The integrals

$$F(y) = \int_0^T f(t)K(y,t)dt \quad \text{or} \quad \int_0^y f(t)K(y,t)dt$$

are evaluated by means of high speed analog computer elements for various values of y. The variable y is given a sequence of values and by a suitable switching procedure the values of $F(y)$ are evaluated at a rate of 60 per second and plotted with a reference value on a cathode ray tube. The functions $f(t)$ and $K(y,t)$ for y fixed are obtained by function generators or by differential analyzer techniques. During the generation of $F(y)$, y which should remain fixed does actually vary. The error due to this cause and the errors due to the multiplier and function generator are discussed.

F. J. M.

1164. J. R. RAGAZZINI & G. REYNOLDS, "The electronic complex plane scanner," *Rev. Sci. Instruments*, v. 24, 1953, p. 523–527.

A rational function $F(z)$ is supposed given in the form of a quotient of products of linear factors $z - \lambda_i$. It is required to obtain $\log |F(jw)|$ and $\arg F(jw)$ where $j = \sqrt{-1}$ and w is real. $\text{Log } |F(jw)|$ is the sum of $\pm \log |(jw - \lambda_i)|$. The quantity $jw - \lambda_i$ is represented as a complex voltage and $\log |(jw - \lambda_i)|$ is obtained by a suitable circuit. The argument of $F(jw)$ is obtained by differentiation using the Cauchy-Riemann equations.

F. J. M.

1165. J. TADAYON, "Measurement of the angle between two curves," *Rev. Sci. Instruments*, v. 24, 1953, p. 871–872.

An optical instrument for measuring the angle between two curves on a graph is based on the mirror principle for finding the normal to a curve.

F. J. M.

NOTES

168.—A PRACTICAL REFUTATION OF THE ITERATION METHOD FOR THE ALGEBRAIC EIGENPROBLEM. In the second part of my paper on algebraic eigenproblems[2] I have proved that the computation by means of the formation of the characteristic equation requires less computational work than the iteration method, and that this holds even when nothing but the first eigenvalue has to be calculated. This advantage grows with every accessory eigenvalue or vector. Further one has no trouble with deflation which requires a lot of multiplications. Also one can compute every eigenvalue and vector apart from the others, and do this to any desired accuracy, by the more powerful methods for algebraic equations. At last, there arise no difficulties from an unfavorable quotient of the two dominant eigenvalues.

The reason these facts are not yet universally acknowledged is that the iteration method seems to be simpler and more mechanical in its application. But one has to consider that iteration does not converge quickly enough in practice, unless the quotient of the two dominant eigenvalues is $\frac{1}{4}$ or less. This last will be rarely the case. For the eigenvalues must lie somewhere between two circles around the origin in the complex plane. The radius of

the inner circle has a value quite different from zero (if the determinant of the matrix is $\neq 0$). In this ring can lie only some of the n eigenvalues having a quotient less than $\frac{1}{4}$. The other eigenvalues must cluster around some circles. In proportion as the situation is more favourable for e.g. the two dominant eigenvalues, the smaller the room will be for the other ones. There is no escape from this situation, and sooner or later the then dominant eigenvalues will lie close together.

That iteration may give trouble in the most simple cases is a well known fact from numerous calculations. Yet the following example of a symmetric 4×4 matrix, the eigenproblem of which is therefore real, is amazing.

$$A = \begin{vmatrix} 2 & 1 & 3 & 4 \\ 1 & -3 & 1 & 5 \\ 3 & 1 & 6 & -2 \\ 4 & 5 & -2 & -1 \end{vmatrix}$$

Let μ be the dominant eigenvalue. Following AITKEN,[1] we begin the iteration with the usual starting vector $v = (1,1,1,1)'$ and so compute $v_m = Av_{m-1}$. The quotient μ_m of the first components of v_m and v_{m-1} yields for μ the "approximations":

$$S: \quad \mu_2 = 7.2,\ \mu_3 = 7.5,\ \mu_4 = 7.85,\ \mu_5 = 7.56,\ \mu_6 = 8.0,\ \mu_7 = 7.6,$$
$$\mu_8 = 8.08,\ \mu_9 = 7.667,\ \mu_{10} = 8.122,\ \mu_{11} = 7.683.$$

The convergence is not impressive, especially if one considers that the sequence S converges to about -8 (minus eight)! On the contrary, S seems to diverge or to "converge" to two limits.

