[2] A. VAN WIJNGAARDEN, Afrondingsfouten MR3, Tevens ZW-(1950)-.001. Math. Centrum Rekenfdeling, Amsterdam (in Dutch). See also A. M. OSTROWSKI. *Two Explicit Formulae for the Distribution Function of the sums of n Uniformly Distributed Independent Variables.* Archiv d. Math., v. 3, 1952, p. 3–11.

# RECENT MATHEMATICAL TABLES

**1165[B,F].**—H. S. UHLER, "On the 16th and 17th Perfect Numbers," *Scripta Math.*, v. 19, 1953, p. 128–131.

This note contains exact values of $2^{n-1}(2^n - 1)$ for $n = 2203$ and 2281, numbers of 1327 and 1373 decimal digits. These are the 16th and 17th perfect numbers. Exact values are given also of $2^n$ for $n = 560, 2202, 2280$ and those digits of $2^{4405}$ and $2^{4561}$ which are not identical with the corresponding digits of the perfect numbers mentioned above.

The author has informed the reviewer of the fact that the 1023rd digit was printed incorrectly: *for* 32633 *read* 32638. This substitution occurred between page proof and printing and would have gone undetected by any author but one having Uhler's indefatigable perspicacity.

D. H. L.

**1166[C].**—NBSCL, *Tables of* $10^x$ (*Antilogarithms to the Base* 10). NBS Applied Math. Series, No. 27, U. S. Gov. Printing Office, Washington, 1953, viii + 543 p., 19.3 × 26.0 cm. Price $3.50.

The main table in the work is Table I, a 500 page table of $10^x$ for $x = 0(.00001)1$. These 100000 values are given to 10D. The arrangement is in four columns of 50 pairs $(x, 10^x)$ each so that consecutive entries lie one under the other making linear interpolation easy. All eleven digits of $10^x$ are given in each entry. No differences are given. Linear interpolation gives 9D accuracy. The effect of the second difference on the 10th decimal may be read from a chart on p. vi. This amounts to at most 7 units in the 10th place.

Table II is a 15D radix table of $10^x$. Specifically it gives $10^y$ where

$$y = n \cdot 10^{-p}, \quad n = 1(1)999, \quad p = 3(3)15.$$

From this table 14 figure antilogarithms can be found by multiplying five entries together. The table can also be made to serve as a table of common logarithms to 15D.

Table II is similar to that of DEPREZ[1] which gives 13D antilogarithms of $x = m \cdot 10^{-r}$ for

$$m = 1(1)999, \quad r = 7(3)13$$

in connection with a basic table for

$$x = 0(.0001)1.$$

Table II will be found very useful in connection with any ordinary radix type logarithm table for very precise work.

Table I is based on DODSON'S[2] rare table of 1742. The entire Dodson table was transferred to punched cards and differenced on a tabulator. After correcting errors the table was checked by summing sets of 50 consecutive entries (as a geometric progression). Finally the printed page proof was subjected to additional differencing. No list of errata in Dodson is given.

Table I is a handy companion to a table of 10 place logarithms since inverse interpolation is avoided in the usual passage from numbers to logarithms and back to numbers. However since interpolation is only linear (at least for 9D work), inverse interpolation presents no greater problem than direct interpolation, especially if one has even the smallest type of desk calculator. From this there are two alternative inferences: (a) One needs no table of antilogarithms; (b) one needs only a table of antilogarithms. Perhaps this table will appeal most to those who want to do no interpolation whatever.

This useful volume is the result of work done by the old New York Mathematical Tables Project.

D. H. L.

[1] F. DEPREZ, *Tables for Calculating, by Machine, Logarithms to 13 Places of Decimals.* Berne, 1939.

[2] J. DODSON, *Antilogarithmic Canon.* London, 1742.

**1167[C].**—NBSCL. *Table of Natural Logarithms for Arguments Between Zero and Five to Sixteen Decimal Places.* NBS Applied Math. Series, No. 31, U. S. Gov. Printing Office, Washington, 1953, [Reissue of MT 10], x + 501 p., 20.1 × 25.9 cm. Price $3.25.

The original NYMTP Table 10 [v. 3 of the original 4v.] $x = 0(.0001)\,5$ is hereby reissued to meet a continued demand. The preface promises also a reissue of the fourth volume for $x = 5(.0001)10$. Although the Introduction states that there has been no revision of the tabular content, there has been a change in the rule for the indication of the signs of the logarithms. Thus in the original edition the logarithm of .0184 is given as 3.995 . . . , the fact that this number is actually negative being understood. Now the minus sign is printed explicitly. This improvement is carried out with a single exception and this occurs at the very first real entry of the table where the logarithm of .0001 is given as 9.21034. . . .

One further change may be noted and this refers to the last entry in the table. The reader who is familiar with the NYMTP tables will recall that arguments are given at the bottom of the page without the corresponding functional values. This tantalizing procedure is followed in the present volume except at the very end where the editor has relented and has given

$$\ln 5.0000 = 1.6094379124341004.$$

Apparently there are no errata known in this monumental table of 1941.

D. H. L.

**1168[F].**—E. S. BARNES & H. P. F. SWINNERTON-DYER, "The inhomogeneous minima of binary quadratic forms," *Acta Math.*, v. 87, 1952, p. 259–320.

Let $f(x, y) = ax^2 + bxy + cy^2$ be an indefinite binary quadratic form with real coefficients and discriminant $D = b^2 - 4ac > 0$. For any point $P$: $(x_0, y_0)$, where $x_0$, $y_0$ are real, define $M(f;P)$ to be the lower bound of

$$|f(x + x_0, \quad y + y_0)|$$

taken for all points $P$ and call $M(f)$ the upper bound of $M(f;P)$. Let $C$ be the set of points $P$ for which $M(f;P) = M(f)$ and $M_2(f)$ the upper bound of $M(f;P)$ over all $P$ not belonging to $C$. Clearly $M_2(f) \leq M(f)$. If the strict inequality holds, $M(f)$ is called an isolated minimum.

The authors list in their table [p. 315–317] the values of $M(f)$ for forms $x^2 - my^2$ for all square-free $m \equiv 2$ or 3 (mod 4), $m \leq 101$, except $m = 46$, 67, 71, 86, 94 and many corresponding values of $M_2(f)$. In many cases complete sets of incongruent (mod 1) points are given for which $M(f;P) = M(f)$ or $M(f;P) = M_2(f)$. All minima given in the table are isolated minima.

A corresponding table is given for forms $f = x^2 + xy - \frac{1}{4}(m - 1)y^2$ where $m \equiv 1$ (mod 4), $m$ is square-free and not greater than 101, except $m = 57$ and 73.

Sources for the results listed, many in the accompanying paper, are given.

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**1169[F].**—A. Gloden, *Table des Solutions de la congruence* $x^4 + 1 \equiv 0$ (mod $p$) *pour* $600000 < p < 800000$. Luxembourg, 1952, published by the author, rue Jean Jaurès, 11, Luxembourg, 22 p., 29.8 × 21.0 cm., mimeographed. Price 120 francs belges.

This table is identical with that described as UMT **158**, *MTAC* v. 7, p. 108 except that the appended factorizations there referred to are wanting.

**1170[F,L].**—A. van Wijngaarden. "On the coefficients of the modular invariant $J(\tau)$," K. Ned. Akad. v. Wetensch. *Proc.*, s.A, v. 56, 1953, p. 389–400.

