and so

$$e^x = 1 + \frac{x}{1 - x/2} + \frac{x^2/(4\cdot 3)}{1} + \frac{x^2/(4\cdot 15)}{1} + \cdots + \frac{x^2/\{4[4(\nu-1)^2 - 1]\}}{1} + \cdots.$$

A convenient equivalent form is

$$e^x = 1 + \frac{x}{1 - x/2} + \frac{x^2/4}{3} + \frac{x^2/4}{5} + \cdots + \frac{x^2/4}{2\nu - 1} + \cdots.$$

These expansions converge quite rapidly for all x. For example, if $-1 \leqslant x \leqslant 1$, we have $e^x - A_4/B_4 < .000084$, $e^x - A_5/B_5 < .000000033$, and $e^x - A_6/B_6 < .000000000081$.

N. Macon [3]

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This note was written while the author was a summer participant at the Oak Ridge National Laboratory, 1954.

1. See, for example, Oskar Perron, *Die Lehre von den Kettenbrüchen*, 2nd ed., Teubner, Leipzig, 1929, p. 201.
2. Perron, *op. cit.*, p. 312. Mr. Garwick also notes that this is credited to Thiele, see, for example, N. E. Nørlund, *Vorlesungen über Differenzenrechnung*, Springer, Berlin, 1924, Chapter 15, p. 415–455, especially p. 454.
3. The referee has noted that this work has been done independently by J. V. Garwick of the Norwegian Defense Research Establishment.

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

78[C, D, E].—Friedrich Lösch, *Siebenstellige Tafeln der elementaren transzendenten Funktionen*. Springer, Berlin-Göttingen-Heidelberg, 1954, viii + 335 p. 27 cm. DM 49.80.

This volume is meant to supersede the 1926 table of K. Hayashi [1], now out of print. The elementary transcendental functions are $\sin x$, $\cos x$, $\tan x$, $\sinh x$, $\cosh x$, $\tanh x$, $\ln x$, e^x, e^{-x}, $\arcsin x$, $\arctan x$, $\text{arcsinh}\, x$, $\text{arctanh}\, x$, together with φ, the angle corresponding to the arc x. In Table I (40 pages) these are given to 9D for $x = 0(.0001)0.1$, together with first differences (except for $\ln x$, and, of course, φ). In Table II (277 pages) these are given to 7D for $x = 0.1(.0005)3.15$, with (*double*) first differences for all functions (but φ) and for $x = 3(.01)10(.1)20$, without differences. For $x > 1$, $\text{arccoth}\, x$ is tabulated in place of $\text{arctanh}\, x$ and $\text{arccosh}\, x$ in place of $\arcsin x$. There are supplementary tables of $\tan x$ to 7D for $x = 1.5680(.0001)1.5730$, $\text{arctanh}\, x$ for $x = 0.9980(.0001)1$, $\text{arccoth}\, x$ for $x = 1(.0001)1.0020$.

In Table III, there are given $\sin x$, $\cos x$, $\ln x$, $\text{arcsinh}\, x$, $\text{arccosh}\, x$ to 7D, $e^{\pm x}$ to 7S, for $x = 0(1)100$. Table IV gives $\frac{1}{2}n\pi$ to 12D for $n = 0(1)100$. Table V gives $\sin \frac{1}{2}\pi x$, $\cos \frac{1}{2}\pi x$ to 7D for $x = 0(.001)0.5$. Table VI gives $\exp(\pm\frac{1}{2}\pi x)$, $\sinh \frac{1}{2}\pi x$, $\cosh \frac{1}{2}\pi x$ to 7D for $x = 0(.01)2$. Table VII gives $\exp(\pm\pi x/180)$, $\sinh(\pi x/180)$, $\cosh(\pi x/180)$ to 7D for $x = 0(1)180$. Table VIII gives φ in degrees

(12D) and in degrees, minutes and seconds (8D) for $x = 0(.0001).001(.001).01(.01).1(.1)1(1)10(10)100$. Table IX gives x in radians to 12D for $\varphi = 0(1°)90°$, for $\varphi = 0(1')1°$ and for $\varphi = 0(1'')1'$. Table X gives a collection of constants to 15D, the exact values of the first twenty Bernoulli numbers and the first 15 factorials; also $(n!)^{-1}$ to 15S, $\log_{10} n!$ to 15D and the binomial coefficients $\binom{n}{r}$ for $n = 1(1)15, r = 1(1)[\frac{1}{2}n]$.