To analyze the sequence S more deeply, all eigenvalues and vectors were computed from the characteristic equation, the vectors by the method described in BODEWIG,[2] part II, section 3, p. 1–3. These are to 8D

$$\mu^{(1)} = -8.02857835,\ \mu^{(2)} = 7.93290472,\ \mu^{(3)} = 5.66886436,\ \mu^{(4)} = -1.57319074.$$
$$x^{(1)} = (1,\ 2.50146030,\ -0.75773064,\ -2.56421169)'$$
$$x^{(2)} = (1,\ 0.37781815,\ 1.38662122,\ 0.34880573)'$$
$$x^{(3)} = (1,\ 0.95700152,\ -1.42046826,\ 1.74331693)'$$
$$x^{(4)} = (1,\ -0.90709211,\ -0.37759122,\ -0.38331238)'.$$

Since S does converge to $-8.0\ldots$ the situation seems to be that the μ_i diverge in the beginning. The μ_{2i} decrease and the μ_{2i+1} increase, and in order to converge to -8, the μ_{2i} go through zero and the μ_{2i+1} through ∞. By a special method the index has been determined where the signs change. This is the case at the 363rd or 364th iteration.

To demonstrate this conclusion we have computed the values of μ_i which would have resulted if the iterations had been really effected. The 100th to 102nd iteration would yield the "approximations":

$$\mu_{100} = 7.28514, \quad \mu_{101} = 8.64677.$$

The 200th to 202nd iteration would yield:

$$\mu_{200} = 5.96936, \quad \mu_{201} = 10.57380.$$

The 300th to 302nd would be:

$$\mu_{300} = 2.86532, \quad \mu_{301} = 22.13220.$$

Our above conjecture concerning the μ_{2i} and μ_{2i+1} is therefore confirmed, as is also apparent from theoretical reasons. Near the critical point 363 we should get

$$\mu_{360} = 0.13587, \mu_{361} = 468.66407, \mu_{362} = 0.04022, \mu_{363} = 1583.336,$$
$$\mu_{364} = -0.055448, \mu_{365} = -1148.735.$$

The 400th to 402nd iteration would give

$$\mu_{400} = -1.75101, \quad \mu_{401} = -36.46902,$$

the 800th to 802nd:

$$\mu_{800} = -7.94348. \quad \mu_{801} - 8.11356,$$

and at last the 1200th to 1202nd:

$$\mu_{1200} = -8.02787279, \quad \mu_{1201} = -8.02927758.$$

Thus 1200 iterations will scarcely yield 4 figures of the dominant eigenvalue! And this for a simple matrix of order 4.

This amazing conduct can afterwards be explained from the knowledge of the eigenvalues and vectors. The slow convergence has two sources. Not only have the dominant eigenvalues nearly the same absolute value, but also the starting vector v is nearly orthogonal to $x^{(1)}$. In fact writing $x^{(1)}$ in the approximate form $(4,10,-3,-10)'$, the cosine between v and $x^{(1)}$ is only $1/30$ which is about cos 88°.

Yet the situation would not considerably improve if another starting vector, e.g. $(1,0,0,0)'$ would be taken. Nor should we waste time by computing the eigenproblem of matrices of the form $A + c$ with varying c's. The time is better spent by computing the characteristic equation as is pointed out in BODEWIG[2] (Part II in "Zusammenfassung"). Even the use of quadratic equations would not be very efficient in our case as $\mu^{(3)}$ lies close to $\mu^{(2)}$. Nor would this be the case for other matrices if one wants to have more than the two dominant eigenvalues, and even then the determination of the vectors is far from agreeable.

We owe to VON MISES[3] the discovery of the iteration method for finding the dominant value and vector and to AITKEN its application to all complicated cases more or less, and to HOTELLING the discovery of deflation. But times have changed. In our days also the higher eigenvalues and vectors must be computed and with extra accuracy. A deeper analysis shows that the iteration method in its present form is not appropriate and that even in cases when, by exception, the first eigenvalue is furnished quickly the later eigenvalues present the greater troubles.

So the whole eigenproblem must be considered anew.