Klein's fundamental modular invariant $J(\tau)$ has an expansion in the form

$$12^3J(\tau) = x^{-1} + 744 + 196884x + 21493760x^2 + \cdots$$

$$= \sum_{n=-1}^{\infty} c(n)x^n$$

in which $x = q^2 = e^{2\pi i\tau}$. These coefficients are positive integers whose properties have only recently been investigated. The present paper gives a two page table of $c(n)$ for $n = -1(1)100$. These values are quite large, $c(100)$ having 53 digits, and the table represents a relatively large amount of computing. Previous tables are those of Berwick[1] for $n \leq 7$ and Zuckerman[2] for $n \leq 24$.

The intimate connections between $J(\tau)$ and other elliptic functions provide a variety of methods for the computation of $c(n)$. That none of these methods can be too easy is pointed out by the author who remarks that "the coefficients grow very rapidly with $n$ and the digits have to come from somewhere."

Zuckerman exploited a connection between $c(n)$ and the number of partitions of $25n$. This becomes ineffective for $n = 25$ since Gupta's[3] tables of the partition function extend only to $n = 600$.

LEHMER[4] had proposed the formula

$$\sum_{k=-1}^{n} c(k)\tau(n-k) = 720\{91\sigma_{11}(n) + 600\tau(n)\}/691$$

and VAN DER POL[5] the more elegant formula

$$\sum_{k=-1}^{n} kc(k)\tau(n-k) = 24\sigma_{13}(n). \tag{1}$$

These formulas are based upon the connection between $J(\tau)$ and the Weierstrassian discriminant

$$\sum_{n=1}^{\infty} \tau(n)x^n = x\prod_{n=1}^{\infty}(1-x^n)^{24}$$

whose coefficients, known as Ramanujan's function, are now tabulated[6] to $n = 2500$. Formula (1) was used by the author up to $n = 50$. At this point another more elaborate method based on the relation

$$27J(\tau) = 2(\theta_2{}^8 + \theta_3{}^8 + \theta_4{}^8)(\theta_2{}^{-8} + \theta_3{}^{-8} + \theta_4{}^{-8})$$

between $J(\tau)$ and Jacobi's theta functions, was used to recompute and extend the table to $n > 100$. Various congruence properties of $c(n)$, including an interesting new one modulo 71, were used to check the calculations.

The table should be a valuable tool for further research on Klein's invariant.

D. H. L.

[1] W. E. H. BERWICK, "An invariant modular equation of the fifth order," *Quart. Jn. of Math.*, v. 47, 1916, p. 94–103.

[2] H. S. ZUCKERMAN, "The computation of the smaller coefficients of $J(\tau)$," Amer. Math. Soc., *Bull.*, v. 45, 1939, p. 917–919.

[3] H. GUPTA, "A table of partitions, I, II," London Math. Soc., *Proc.*, v. 39, 1935, p. 142–149, v. 42, 1937, p. 546–549.

[4] D. H. LEHMER, "Properties of the coefficients of the modular invariant $J(\tau)$," *Amer. Jn. of Math.*, v. 64, 1942, p. 488–502.

[5] B. VAN DER POL, "On a non-linear partial differential equation satisfied by the logarithm of the Jacobian theta-functions, with arithmetical applications, I, II," K. Ned. Akad. v. Wetensch., *Proc.*, s.A, v. 54, 1951, p. 261–284.

[6] See *MTAC*, v. 4, 1950, p. 162, UMT **101**.

**1171[I].**—H. E. SALZER, *Tables of Coefficients for the Numerical Calculation of Laplace Transforms*. NBS Applied Math. Series, No. 30, U. S. Gov. Printing Office, Washington, 1953, ii + 36 p., 20.1 × 25.9 cm. Price 25 cents.

These tables are intended to be used in the approximation of the transform

$$F(p) = \int_0^\infty e^{-pt}f(t)dt$$

by means of the sum

$$F(p) = \sum_{i=1}^{n-1} A_i f(i)$$

where

$$A_i = A_i^{(n)}(p)$$

depend on both $p$ and $n$, the number of ordinates. The main tables are arranged by $n$ which takes the values 2(1)11. The values of $p$ are .1(.1)$n - 1$ up through $n = 7$. For $n = 8$ and 9 the interval is .2 and for $n = 10$ and 11 it is 1. Values of the $A$'s are given to 9S. There are auxiliary tables as follows.

Table II (p. 27–36) gives values of $n!/p^{n+1}$ for $n = 0(1)10$, $p = .1(.1)10$ to 8*s*. This function is the Laplace transform of $t^n$ and the table is intended for use in transforming polynomials up through those of degree 10.

The function $(n - 1)!p^nA_i^{(n)}(p)$ is a polynomial in $p$ with integer coefficients. These polynomials are listed in the introduction (p. 7–8) together with the corresponding Lagrange interpolation coefficients (polynomials in $t$) for $n = 2(1)11$.

Four explicit examples are worked out including one in which the error is estimated.

These tables should be quite useful in dealing with functions whose Laplace transform is unfamiliar or intractable.

D. H. L.

**1172[K].**—Hirojiro Aoyama, "On a test in paired comparisons," Tokyo Inst. Math., *Annals*, v. 4, 1953, p. 83–87.

Let each of $n$ persons, no two of which have the same occupation, rate his own occupation in comparison with each of the others. This gives rise to $k = \binom{n}{2}$ paired comparisons, to which we assign the following scores: +1 in those cases in which each subject in a pair rates his own occupation higher than the other, −1 in those cases in which each subject rates his own occupation lower than the other, 0 in all other cases. Let $S$ be the total of the $k$ scores. This is proposed as a test criterion for the null hypothesis that each subject is as likely as not to rate his own occupation higher than any other of the $n$ occupations. Under this model for $k$ throws of a pair of coins, $S$ is the excess of the number of cases of both heads over the number of cases of both tails. A table is given for $\Pr(|S| \geqslant S_0)$ for $n = 3(1)9$ to at least 4D always giving at least 2S.

C. C. C.

**1173[K].**—Joseph Berkson, "A statistically precise and relatively simple method of estimating the bio-assay with quantal response, based on the logistic function," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 565–599.

This paper sets forth very clearly and succintly the author's "minimum logit chi-square" method in bio-assay. To facilitate the necessary computations three tables are provided. Table 1 gives $l = \ln [p/(1 - p)]$ to 5D for $p = .001\ (.001)\ .999$. Table 2 inverts Table 1, giving $p$ to 5D for $l = 0(.01)$-4.99. Table 3 gives the weights needed in the logistic calculation, namely $p(1 - p)$ and $p(1 - p)l$, to 4D, for $p = 0(.001)1$.

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**1174[K].**—K. N. CHANDLER, "The distribution and frequency of record values," Roy. Stat. Soc., *Jn.*, s.B, v. 14, 1952, p. 220–228.

The frequency and interval of occurrence of record values is of considerable interest whether we are dealing with weather data or sampling other types of populations. The lowest record value at any given time is defined as that member of the sample which is less than or equal to all previous members and the greatest record value similarly is defined as that value which is greater than or equal to all previous values. In this paper Chandler considers the random series $x_u$, $u = 1, 2, 3$, etc. with the first record value $X_1$ equal to $x_1$. Let $X_i$ be the first occurring value of $x$ which is less than $X_{i-1}$, $i = 2, 3$, etc. Then $X_r$ is called the $r$th lower record value (similar considerations apply to the higher record values). Assuming that the distribution function of $x$ is either normal or rectangular, Chandler derives the probability distribution of the $r$th lower record value $X_r$ (or $r$th higher record value), the distribution of the serial number, $u_r$, of the lower record value and also the probability distribution for the interval of occurrence (number of observations) between the $r$th lower record value and the $(r - 1)$st lower record value.