The volume is excellently printed on rather large pages; it is, however, expensive. The Introduction gives worked examples, which cover typical cases. In general the error in linear interpolation (in Table I, II) is at most two units; in cases where it is larger, attention is called by use of italics for differences or by their omission.

It is stated that considerable parts of the tables were computed especially for this volume; for the remainder use was made of available tables and interpolation in such tables; the calculations were made to 9 or 10D and checked by differencing before rounding to 7 or 9D. The new material is mainly that concerned with the inverse hyperbolic functions and the inverse tangent; some of the hyperbolic material is available in the Harvard volume [2].

Two pages of Table I and two of Table II were checked by comparison with recognized sources and new calculations (made on SEAC): no discrepancies were noted.

Users will have to be wary of noting the leading figures of values and differences at the head (and foot) of the columns and the changes which occur in them.

J. T.

1. K. Hayashi, *Sieben- und mehrstellige Tafeln der Kreis- und Hyperbel-funktionen und deren Produkte sowie der Gammafunktion.* Springer, Berlin, 1926.
2. Harvard University, Computation Laboratory, Annals, vol. 20, *Tables of inverse hyperbolic functions*, 1949.

79[D, I].—Owen R. Mock, *A table of sines and cosines of* $\frac{\pi}{2}\cdot\frac{k}{256}$. 13 typewritten leaves deposited in UMT File.

These tables, constructed for Fourier analysis and synthesis with data from binary sources list sin and cos to 25D for $k = 1(1)255$. There is no introduction or description. The table is believed to have been computed on a Model 701 International Business Machine. The author believes that all digits are correct.

C. B. T.

80[F].—Emma Lehmer, "On the cyclotomic numbers of order sixteen," *Canadian Jn. Math.*, v. 6, 1954, p. 449–454.

Dickson proved that $(i, j)_8$, the number of solutions of

$$g^{8\nu+i} + 1 \equiv g^{8\mu+j}(p)$$

can be expressed for every i, j as a linear combination, with rational coefficients whose denominators are divisible only by factors of 64, of p, x, y, a, b, where $p = x^2 + 4y^2 = a^2 + 2b^2$ and p is a prime of the form $8n + 1$. The signs of x, y, a, b have to be chosen in a special way which was discussed in another paper by the author "On the number of solutions of $u^k + D \equiv w^2 \pmod{p}$," *Pac. J. Math.*, v. 5, 1955, p. 103–118.

In order to decide whether analogous relations hold for $(i, j)_{16}$, with coefficients whose denominators are divisible by factors of 256, computations were carried out on SWAC for a set of prime numbers of the form $32n + 1$ and $32n + 17$. These computations lead to the conclusion that Dickson's result cannot be extended to 16, at least not with the same sign rules which worked for $(i, j)_8$.

Some of the numerical results obtained for this investigation are published here as a table. It is possible to obtain 205 of the 256 values of $(i, j)_{16}$ from the 51 values of $(i, j)_{16}$ for which $i \leq j$, $j - i \geq i$, $16 - j > i$. Only these $(i, j)_{16}$ are tabulated for the eight primes of the form $32n + 1$:

97, 193, 449, 641, 673, 769, 929, 1409

and for the six primes of the form $32n + 17$:

241, 401, 433, 977, 1009, 1297.

Also the values of x, y, a, b are given for each p, with the signs needed for the relations concerning $(i, j)_8$.

Printing error: p. 449, line 7 from top *should read*: $x \equiv a \equiv 1(4)$.

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81[F].—FRANCIS L. MIKSA, "A table of integral solutions of $a^2 + b^2 + c^2 = r^2$," *Mathematics Teacher*, v. 48, 1955, p. 251–255.

Solutions for $r = 3(2)207$. See Review 82 in this issue.