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[1] A. C. AITKEN, "The evaluation of the latent roots and latent vectors of a matrix," Roy. Soc. Edinburgh, *Proc.*, v. 57, 1936–1937, p. 269–304.
[2] E. BODEWIG, "Bericht über die Methoden zur numerischen Lösung algebraischer Eigenwertprobleme," Seminario matematico e fisico dell' Universita di Modena, *Atti*, v. 4, 1949–1950, v. 5, 1950–1951.
[3] R. VON MISES, "Praktische Methoden zur Gleichungsauflösung," *Zeit. angw. Math. Mech.*, v. 9, 1929, p. 62–77.

169.—ANALYTICAL APPROXIMATIONS. [See also NOTE **157.**] The following are approximations for the exponential integral and certain Bessel functions.

(56) $0 \leq x \leq 1 \quad |\epsilon|_{\max} = 2 \times 10^{-7}$

$$-\mathrm{Ei}(-x) + \log_e x \doteq -.57721566 + .99999193\,x - .24991055\,x^2 + .05519968\,x^3 - .00976004\,x^4 + .00107857\,x^5$$

(57) $-3 \leq x \leq 3 \quad |\epsilon|_{\max} = 10^{-7}$

$$J_0(x) \doteq 1 - 2.2499997\,(x/3)^2 + 1.2656208\,(x/3)^4 - .3163866\,(x/3)^6 + .0444479\,(x/3)^8 - .0039444\,(x/3)^{10} + .0002100\,(x/3)^{12}$$

(58) $0 \leq x \leq 3 \quad |\epsilon|_{\max} = 2 \times 10^{-8}$

$$Y_0(x) - \frac{2}{\pi}\log_e\frac{x}{2}J_0(x) \doteq .36746691 + .60559366\,(x/3)^2 - .74350384\,(x/3)^4 + .25300117\,(x/3)^6 - .04261214\,(x/3)^8 + .00427916\,(x/3)^{10} - .00024846\,(x/3)^{12}$$

(59, 60) $3 \leq x < \infty$

$$J_0(x) = x^{-\frac{1}{2}} f_0(3/x)\cos\{x - \varphi_0(3/x)\}$$
$$Y_0(x) = x^{-\frac{1}{2}} f_0(3/x)\sin\{x - \varphi_0(3/x)\}$$

$|\epsilon|_{\max} = 10^{-8}$

$$f_0(3/x) \doteq .79788456 - .00000077\,(3/x) - .00552740\,(3/x)^2 - .00009512\,(3/x)^3 + .00137237\,(3/x)^4 - .00072805\,(3/x)^5 + .00014476\,(3/x)^6$$

$|\epsilon|_{\max} = 5 \times 10^{-8}$

$$\varphi_0(3/x) \doteq .78539816 + .04166397\,(3/x) + .00003954\,(3/x)^2 - .00262573\,(3/x)^3 + .00054125\,(3/x)^4 + .00029333\,(3/x)^5 - .00013558\,(3/x)^6$$

(61) $0 \leq x \leq 3 \quad |\epsilon|_{\max} = 5 \times 10^{-9}$

$$J_1(x)/x \doteq .5 - .56249985\,(x/3)^2 + .21093573\,(x/3)^4 - .03954289\,(x/3)^6 + .00443319\,(x/3)^8 - .00031761\,(x/3)^{10} + .00001109\,(x/3)^{12}$$

(62) $0 \leq x \leq 3 \quad |\epsilon|_{\max} = 5 \times 10^{-8}$

$$\left\{Y_1(x) - \frac{2}{\pi}\log_e\frac{x}{2}J_1(x)\right\}x \doteq -.6366198 + .2212091\,(x/3)^2 + 2.1682709\,(x/3)^4 - 1.3164827\,(x/3)^6 + .3123951\,(x/3)^8 - .0400976\,(x/3)^{10} + .0027873\,(x/3)^{12}$$

(63, 64) $3 \leq x < \infty$

$$J_1(x) = x^{-\frac{1}{2}} f_1(3/x)\sin\{x - \varphi_1(3/x)\}$$
$$Y_1(x) = -x^{-\frac{1}{2}} f_1(3/x)\cos\{x - \varphi_1(3/x)\}$$