Table 1 and Table 2 of the paper give the .005, .01, .1, .5, .9, .99 and .995 probability points for the distribution of $X_r$ for the normal distribution in standard units to 3D and for the rectangular distribution to 4S for $r = 2(1)9$ in both cases. Table 3 gives the probability that the serial number, $u_r$, of the lower record value, will be $\leqq n$ to 6D for $r = 3(1)9$ and $n = 3(1)30(5)60(10)100$, 200, 500, 1000, 2000, 5000, 10000, 20000, 50000, 100000, 200000, 500000, 1000000. Table IV gives the probability that the interval of occurrence, $u_r - u_{r-1}$, the number of observations between the $r$th lower record value and the $(r - 1)$st lower record value $\geqslant n$ to 6D for $r = 2(1)9$ and the same set of values of $n$ as in Table 3. The serial numbers $u_r$, and their differences, $u_r - u_{r-1}$, do not have finite means and are independent of the parent population provided it has finite or zero probability density at all points!

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**1175[K].**—W. J. DIXON, "Power functions of the sign test and power efficiency for normal alternatives," *Annals Math. Stat.*, v. 24, 1953, p. 467–473.

The two sided sign test has long been used as a non-parametric test of significance. In using such a test one generally compares the number of changes in sign that occur in his observations with the expected number under a given hypothesis. In using such a non-parametric technique, one is interested in its power as well as how this power compares with the power of corresponding parametric tests. This paper contains tables which help the research worker answer such questions.

Tables I and II give respectively the power of the sign test to 5D for the 5% and 1% level of significance for sample sizes $N = 5(1)20(5)50(10)100$ in testing the null hypothesis that the proportion of objects in the population is .5 against varying proportions, $p = .05(.05).95$ in the alternative population.

To compare the power of the sign test with normal alternatives the author introduces a power efficiency function which gives the power efficiency of the sign test as compared with the normally based test for each alternative. Tables for selected values of the parameters and representative curves of this power efficiency function are given. The results exhibited by this function are compared with various asymptotic and approximate efficiency estimates that have been obtained by various authors in the past.

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**1176[K].**—BENJAMIN EPSTEIN & MILTON SOBEL, "Life testing," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 486–502.

For a characteristic $X$ with density function

$$f(x;\theta) = (\exp(-x/\theta))/\theta$$

the maximum likelihood estimate of $\theta$ based on the first $r$ order statistics of a sample of $n$ is

$$\hat{\theta}_{r,n} = [x_{1,n} + \cdots + x_{r,n} + (n-r)x_{r,n}]/r. \quad (r \leqq n)$$

Furthermore, $2r\hat{\theta}_{r,n}/\theta$ is distributed as $\chi^2$ with $2r$ degrees of freedom. Since this result is independent of $n$, an appreciable saving of time can be made in situations, such as life testing, where the observations become available in order by basing the estimate of $\theta$ on the first $r$ available results from a larger sample. Table I tabulates to 2D the ratio $E(X_{r,n})/E(X_{r,r})$ of the expected time to obtain the first $r$ results from a sample of $n$ to the expected time to obtain all $r$ results from a sample of $r$ for $r = 1(1)5, 10$ and $n = 1(1)5(5)20$. A simple derivation of previously known forms for $E(X_{r,n})$ and $\mathrm{Var}(X_{r,n})$ is given.

It is also shown that the region of rejection for the best test of $H_1(\theta) = \theta_1$ against the alternative $H_2(\theta) = \theta_2 < \theta_1$ is given by $\hat{\theta}_{r,n} < C$. Table II tabulates for $\theta_1/\theta_2 = 1.5(.5)3(1)5(5)10$; $\alpha$, $\beta = .01$ and $.05$ the minimum value of $r$, and the corresponding upper and lower limits for $C/\theta_1$ to 4D, such that the errors of Type I and Type II will be less than $\alpha$ and $\beta$, respectively. This table represents a rearrangement and, in some respects, an extension of tables published by EISENHART[1] which (in the notation of the present paper) gives for fixed $r$, $\alpha$, and $\beta$ the maximum value of $\theta_1/\theta_2$ such that the test conditions are satisfied. It is also directly related to other tables dealing with the operating characteristics of the one-sided tests of a hypothetical variance $\sigma_0^2$ against the alternatives $\sigma^2 < \sigma_0^2$.

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[1] C. EISENHART, M. W. HASTAY, W. A. WALLIS, editors, *Selected Techniques of Statistical Analysis*. New York and London, 1947, ch. 8, p. 267–318.

**1177[K].**—H. L. JONES, "Approximating the mode from weighted sample values," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 113–127.

This paper presents a method of estimating the population mode by a weighted sum of the order statistics of a sample. The derivation of the values of the weights is based on an approximation to the maximum likelihood solution. The parametric form of the population is assumed to be known. The method is of a general nature but does not necessarily yield a reasonable approximation to the mode unless certain favorable conditions are satisfied. The case of a $t$-distribution with varying degrees of kurtosis is analyzed to illustrate application of the method. Table 1 contains weights to 2D to be used in approximating the mode of a $t$-distribution for $n$ (sample size) = 3(1)10 and $\alpha_4 = \mu_4/\mu_2^2 = 3(.5)5, 6, 9, \infty$.

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**1178[K].**—R. KAMAT, "On the mean successive difference and its ratio to the root mean square," *Biometrika*, v. 40, 1953, p. 116–127.

Given a sequence of $n$ normal variates $\{x_i\}$ with common mean and variance. The mean square successive difference is

$$d = (n-1)^{-1} \sum_{i=1}^{n-1} |X_i - X_{i+1}|.$$

Table 1 presents the standard deviation, $\beta_1$ and $\beta_2$ of $d/\sigma$, for $n = 3(1)$ 10(5) 30, 40, 50 to 4D. Table 2 presents approximate upper and lower .5, 1, 2.5 and 5% percentage points of $d/\sigma$, to 2D, using a Pearson Type I curve; exact results are given for $n = 3$.

The ratio $W = d/s$, where $s$ is the sample standard deviation, is also considered. Table 4 presents the mean to 4D, the standard deviation to 4D, $\beta_1$ to 3D and $\beta_2$ to 2D of $W$ for $n = 5(5)$ 30, 40, 50. Upper and lower 0.5, 1, 2.5 and 5% points for $W$ are given in Table 5 for $n =$ 10(5)30(10)50 to 2D.

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**1179[K].**—TOSIO KITAGAWA, TEISUKE KITAHARA, YUKIO NOMACHI & NOBUO WATANABE, "On the determination of sample size from the two sample theoretical formulation," *Bulletin Math. Stat.*, v. 5, 1953, p. 35–46.

