of J. S. Frame[1] is one proposed by Frank T. Freeland.[2] The machine is based upon a method for formation of linkages for the successive powers, published by Freeland in American Journal of Mathematics, v. 3, No. 2.

The machine provides n levers L_k, so connected by linkages that a displacement x of L_1 produces a displacement x^k of L_k, $k = 2$ to n. On each L_k is a pulley P_k with axis parallel to and distance $\frac{1}{2}c_k$ from the axis of L_k, where c_k is the coefficient of x^k. An inextensible cord is passed around the pulleys with the two portions of the cord on P_k parallel, perpendicular to L_k and on the same side of it, using fixed pulleys to change cord direction if necessary. To the end of the cord is fixed a pointer brought parallel to a scale marked off in units of x.

The pointer is set at c_0 with $x = 0$. As L_1 is displaced through values of x, the pointer indicates on the scale the corresponding values of the polynomial. A real root is indicated by zero and the real part of a pair of imaginary roots by a change of direction of the pointer.

Another machine is described for quadratic equations based on a linkage for x^2 suggested by Sylvester. A historical sketch is presented in an appendix, describing machines proposed by Clairaut in 1820 and de Roos in 1879 and mentioning the analytic engine of Charles Babbage. Finally, linkages are described, one for transporting a dimension parallel to itself and one for forming a product based on the differences of squares.

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[1] J. S. Frame, "Machines for solving algebraic equations," *MTAC*, v. 1, 1945, p. 337–353.

[2] Frank T. Freeland, "A machine for the solution of the equation of the n-th Degree," Engineers' Club of Philadelphia, *Proc.*, v. 2, Feb. 23, 1880.

QUERIES—REPLIES

50. Incomplete Hankel Function. (**Q. 43,** v. 8, p. 51).

Put

$$z = (x^2 - 1)^{\frac{1}{2}}, \quad s = y(t^2 - 1)^{\frac{1}{2}}, \quad u = yt = (s^2 + y^2)^{\frac{1}{2}}$$

to obtain

$$\begin{aligned} f(x, y) &= \int_x^\infty (t^2 - 1)^{-\frac{1}{2}} e^{-iyt} dt \\ &= \int_{yz}^\infty u^{-1} e^{-iu} ds \\ &= \int_0^\infty u^{-1} \cos u ds - i \int_0^\infty u^{-1} \sin u ds - \int_0^{yz} u^{-1} ds \\ &+ \int_0^{yz} u^{-1}(1 - \cos u) ds + i \int_0^{yz} u^{-1} \sin u ds. \end{aligned}$$