$|\epsilon|_{\max} = 3 \times 10^{-8}$

$$f_1(3/x) \doteq .79788456 + .00000156\,(3/x) + .01659667\,(3/x)^2 + .00017105\,(3/x)^3 - .00249511\,(3/x)^4 + .00113653\,(3/x)^5 - .00020033\,(3/x)^6$$

$$|\epsilon|_{\max} = 8 \times 10^{-8}$$

$$\varphi_1(3/x) \doteq .78539816 - .12499612\,(3/x) - .00005650\,(3/x)^2 + .00637879\,(3/x)^3 - .00074348\,(3/x)^4 - .00079824\,(3/x)^5 + .00029166\,(3/x)^6$$

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170.—A SIEVE PROBLEM ON "PSEUDO-SQUARES." The following problem originated by KRAITCHIK,[1] and extended by LEHMER[2] by a special sieve, has recently been further extended by the SWAC. Let p be a prime. Let N_p be the least positive non-square integer of the form $8x + 1$ that is a quadratic residue of all primes $\leqslant p$. In this definition, zero is not counted as a quadratic residue so that N_p is not allowed to be divisible by any primes $\leqslant p$. Since squares are quadratic residues of any prime, the numbers N_p behave like squares and may be called pseudo-squares. This fact makes the problem of discovering pseudo-squares not only a sifting problem but also one of rejecting squares. Thus the problem is unsuitable for a high speed sieve alone since the output would be mostly squares, each of which would have to be tested by a more elaborate arithmetic unit. The problem is therefore one for an all-purpose computer programmed for sifting.[3]

Since for $p > 3$, N_p must be of the form $24x + 1$. One may proceed to exclude values of x using prime moduli between 5 and p inclusive. For every value of x not excluded the machine is programmed to extract the square root of $24x + 1$. If this is a perfect square, the machine returns to the sifting program for the next value of x. Fortunately the early part of the program, where the squares come thick and fast, had already been carried[2] as far as $N_{61} = 48473881$ in 1928 so that when programmed for the SWAC the routine spends most of its time sifting. Actually, for the record, the SWAC was instructed to print out every 64th square it produced. The complete table of N_p for $p < 83$ is as follows.

p	N_p	p	N_p	p	N_p
2	17	19	53881	47	9257329
3	73	23	87481	53	22000801
5	241	29	117049	59	48473881
7	1009	31	515761	61	48473881
11	2641	37	1083289	67	175244281
13	8089	41	3206641	71	427733329
17	18001	43	3818929	73	427733329
				79	898716289

All N_p above are primes except for

$$N_{11} = 19 \cdot 139$$
$$N_{17} = 47 \cdot 383$$
$$N_{29} = 67 \cdot 1747$$
$$N_{41} = 643 \cdot 4987$$

The interest in pseudo-squares is heightened by the fact that they may be used in tests for primality, as shown by MARSHALL HALL.[4] The operation of the SWAC and the reduction and checking of the output data was done by JOHN SELFRIDGE.

D. H. L.

[1] M. KRAITCHIK, *Recherches sur la Theorie de Nombres*, v. 1, Paris, 1924, p. 41–46.
[2] D. H. LEHMER, "The mechanical combination of linear forms," *Amer. Math. Monthly*, v. 35, 1928, p. 114–121.
[3] D. H. LEHMER, "The sieve problem for all-purpose computers," *MTAC*, v. 7, 1953, p. 6–14.
[4] M. HALL, "Quadratic residues in factorization," Amer. Math. Soc., *Bull.*, v. 39, 1933, p. 578–763.

171.—L. F. RICHARDSON (1881–1953). This English mathematician made several notable contributions to numerical analysis. A brief account of his life by P. A. SHEPPARD appears in *Nature* (v. 172, 1953, p. 1127–8).

His work on numerical analysis (apart from that appearing incidentally in his book[1]) was mainly contained in three long papers:

A: "The approximate arithmetical solution by finite differences of physical problems, involving differential equations with an application to the stresses in a masonry dam." Royal. Soc. *Phil. Trans.*, v. 210 A, 1910, p. 307–357.

B: "The deferred approach to the limit"—Part I, L. F. RICHARDSON, Part II, J. A. GAUNT, Royal Soc. *Phil. Trans.*, v. 226 A, 1926, p. 299–361.

C: "A purification method for computing the latent columns of numerical matrices and some integrals of differential equations," Royal Soc. *Phil. Trans.*, v. 242 A, 1950, p. 439–491.

There were minor contributions in

D: "Theory of the measurement of wind by shooting spheres upward," Royal Soc. *Phil. Trans.*, v. 223 A, 1923, p. 345–361.