The authors consider a two-sample procedure alternative to that of Stein[1,2] for determining a confidence interval of fixed length $2d$ for the mean of a normal population with unknown variance. They give a table for the following function connected with the probability distribution of the second sample size:

$$I(n_2;n_1|d^2\sigma^{-2},\alpha,\beta) = \int_{b(n_2-1)}^{b(n_2)} \phi_{n_1}(s_1;\sigma)ds_1 \int_0^{c(n_2)} \phi_{n_2}(s_2;\sigma)ds_2,$$

where

$$b(n) = dn^{\frac{1}{2}}\{t_{n-1}(\alpha)\}^{-1}\{F_{n_1-1}^{n-1}\ (\beta)\}^{-\frac{1}{2}},$$

$$c(n) = dn^{\frac{1}{2}}\{t_{n-1}(\alpha)\}^{-1},$$

$t_{n-1}(\alpha)$ is the $100\alpha$-percentage point of the $t$-distribution with $n - 1$ degrees of freedom, $F_{n_1-1}^{n-1}\ (\beta)$ is the $100\beta$-percentage point of the $F$-distribution with

$n - 1$ and $n_1 - 1$ degrees of freedom,

$$s_i = \left\{ (n_i - 1)^{-1} \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2 \right\}^{\frac{1}{2}}, \quad i = 1, 2,$$

is the square root of the unbiased estimate of the population variance given by the first, respectively, second sample, of sizes $n_1$ and $n_2$, and $\phi_{n_i}(s_i;\sigma)$ denotes the density function of $s_i$. The tables are for only the selected combinations of the arguments $n_1 = 10, 15, 21, 25, 31$; $n_2 = 3(1)60$; $\alpha = .01, .05$; $\beta = .01, .05$; $d^2\sigma^{-2} = .75, .5, .25$.

JULIUS LIEBLEIN

NBSSEL

[1] W. G. COCHRAN, *Sampling Techniques*. New York, 1953, p. 59–60.
[2] B. M. SEELBINDER, "On Stein's two-stage sampling scheme," *Ann. Math. Stat.*, v. 24, 1953, p. 640–649.

**1180[K].**—J. LEFÈVRE, "Application de la théorie collective du risque à la réassurance 'Excess-Loss,'" *Skandinavisk Aktuarietidskrift*, v. 35, 1952, p. 161–187.

This paper contains a table of values of

$$B_n(u) = \int_0^\infty te^{-tu} d\phi^{(n)}(t)$$

in which $\phi(t)$ is the cumulative normal frequency function in standard units. Values are given for $n = 0, 3, 4, 6$ for $u = 0(.1)3$ to 5D.

C. C. C.

**1181[K].**—JACK MOSHMAN, "Critical values of the log-normal distribution," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 600–609.

The three-parameter log-normal distribution function may be written as

$$f(x) = \frac{(2\pi)^{-\frac{1}{2}}}{c(x-a)} \exp\left\{ -\frac{1}{2c^2}\left(\log\frac{x-a}{b}\right)^2 \right\}.$$

In terms of parameters $a$, $b$, and $c$, the mean, variance and skewness (third standard moment) of this distribution may be expressed as $\mu = b\omega^{\frac{1}{2}} + a$, $\sigma^2 = b^2\omega(\omega - 1)$, and $\alpha_3 = \pm (\omega - 1)^{\frac{1}{2}}(\omega + 2)$ where $\omega = \exp c^2$ and $\alpha_3$ takes the same sign as $b$. The author tabulates selected critical values $\tau_\beta$ of the standardized log-normal variate such that $P(\tau \geqq \tau_\beta) = \beta$, where $\tau = (x - \mu)/\sigma$. Tabulations are to 3D for $\beta = .005, .01, .025, .05, .10, .90, .95, .975, .99$, and $.995$ with $\alpha_3 = 0(.05)3$. Accuracy to within one or two digits in the last decimal is claimed for all table entries and the author indicates that three point Lagrangian interpolation will give similar accuracy for intermediate values of $\alpha_3$.

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**1182[K].**—K. R. NAIR, "Tables of percentage points of the 'Studentized' extreme deviate from the sample mean," *Biometrika*, v. 39, 1952, p. 189–191.

Let $x_\nu(\nu = 1(1)n)$ be the $\nu$th ordered variate in a sample of size $n$ taken from a normal population with unknown standard deviation $\sigma$. If an estimate $s_r$ of $\sigma$ is available with $r$ degrees of freedom, independent of the sample, the author suggested the use of the Studentized extreme deviation $(x_n - \bar{x})/s_r$ or $(\bar{x} - x_1)/s_r$ as a test criterion for a single outlier. In a previous paper[1] he gave the lower and upper 5% and 1% points of this deviate. These tables have now been extended to cover four more percent points, namely 10, 2.5, .5 and .1%. Table 1A gives the 6 lower percentage points mentioned to 2D for $n = 3(1)9$ and $r = 10, 15, 30, \infty$. Table 1B gives the same six upper percentage points to 2D for $n = 3(1)9$ and $r = 10(1)20, 24, 30, 40, 60, 120, \infty$.

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[1] K. R. NAIR, "The distribution of the extreme deviate from the sample mean and its Studentized form," *Biometrika*, v. 35, 1948, p. 118–144.

**1183[K].**—E. S. PEARSON & H. O. HARTLEY, "Charts of the power function for analysis of variance tests, derived from the non-central $F$-distribution," *Biometrika*, v. 38, 1951, p. 112–130.

Let $u_i(i = 1, 2, \cdots, \nu)$ be $\nu$ normally distributed independent variables with unit variance and zero mean and let $a_i(i = 1, 2, \cdots, \nu)$ be $\nu$ fixed constants, then the distribution of

$$\chi'^2 = \sum_{i=1}^{\nu} (u_i + a_i)^2,$$

which is a Bessel function, is called the non-central chi-square distribution with $\nu$ degrees of freedom and $\lambda = \sum_{1}^{\nu} a_i^2$ is called the non-centrality parameter. If $\chi_1'^2$ is such a value with $\nu_1$ degrees of freedom and $\chi_2^2$ another independent central chi-square with $\nu_2$, then $F' = (\chi_1'^2\nu_2)/(\chi_2^2\nu_1)$, called the non-central variance ratio, has a known distribution. Its numerical values are obtained with the help of the incomplete $\beta$ functions. The probability $\beta(\lambda | \alpha, \nu_1, \nu_2) = Pr\ (F' > F_\alpha)$ regarded as a function of $\lambda$ is the power function of the analysis of variance test with significance level $\alpha$. On the basis, mainly, of TANG's table[1] eight charts are given corresponding respectively to $\nu_1 = 1(1)8$ for $\beta$ at the levels $\alpha = .05$ and $\alpha = .01$. Here the non-centrality parameter $\varphi = (\lambda/(\nu_1 + 1))^{\frac{1}{2}}$ is used as the abscissa instead of $\lambda$ on a linear scale. Each chart gives two families of eleven power curves corresponding to $\nu_2 = 6(1)10, 12, 15, 20, 30, 60, \infty$ for the two values of $\alpha$. The use of a logarithmic scale for the ordinate $\beta$ straightens the curves and expands them in the region of high power $.80 \leq \beta \leq .99$. The $\beta$ grid, .1(.1).5(.05).7(.02).90(.01).99, and the $\varphi$ grid of .2 for $\alpha = .01$ and of .1 for $\alpha = .05$ are sufficiently fine to allow interpolation by sight. The calculation of $\varphi$ is shown for the one way classification into $k$ groups with $n$ observations in each, for double classification with one observation in each cell and for the latin square arrangement.

The use of the charts is explained for the analysis of the effect of machine variations on the standard deviation of manufactured bulk product and of the effect of personal factors introduced in routine tests.

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[1] P. C. TANG, "The power function of the analysis of variance test with tables and illustrations of their use," *Stat. Res. Memoirs*, v. 2, 1938, p. 126–157.

**1184[K].**—FRANK PROSCHAN, "Confidence and tolerance intervals for the normal distribution," Amer. Stat. Assn., *Jn.*, v. 48, 1953, p. 550–564.