An introduction to some of the material of A, B appeared in

E: "How to solve differential equations approximately by arithmetic," *Math. Gazette* v. 12, 1925, p. 415–421.

and one to some of the material of C in

F: "A method for computing principal axes," *British Jn. of Psychology*, v. 3, 1950, p. 16–20.

His work is highly individualistic and his language and symbolism picturesque. For instance he introduced the terms "marching" problem, for initial value problems of the form

$$y'' = ky, \quad y(0), \quad y'(0) \text{ given},$$

and "jury" problem for a problem of the form

$$y^{vi} - 3y^{iv} + 3y'' - y = \lambda y, \; y = y'' = y''' = 0 \text{ for } x = \pm 1.$$

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We shall discuss briefly two of Richardson's contributions to numerical analysis.

Richardson early in his life recognized the importance of the use of central difference operators in numerical applications, a fact which had been pointed out somewhat earlier by W. F. SHEPPARD. The use of central-difference approximation to derivatives suggested that the (local) difference between the solution to a discrete problem and that of the continuous problem would be a power series in h^2, h being the mesh-length. Taking account of the first term only, it is possible by solving the discrete problem for two values of h, and then eliminating the h^2-contribution to obtain a better approximate solution. This he called the "deferred approach to the limit." Among his examples were the extrapolation from the perimeter of a square and hexagon to that of a circle (see *MTAC* v. 2 1946, p. 114 and p. 223–4) and the evaluation of e. He also indicated the passage to the fundamental frequency of a continuous string from that of strings of beads. Richardson is mainly concerned with applications of his method to the numerical solution of differential equations; there are, however, discussions in D of a quadrature and the solution of a Volterra type integral equation.

In B he examines in some detail the justification of this process, considering two main questions: (1) Are the odd-powers of h always absent? (2) How small must h be in order that h^2-extrapolation may make an improvement? We shall not discuss this paper in detail: Undoubtedly the process is a valuable practical tool, but there are certainly cases where it is unreliable.[2] Richardson was fully aware of the possible failures and difficulties which might be encountered in its application and discussed various bad examples.

The latter part of A is concerned with a detailed study of the stresses in a dam, with particular reference to a model of the Assuan dam.

An interesting remark in A concerns the cost of computation about 1910. The unit operation was the evaluation of

$$v_N + v_W + v_S + v_E - 4v_0$$

and for this the rate was $n/18$ pence where n was the number of digits carried. An average output in the case $n = 3$ was of the order of 2,000 *correct* units per week, paying about 28 shillings or about 5 dollars at the then current rate.

The second contribution of Richardson which we shall discuss is an algorithm for the solution of a system of n linear equations, $A\mathbf{x} = \mathbf{b}$. This is to choose an arbitrary $\mathbf{x}^{(0)} = \{x_i^{(0)}\}$ and then put

$$\mathbf{x}^{(r+1)} = \mathbf{x}^{(r)} + \beta_r\{A\mathbf{x}^{(r)} - \mathbf{b}\},$$

where the factors β_r are to be chosen. Some suggestions for their choice was given in A; an up-to-date study has been given by D. M. YOUNG.[3]

The success of this algorithm can be established by expressing the errors in terms of the characteristic values of A. Following Young we assume that A has linearly independent characteristic vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$ with characteristic values $\lambda_1, \cdots, \lambda_n$. Then the error vector $\boldsymbol{\varepsilon}^{(r)} = \mathbf{x}^{(r)} - \mathbf{x}$ satisfies

$$\boldsymbol{\varepsilon}^{(r+1)} = \boldsymbol{\varepsilon}^{(r)} + \beta_r A\boldsymbol{\varepsilon}^{(r)}.$$

If we expand $\epsilon^{(0)}$ in terms of the $\mathbf{v}_i$ as: $\epsilon^{(0)} = \sum_{i=1}^{n} c_i\mathbf{v}_i$ we find

$$\epsilon^{(r)} = \sum_{i=1}^{n} c_i\mathbf{v}_i \prod_{k=1}^{r} (1 + \beta_k\lambda_i).$$

This gives

$$\|\epsilon^{(r)}\|^2 = \sum_{i=1}^{n} c_i^2\|\mathbf{v}_i\|^2 \left\{ \prod_{k=1}^{r} (1 + \beta_k\lambda_i)^2 \right\} \leqslant \|\epsilon^{(0)}\|^2 M^{(r)}$$

where

$$M^{(r)} = \max_{1\leq i\leq n} \left\{ \prod_{k=1}^{r} (1 + \beta_k\lambda_i)^2 \right\}.$$

For convergence we have to show that $M^{(r)} \to 0$. In practice the sequence of β_r will often be taken as periodic and then it will be sufficient if the product taken over a period, is less than unity. In this case, if we know, for instance, that

$$0 < \lambda_r \leqslant b, \quad r = 1,2,\cdots,n$$

and choose β_r so that $0 > \beta_r > -2b^{-1}$, then each factor will be less than unity and convergence is assured.