The author presents an excellent summary of confidence and tolerance intervals for the normal distribution for the various combinations of known and unknown mean and standard deviation. Let $x$ be normally distributed with mean $\mu$ and standard deviation $\sigma$. Define

$$\bar{X} = \sum_{i=1}^{n} X_i/n, \quad s^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1).$$

Let $m$ represent either $\mu$ or $\bar{X}$; let s.d. represent either $\sigma$ or $s$. Then either confidence interval statements or tolerance interval statements may be made about $m \pm k$ s.d., where the value of $k$ depends on the particular type of interval and whether or not $\mu$ and $\sigma$ are known or unknown. All tables of $k_i$, $i = 1(1)9$ are given for $n = 2\ (1)\ 30, 40, 60, 120, \infty$, to 3D.

Given $\sigma$ known, the 50% confidence interval for $\mu$ is given by $\bar{X} \pm k_1\,\sigma$, $k_1 = .6745/\sqrt{n}$; if both $\mu$ and $\sigma$ are unknown, the 50% confidence interval is given by $\bar{X} \pm k_2 s$; if both $\mu$ and $\sigma$ are unknown, $\overline{X_1} \pm k_3 s$ provide 50% confidence interval for the second sample mean $\overline{X_2}$, when $n_1 = n_2$.

The next $k_i$, $i = 4(1)7$, refer to tolerance limits. If both $\mu$ and $\sigma$ are known $\mu \pm k_4\sigma$, $k_4 = .6745$ provide tolerance (probability) limits such that the proportion $p$ of the population included by the interval is .50. In case $\mu$ and $\sigma$ are unknown the factors $k_{5,a}$ in the tolerance limits $\bar{X} \pm k_{5,a}s$ are given for $a = .50, .75, .95, .99, .999$, i.e. the average $p$ contained in $\bar{X} \pm k_{5,a}s$ will be $a$. If $\mu$ is unknown and $\sigma$ known, tolerance limits are found by use of $\bar{X} \pm k_6\sigma$, and $k_6$ is tabulated for $a = .50$; for $\mu$ known, $\sigma$ unknown, tolerance limits are given by $\mu \pm k_7 s$ and $k_7$ is tabulated for $a = .50$.

The final cases $k_8$ and $k_9$ refer to confidence statements about tolerance limits. BOWKER[1] has tabulated the values of $k$ such that the probability is $\gamma$ that $\bar{X} \pm ks$ will include $p$ or more of the population. In Bowker's tables $p$ and $\gamma$ are listed for all combinations of .75, .90, .95, .99 and .999, $\mu$ and $\sigma$ unknown. The author assumes $\mu$ known and $\sigma$ unknown and gives values of $k_8$ for $p = \gamma = .50$; and $\mu$ unknown, $\sigma$ known, $k_9$ for $p = \gamma = .50$.

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[1] C. EISENHART, M. W. HASTAY, W. A. WALLIS, editors, *Selected Techniques of Statistical Analysis*. New York and London, 1947, ch. 2 by A. H. BOWKER, p. 102–107.

**1185[K].**—S. RUSHTON, "On sequential tests of the equality of variances of two normal populations with known means," *Sankhyā*, v. 12, 1952, p. 63–78.

Tables are given for sequential tests of the equality of variances in two samples from normal populations using sums of squares (a) about population means, (b) about sample means and using ranges of four and eight observations. $U$ is the ratio of sums of squares from the two samples. Sampling is continued unless $U \leqslant U'$ (accept $\sigma_1 = \sigma_2$) or $U \geqslant U''$ (accept $\sigma_1 = \delta\sigma_2$). Tables list $U'$ and $U''$ for degrees of freedom $n = 1(1)10(2)20,25,30$ and $\delta = 1.5, 2.0, 3.0$ for all combinations of $\alpha$ and $\beta = .01, .05, .10$. The test using ranges is based on $R_n$, the ratio of sums of ranges from the two samples. The tables list $R_n'$ and $R_n''$ for the test as above and for the same range of parameters except that $\alpha$ and $\beta$ are restricted to .01 and .05 and $n$ refers to the number of ranges of 4 or 8 observations so that the sample sizes are $4n$ and $8n$.

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**1186[K].**—E. S. SMITH, *Binomial, Normal and Poisson Probabilities.* Published by the author, Box 224C, RD2, Bel Air, Md., 1953. 71 p., 21.6 × 27.9 cm. $2.50.

This is a small set of tables and charts, centering on the cumulative binomial distribution, with extended discussion of actual (some novel) as well as alternative methods of computation. Many of the tables shown are obtainable from larger tables constructed by others (referred to in the text). Tables are:

(1) Maximum Poisson probabilities $p_i(x, a)$ for various $a$ ($x$, of course, $= 0, a, a - 1$), a monotone function falling from .999001 (at $x = 0$) for $a = .001$ to .039861 for $a = 100$ to 6D

| | | |
|---|---|---|
| (2) $n!$ and $\log n!$, | $n = 0(1)200$ | 10S and 10D respectively |
| (3) $\log n$, | $n = 1(.01)10$ | 10D |
| (4) $C_x^n$ and $\log C_x^n$, | $n = 1(1)50$ | 5S and 5D respectively |
| (5) $e^{-x}$, | $x = 0(.001)1(1)100$ | 10D |
| (6) $B(c,n,p)$, | $n = 1(1)20; p = .01(.01)5$ | 5D |

where $B(c,n,p) = \sum_{x=c}^{n} C_x^n p^x (1 - p)^{n-x}$

(7) Normal (Gaussian) integrals $\int_0^t \phi(y)dy$, density functions $\phi(t)$, second derivatives $\phi^{(2)}(t)$, $t = 0(.01)4$, to 5D.

(8) $p(c,a)$, $c=1(1)22$; $a=.001(.001).01(.01).1(.1)2(1)10$ and $\frac{c}{a} =.1(.1)2.2$; $a = 10(10)100$ to 5D,

where $p(c,a) = \sum_{x=c}^{\infty} e^{-a}a^x/x!$

The cumulative binomial probabilities are obtained (a) directly, (b) from the Poisson cumulatives (singly or the two-term Gram-Charlier Type B), (c) from the normal (singly or the two-term Gram-Charlier Type A, with or without remainder modifications) depending on the ranges of the binomial parameters. There are a number of useful charts, not, to my knowledge, to be found elsewhere. These include (1) the probability of $n$ successes in $n$ trials, with constant probability of success in single trial, (2) the expected number ($np$) of binomial successes, (3) (most useful of all) a set of charts, some expressed in terms of correction factors, comparing the normal, binomial, and Poisson probabilities for various ranges of the parameters.

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**1187[K].**—H. THEIL, "On the time shape of economic microvariables and the Munich business test," Inst. International Stat., *Revue*, v. 20, 1952, p. 105–120.

This paper contains a table giving the frequency distributions of the difference between two independent random variables each obeying Poisson distributions with means $m$ and $\mu$ respectively. Values are given to 3D for $m,\mu = \frac{1}{2}, 1, 2, 4$.

C. C. C.

**1188[K].**—H. URANISI, "The distribution of statistics drawn from the Gram-Charlier Type A population," *Bull. Math. Stat.*, v. 4, 1950, p. 1–14.

From the expansion of a frequency function in Gram-Charlier Type A series, the author obtains early terms of a similar expansion of the distribution of the $t$-statistic for a sample of $n$. For both the one-tailed and two-tailed cases, four coefficients of this series are tabled to 6D for $n = 5, 10, 15, 21$ for values of the argument $u = (1 + t^2/[n - 1])^{-1} = .05, .1(.1).9, .95, 1$. With the aid of these tables it is possible to estimate the tail probabilities for given values of $\beta_1$, $\beta_2$, $\beta_3$, and $\beta_4$, and this is done for several examples. The results are not entirely comparable with those of GAYEN,[1] in that the latter employed the Edgeworth series.

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[1] A. K. GAYEN, "The distribution of 'Student's' $t$ in random samples of any size drawn from non-normal universes," *Biometrika*, v. 36, 1949, p. 353–369.

**1189[K].**—J. WESTENBERG, "A tabulation of the median test with comments and corrections to previous papers," K. Ned. Akad. v. Wetensch., *Proc.*, v. 55, s.A, 1952, p. 10–15.

Let $X_1, \cdots, X_{N_1}$ and $Y_1, \cdots, Y_{N_2}$ be samples of $N_1$, and $N_2$ observations drawn at random from populations having continuous distribution functions $F_1(x)$ and $F_2(y)$ respectively. Let $\delta$ be the number of observations belonging to one of the samples that lies between the median of that sample and the median of the combined sample. The hypothesis $H_0$ that $F_1 \equiv F_2$ is rejected when $\delta$ exceeds a critical value $\delta_0$. Tables of $\delta_0$ to 1D are given for significance levels .001, .005, .01, (.01), .05 when considering two-sided alternatives to $H_0$ and .0005, .0025, .005, .01, .015, .02, .025 when considering one-sided alternatives to $H_0$ for $N_1, N_2 = 6, 10, 20, 50, 100, 200, 500, 1000, 2000$.

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**1190[K].**—J. W. WHITFIELD, "The distribution of total rank value for one particular object in $m$ rankings of $n$ objects," *British Jn. of Stat. Psychology*, v. 6, 1953, p. 35–40.

The problem considered is, essentially, that of the distribution of the sum of $m$ independent random variables, each with uniform distribution on the integers $1, 2, \cdots, n$. The distributions, of course, are symmetric about $\frac{1}{2}m(n+1)$. Three pages of tables give the lower halves of the cumulative distribution functions to 5D for $m = 2$ (1) 8 and $n = 3$ (1) 8.

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**1191[L].**—C. DOMB, "Tables of functions occurring in the diffraction of electromagnetic waves by the earth," *Advances in Physics*, v. 2, 1953, p. 96–106.

The tables given in the paper are related to the Airy integral

$$\mathrm{Ai}(z) = \pi^{-1} \int_0^\infty \cos\left(\tfrac{1}{3}t^3 + zt\right) dt$$

in the complex plane.

The numbers $-a_n$ being the zeros of $\mathrm{Ai}(z)$, the author puts

$$f_n(y) = \exp(\lambda_n + i\mu_n) = \frac{\mathrm{Ai}[-a_n + y\exp(\pi i/3)]}{\mathrm{Ai}'(-a_n)\exp(\pi i/3)};$$

the numbers $b_n$ being roots of the equation

$$\mathrm{Ai}(z) = t\mathrm{Ai}'(z)\exp(-5\pi i/12),$$

in which $t$ is a parameter, he also puts

$$\xi_n + i\eta_n = b_n \exp(\pi i/6)$$

and

$$\exp(\gamma_n + i\delta_n) = \tfrac{1}{2}\pi^{-\frac{1}{2}} \exp(-5\pi i/12)[\mathrm{Ai}'(b_n)]^{-1} \times [1 - t^2 b_n \exp(-5\pi i/12)]^{-\frac{1}{2}}.$$

Tables of the Airy integral for real argument have been reviewed in *MTAC*, v. 2, p. 302–305, RMT **413**, and for complex argument, in v. 2, p. 309, RMT **420**.

The author tabulated the functions $f_n(y)$ for $n = 1(1)5$ in 1942, under the guidance of J. C. P. MILLER. These tables were subsequently checked, corrected, and sub-tabulated by the Mathematics Division of the National

Physical Laboratory of Great Britain. Photostatic copies are available from H. M. Nautical Almanac Office, Great Britain. The computations of Admiralty Computing Service are described in *MTAC*, v. 2, p. 35, RMT **260.**

Table 1 of the present paper gives 3D values of $\lambda_n$ and $\mu_n$ for $n = 1(1)5$, $y = 0(.2)3(1)10$ and 3D values of $\lambda_n$ and $\mu_n + \frac{2}{3}y^{\frac{3}{2}}$ for $n = 1(1)5$ and $y = 10(10)100$.

Table 2 gives 3D values of $\xi_n$ and $\eta_n$ for $n = 1(1)5$, $t = 0(.1)1$, $t^{-1} = 1(-.1)0$.

Table 3 gives 3D values of $\gamma_n$ and $\delta_n$ for $n = 1(1)5$ and for $t$ ranging from 0 to $\infty$; the number of selected values of $t$ varies with $n$.

There is a brief description of the computations, and an indication of the application of the functions tabulated here.

A. E.

**1192[L].**—M. Mashiko, *Tables of Generalized Exponential-, Sine-, and Cosine Integrals* Ei$(x + iy)$, Si$(x + iy)$, Ci$(x + iy)$. Numerical Computation Bureau, Tokyo, Japan, Report No. 7, March 1953, 43 p.

Let

$$I(z) = \int_z^\infty t^{-1}e^{-t}dt.$$

For $z = \xi e^{i\alpha}$ $(0 \leq \xi \leq 5)$ put

$$I(\xi e^{i\alpha}) = C_\alpha(\xi) - iS_\alpha(\xi).$$

For $z = \frac{1}{\eta} e^{i\alpha} (\eta \leq 0.2)$ put

$$I\left(\frac{1}{\eta} e^{i\alpha}\right) = \frac{e^{-z}}{z} A_\alpha(\eta) \exp(i\Phi_\alpha(\eta)).$$

The report contains values of

(a) $C_\alpha(\xi) + \log \xi$ and $S_\alpha(\xi)$ to six decimal places with second differences for $\xi = 0(.05)5.00$, $\alpha = 0°(2°)60°(1°)90°$.

(b) $A_\alpha(\eta)$ to six decimal places, $\Phi_\alpha(\eta)$ to five decimal places, each with second differences, for the above range of $\alpha$ and $\eta = 0(.01).20$. In defining the ranges, the reversals of the inequality signs are presumably misprints.

The present table is therefore concerned with the exponential integral for complex argument, not the generalized exponential-integral which has been tabulated by the Harvard University Computation Laboratory [*MTAC*, v. 4, p. 92–93]. It breaks new ground in covering, for complex arguments in polar form, the whole first quadrant.

No statement is made as to the accuracy of the table. Spotchecking a few values against the forthcoming NBS *Tables of Exponential Integrals for Complex Arguments* (cartesian form) did not reveal any discrepancy.

Everett's interpolation formula involving second differences gives the maximum attainable accuracy in both directions. In order to facilitate interpolation, a four decimal place table of Everett second-order interpolation coefficients for arguments at intervals of .01 is also given.

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**1193**[L].—NBSCL, "Struve function of order three-halves," NBS, *Jn. Research*, v. 50, 1953, p. 21–29.

Tables of

$$h_{\frac{3}{2}}(x) = \left(\frac{2\pi}{x}\right)^{\frac{1}{2}} H_{\frac{3}{2}}(x)$$
$$= 1 + \frac{2}{x^2} - \frac{2}{x}\left(\sin x + \frac{\cos x}{x}\right),$$

to 10D, with second central differences, modified when necessary (modification being indicated by a letter $C$ placed after the entry); $x = 0(.02)15$. "The values are expected to be correct to within one unit of the last place."