It is clear that these ideas can also be used in the practical determination of the characteristic vectors of a matrix. Suppose we have an approximation $\mathbf{v}$ to a characteristic vector $\mathbf{v}_1$ of a matrix A and, for simplicity, assume that the only contamination is a component of the characteristic vector $\mathbf{v}_2$. Suppose we have $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Let λ_1,λ_2 be the characteristic values corresponding to $\mathbf{v}_1,\mathbf{v}_2$. Then, for any $\alpha(\neq\lambda_1)$

$$\begin{aligned}\mathbf{v}^{(1)} &= (A - \alpha I)\mathbf{v} = c_1(\lambda_1 - \alpha)\mathbf{v}_1 + c_2(\lambda_2 - \alpha)\mathbf{v}_2 \\ &= (\lambda_1 - \alpha)[c_1\mathbf{v}_1 + c_2(\lambda_2 - \alpha)(\lambda_1 - \alpha)^{-1}\mathbf{v}_2].\end{aligned}$$

The strength of the component of $\mathbf{v}_2$ will therefore be reduced if $|(\lambda_2 - \alpha| < |\lambda_1 - \alpha|$, i.e. if α is nearer λ_2 than λ_1. If this is so successive repetition of the multiplication $A - \alpha I$ will purify $\mathbf{v}$. This method is a generalization of the familiar "power" method for the determination of the characteristic value of largest absolute value.

In C, Richardson exploits this idea, with a wealth of numerical examples, including cases when the matrix is unsymmetric or has non-linear elementary divisors. He discusses the use of purifiers of the form

$$(A - \alpha_1 I)(A - \alpha_2 I)\cdots(A - \alpha_k I)$$

and the optimal choice of the α_i. It is clear that information about the location of the characteristic roots is essential for satisfactory choice of the α_i. Richardson makes use of the bounds given by HIRSCH, Rayleigh's quotient, and the comparison of the ratios of corresponding components of $A\mathbf{v}$ and $\mathbf{v}$. The latter is used in an intuitive way, no mention being made of the result of COLLATZ, that there is always at least one characteristic root between the greatest and least of the ratios of the components.

J. T.

[1] L. F. RICHARDSON, *Weather Prediction by Numerical Processes.* Cambridge, England, 1923. For the following comment on this, I am indebted to G. E. FORSYTHE, "It is a monumental attempt to forecast for six hours, from almost no initial condition, and (I understand) a poor balance of Δt and Δx, Δy, Δz. It is superbly written and the author has (in my opinion) the most elegant English style of any mathematical writer of the century. [See p. 219 of this book, or the first page or two of C.] The Preface speaks for itself of the troubles encountered by the author."

[2] Cf. W. WASOW, "Discrete approximations to elliptic differential equations." *Zeit. f. ang. Math. u. Phys.*, v. 5, 1954.

[3] D. M. YOUNG, "On Richardson's method for solving linear equations with positive definite matrices." *Jn. Math. and Physics*, v. 32, 1953, p. 243–255. Experiments on the solution of the Laplace equation by this method, on ORDVAC, have been carried out by D. M. Young, and C. H. WARLICK.

CORRIGENDA

V. 8, p. 93, l.-3, *for* $12\mu = \mu^3$ *read* $12\mu + \mu^3$.

V. 8, p. 106, l. 8, *for* PEARCY *read* PEARCEY.

V. 8, p. 121, l. 20 *for* $+3(2 + i)$ *read* $-3(2 + i)$.

EDITORIAL NOTE. With this issue of *MTAC* the present Editorial Committee rounds out its fifth year and resigns. It is a pleasure to thank our many contributors, reviewers and referees for their cooperative assistance to the Committee and to *MTAC*. Future editorial correspondence should be addressed to

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