The entire computation of this table was done on the SEAC, under the supervision of ETHEL MARDEN, by KATHRYN CHRISTOPH, ANNE FUTTERMAN, RENEE JASPER, SALLY TSINGOU, and BERNARD URBAN.

The table is also available on IBM cards.

A. E.

**1194**[L].—HERBERT E. SALZER, RUTH ZUCKER, & RUTH CAPUANO, "Table of the zeros and weight factors of the first twenty Hermite polynomials," *Jn. Research*, NBS, v. 48, 1952, p. 111–116.

The definition of Hermite polynomials used in this paper is

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2}).$$

$x_i^{(n)}$ is the $i$-th positive zero of $H_n(x)$,

$$\alpha_i^{(n)} = \pi^{\frac{1}{2}} 2^{n+1} n! [H_n'(x_i^{(n)})]^{-2}$$

is the corresponding Christoffel number, and

$$\beta_i^{(n)} = \alpha_i^{(n)} \exp [(x_i^{(n)})^2].$$

The present paper gives 15D values of $x_i^{(n)}$ and 13S values of $\alpha_i^{(n)}$ and $\beta_i^{(n)}$ for $n = 1(1)20$, $i = 1(1)n$. A list of references (28 items) is appended.

Other tables of zeros of Hermite polynomials are referred to in *MTAC*, v. 1, p. 152, RMT **131**; v. 3, p. 26, RMT **466**; v. 3, p. 416, RMT **619**; v. 3, p. 473, RMT **641**; v. 6, p. 232, RMT **1034**. The present tables were compared with those of REIZ [RMT **466**], GREENWOOD & MILLER [RMT **619**] and KOPAL [RMT **641**], and with the HARVARD tables [RMT **1034**].

A. E.

**1195**[L].—K. M. SIEGEL, J. W. CRISPIN, R. E. KLEINMAN & H. E. HUNTER, "Note on the zeros of $(dP_{m_i}^{(1)}(x)/dx)|_{x=x_0}$," *Jn. Math. Phys.*, v. 32, 1953, p. 193–196.

The authors apply the identical technique of a previous paper to obtain approximate values $m_i$ such that $dP^{(1)}{}_{m_i}(x_0)/dx = 0$ and values of

$$\int_{x_0}^{1} x_0 [P_{m_i}^{(1)}(x)]^2 \, dx$$

[see *MTAC*, v. 7, p. 183]. Again, the theory is demonstrated for $x_0 =$

cos 165°. For this case, the first 15 approximate values of $m_i$ and corresponding values of the integral are tabulated.

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**1196[L].**—N. B. SLATER, "Gaseous unimolecular reactions: theory of the effects of pressure and of vibrational degeneracy," Roy. Soc. London, *Philos. Trans.*, s.A, v. 246, 1953, p. 57–80.

Table 3 (p. 71) gives values to 2 or 3S of

$$I_n(\theta) = [\Gamma(m+1)]^{-1} \int_0^\infty x^m e^{-x}(1 + \theta^{-1}x^m)^{-1}dx \qquad m = (n + 1)/2$$

for $n = 3(2)13$ and some of the values $\log_{10}\theta = -2(1)8$.

Table 4 (p. 71) gives 2 or 3S values for $n = 3(2)13$ of $\theta_5$ and $\theta_{50}$ such that $I_n(\theta_5) = .95$, $I_n(\theta_{50}) = .50$, and also $\theta_5/\theta_{50}$.

A. E.

**1197[L].**—R. C. T. SMITH, "Conduction of heat in the semi-infinite solid, with a short table of an important integral," *Australian Jn. Phys.*, v. 6, 1953, p. 127–130.

Table 1 (p. 128–129) gives 5D values of

$$\int_0^U (1 + u^2)^{-1} \exp [-\alpha(1 + u^2)]du$$

for $\alpha = .1(.1)2$ and $U = .1(.1)2, 2.5, 3, \infty$, and also for $\alpha = 2.5, 3, 4, 5$ and a shorter range of $U$.

A. E.

**1198[L].**—MICHAEL TIKSON, "Tabulation of an integral arising in the theory of cooperative phenomena," NBS, *Jn. Research*, v. 50, 1953, p. 177–178.

Table 1. Values of the coefficients $c_{2m}$ in the expansion

$$[I_0(x)]^3 = \sum_0^\infty c_{2m}x^{2m}$$

for $m = 0(1)20$. Here $I_0(x)$ is the modified Bessel function of order zero.

Table 2. Values of

$$I(b) = \pi^{-3} \int_0^\pi \int_0^\pi \int_0^\pi [3b - (\cos x + \cos y + \cos z)]^{-1}dx\, dy\, dz$$

$$= \sum_0^\infty (2m)!\, c_{2m}\, (3b)^{-2m-1}$$

to 5D for $b^{-1} = .01(.01)1$. For $b^{-1} \leq .8$, the expansion in powers of $b^{-1}$ was used to compute $I(b)$, at most 21 terms of this expansion being required. The remaining values of $I(b)$ were obtained by numerical integration.

A. E.

**1199[L,V].**—C. TRUESDELL, "Precise theory of the absorption and dispersion of forced plane infinitesimal waves according to the Navier-Stokes equations," *Jn. Rational Mech. and Analysis*, v. 2, 1953, p. 643–741. Tables (p. 723–734) computed by HARRISON HANCOCK. Graphs (p. 735–741).

These tables give results on the calculation of the propagation of plane infinitesimal pressure waves (sound waves) in a uniform fluid governed by the Navier-Stokes equations. The fluid is characterized by the viscosity coefficients $\mu$ and $\lambda$, the coefficient of heat conduction $\kappa$, and the specific heats $c_p$, and $c_v$. The tables are divided according to the two parameters, the ratio of specific heats

$$\gamma = c_p/c_v$$

and the thermoviscous number $Y$,

$$Y = \frac{\kappa}{(\lambda + 2\mu)c_p}.$$

Tables given are for $(\gamma, Y)$ of the following values $(1, Y)$ (piezotropic fluids); (1.10, .5), (1.10, .8), (1.25, .25), (1.25, .5), (1.25, .8), (1.40, .25), (1.40, .5), (1.40, .8), $\left(\frac{5}{3}, .25\right)$, $\left(\frac{5}{3}, .5\right)$, $\left(\frac{5}{3}, .6\right)$, $\left(\frac{5}{3}, .8\right)$, $\left(\frac{5}{3}, 1.05\right)$, $\left(\frac{5}{3}, 1.25\right)$, (1.8, .3), (1.8, .5), (2, .3), (2, .5), (2.2, .3), (2.2, .5) (weak and moderate conductors); (1.10, 20), (1.10, 30), (1.10, 40), (1.15, 10), (1.15, 20), (1.15, 25), (1.15, 30), (1.15, 35), (1.15, 40), (1.20, 20), (1.20, 30), (1.20, 40), (1.50, 50) (strong conductors). In each table, the various dimensionless quantities characterizing the speed, the absorption and the dispersion of a plane sound wave of circular frequency $\omega$ are listed against the argument $X$, the frequency number

$$X = \frac{(\lambda + 2\mu)}{\gamma p}$$

where $p$ the undisturbed pressure of the fluid. The ranges of argument is $X = .1(.2).7, 1(.5)5$.

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**1200[L].**—G. ZARTARIAN & H. M. VOSS, "On the evaluation of the function $f_\lambda(M, \omega)$," *Jn. Aeron. Sciences*, v. 20, 1953, p. 781–782.

9D table of the coefficients

$$a_{2n}(M) = \sum_{j=0}^{n} \frac{(2M)^{-2j}}{(j!)^2[2(n-j)]!},$$

$$b_{2n}(M) = \sum_{j=0}^{n} \frac{(2M)^{-2j}}{(j!)^2[2(n-j)+1]!}$$

for $n = 0(1)6$, $M = 5/4, 10/7, 3/2, 5/3, 2, 5/2$. These coefficients occur in the expansion

$$\int_0^1 e^{-i\omega u} J_0(\omega u/M) u^\lambda du$$

$$= \sum_{n=0}^{\infty} (-1)^n \left( \frac{a_{2n}(M)}{\lambda + 2n + 1} - i\omega \frac{b_{2n}(M)}{\lambda + 2n + 2} \right) \omega^{2n},$$

and may be used for computing the integral.

A. E.

**1201[Q].**—H. Wood, (a) Kepler's problem, Roy. Soc. of New South Wales, Jn. and Proc., v. 83, 1950, p. 150–163; (b) Kepler's problem—the parabolic case, v. 83, 1950, p. 181–194; (c) Tables for nearly parabolic elliptic motion, v. 84, 1951, p. 134–150; (d) Tables for hyperbolic motion, v. 84, 1951, p. 151–164; (e) Five figure tables for the calculation of ephemerides in parabolic and nearly parabolic motion, Sydney Observatory Papers No. 16, 1951. (*a*), (*b*), (*c*) and (*d*) are also Sydney Observatory Papers Nos. 10, 11, 14 and 15 respectively

The first four papers give the theory and seven-figure tables which are especially applicable to the problem of calculating the ephemerides of comets in parabolic or nearly parabolic orbits about the sun. The fifth paper gives five-figure tables which are sufficiently accurate for finding purposes. Some of these tabulations may be of more general mathematical interest.

The two body problem was solved (kinematically) by Kepler with the enunciation of his three laws of planetary motion. The problem of finding the coordinates of a planet, or comet, in the plane of its orbit in unperturbed motion is Kepler's problem. If $M$, $E$ and $v$ are the mean, eccentric and true anomalies, respectively, and $e$ the eccentricity, then: $M = E - e \sin E$, where $\tan E/2 = ((1 - e)/(1 + e))^{1/2} \tan v/2$. This equation is Kepler's equation; the necessity for its frequent, accurate solution, and the difficulties, both numerical and analytical which it presents, have kept alive interest in the equation for the past 300 years. In general, $e$ is quite small for the orbits of planets, asteroids and most binary stars and there are numerous satisfactory methods of solution. Cometary orbits, on the other hand, are very often parabolic, or nearly so, and are sometimes even hyperbolic because of planetary perturbations, or fictitiously hyperbolic because of observational errors. If $e$ is close to unity and $E$ small, special methods must be devised to solve Kepler's equation accurately. Wood writes the equation as follows:

$$D = 12k\,(1 + e)^{1/2} q^{-3/2} t = 12\mu + \mu^3(1 + \epsilon)6 \left( \frac{\sin^{-1}\epsilon^{1/2}\mu - \epsilon^{1/2}\mu}{\epsilon^{3/2}\mu^3} \right),$$

where $k$ is the Gaussian gravitational constant, $q$ the perihelion distance, $t$ the time ($t = 0$ at perihelion), $\epsilon = (1 - e)/(1 + e)$ and $\mu = y_0/q$ where $x_0$ and $y_0$ are the rectangular coordinates in the orbital plane with the $x_0$-axis directed towards perihelion.

For $\epsilon = 0$ ($e = 1$) we have the parabolic case and

$$D = 12\mu = \mu^3, \quad \text{where } \mu = 2 \tan v/2.$$

Table 1 in (*b*) gives $\mu$ to 7D for $D = 0(0.1)100$; to 6D for $D = 100(1)1000$. For $D > 1000$ use $\mu = D^{1/3} - 4/D^{1/3} + R$. Table 2 in (*b*) gives $R$ to 6D for

$4/D^{\frac{1}{3}} = 0(.01)0.41$. Previous tables are useful only up to $D = 88$ and Wood states that there have been 36 comets observed outside this range of $D$, with the probability that the future will see an increasingly higher percentage of such observations.

The above tables are also useful in the nearly parabolic case. For $e \neq 1$, $D$ is redefined as:

$$D = 12k(1 + e)^{\frac{1}{2}}q^{-\frac{3}{2}}ct = 12c\sigma + c^3\sigma^3,$$

and

$$\mu = \frac{c\sigma}{c}\{J - hK + R\},$$

where $c^2$ and $h$ are certain power series in $\epsilon$, and $J$ and $K$ are certain power series in $\epsilon\sigma^2$. The coefficients of the last terms used in the $K$ and $h$ series have been adjusted so that $R$ is negligible in the seventh decimal place. $\sigma$ is defined by the top equation and $c\sigma$ can be evaluated from Table 1 in (*b*). $c$ is tabulated to 7D and $h$ to 5D for $\epsilon = 0(.001)0.100$ in Table 2 in (*c*) (ellipse) and for $\alpha(= -\epsilon) = 0(.001)0.100$ in Table 2 in (*d*) (hyperbola). $J$ and $K$ are given to 7D for $\epsilon^{\frac{1}{2}}\sigma = 0(.001)0.600$ in Table 3 in (*c*) and similarly for $\alpha^{\frac{1}{2}}\sigma$ in Table 3 in (*d*).

Other useful quantities tabulated by Wood are of the form:

$$A = 6\left\{\frac{\sin^{-1}w - w}{w^3}\right\}, \quad I = 6\left\{\frac{1-\sqrt{1-w^2}}{w^2\sqrt{1-w^2}}\right\}, \quad \text{and } N = 2\left\{\frac{1-\sqrt{1-w^2}}{w^2}\right\},$$

where $w = \epsilon^{\frac{1}{2}}\mu$ for the ellipse and:

$$A = 6\left\{\frac{w - \sinh^{-1}w}{w^3}\right\}, \quad I = 6\left\{\frac{\sqrt{1+w^2}-1}{w^2\sqrt{1+w^2}}\right\}, \quad \text{and } N = 2\left\{\frac{\sqrt{1+w^2}-1}{w^2}\right\},$$

where $w = \alpha^{\frac{1}{2}}\mu$ for the hyperbola. If we define $D_1$ as:

$$D_1 = (1 + e)\,6\mu + A\mu^3, \quad \text{then } \frac{dD_1}{d\mu} = 6(1 + e) + \mu^2 I.$$

$A$ and $I$ are useful in the iterative computation of $\mu$ and of velocities. The other rectangular coordinate in the orbital plane is $\lambda = x_0/q = 1 - \mu^2N/2(1 + e)$. This becomes $\lambda = 1 - \mu^2/4$ for the parabolic case. Appropriate formulae are given to calculate the equatorial heliocentric coordinates $x$, $y$ and $z$, from $N$ and $\mu$.

$A$ and $N$ are tabulated to 7$D$ and $I$ to 4D for $\epsilon^{\frac{1}{2}}\mu = 0(.001)0.600$ in Table 1 of (*c*), and similarly for $\alpha^{\frac{1}{2}}\mu = 0(.001)0.600$ in Table 1 of (*d*).

Paper (*e*) gives 5D tables applicable to the three cases considered. There has been some slight rearrangement of the formulae and tabulated quantities but the details are probably not of sufficient general interest and will not be given here. The author states that, in general, the tables have been calculated to two extra decimal places. A very useful bibliography will be found in paper (*a*).

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