This paper lists 14 errata which the author has found in the earlier table by R. H. BACON [1], which listed solutions for $r = 3(2)99$.

C. B. T.

1. R. H. BACON, "Integral solutions of $x^2 + y^2 + z^2 = r^2$," *School Science and Mathematics*, v. 47, 1947, p. 155–164.

82[F].—FRANCIS L. MIKSA, *A table of integral solutions of $A^2 + B^2 + C^2 = R^2$ for all odd R from $R = 3$ to $R = 325$, in two sets, $R = 3$ to 207 and an extension from $R = 207$ to $R = 325$.* 32 leaves, reproduced from typewritten manuscript. On deposit in UMT FILE.

This table extends earlier tables by R. H. BACON [1] and the author [2] to $R = 325$. Bacon's table listed decompositions of odd numbers into the sum of three squares for numbers $R = 3(2)99$, and the paper by MIKSA presented the list for $R = 3(2)207$. The calculation follows the lines of the two described; the present table has a clear introduction telling how it was calculated and listing a few pertinent identities. For $R \geq 209$ non-primitive sets are marked.

C. B. T.

1. R. H. BACON, "Integral solutions of $x^2 + y^2 + z^2 = r^2$, *School Science and Mathematics*, v. 47, 1947, p. 155–164.
2. FRANCIS L. MIKSA, "Table of integral solutions of $a^2 + b^2 + c^2 = r^2$," *Mathematics Teacher*, v. 48, 1955, p. 251–255.

83[F].—FRANCIS L. MIKSA, *Table of quadratic partitions* $x^2 + y^2 = N$. Manuscript of 102 leaves (reproduced from typewritten masters) deposited in the UMT FILE.

The table attempts a complete description of the possible decomposition of all odd numbers $N = 1(2)50{,}000$ into the sum of two squares. Some pertinent information concerning this is contained in [1]. Rules for deriving similar decomposition of even numbers are given in the introduction. No details of the calculation are included.

C. B. T.

1. J. V. USPENSKY & M. A. HEASLET, *Elementary Number Theory*, McGraw-Hill, New York, 1939, p. 337–341.

84[F].—R. J. PORTER, *Irregular Negative Determinants of Exponent 3n with their critical classes. Part I, from* $-D = 1$ *to* 50,000. 174 typewritten pages, 25.5 × 10 cm., deposited in UMT FILE.

This table is a supplement to *Table of irregular negative determinants of exponent 3n up to* $-2 = 150{,}000$, which were previously deposited in the UMT FILE [*MTAC*, v. 9, 1955, p. 26].

R. J. PORTER

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85[F, I, J, K].—FRANCIS L. MIKSA, *Stirling numbers of the second kind*. Typewritten manuscript (reproduced from typewritten masters) of 32 p., on deposit in UMT FILE.

This table gives ${}_rS_n$ for $n = 1(1)50$, a table of G_n for $n = 1(1)51$, an excellent introduction, and a bibliography prepared by HENRY FINLAYSON. The number ${}_rS_n$ is the Stirling number of the second kind defined by the expansion formula. It is also given as

$$\frac{\Delta^r 0^n}{r!}.$$

The number $G_n = \sum_r {}_rS_n$.

The author gives the recurrence relations used in computing and in checking the tables. The numbers are used in number theory, some aspects of analysis dealing with series and with probability or distribution problems involving arrangements of n objects.

Other published lists are reported by FLETCHER, MILLER and ROSENHEAD [1].

C. B. T.

1. FLETCHER, MILLER & ROSENHEAD, *An Index of Mathematical Tables*, Scientific Computing Service, Ltd., London, p. 72.

86[H].—ANDRES ZAVROTSKY, *Tablas para la Formacion de las Ecuaciones Cubicas*, Publicaciones de la Direccion de Cultura de la Universidad de los Andes, No. 44, Merida, Venezuela, 1955, iv + 48 pp., 21.5 × 15.8 cm.

This table gives the exact values of the coefficients of the cubic equation $x^3 + px + q = 0$ as a function of two of its roots x_1 and x_2 (the third root x_3 being $-x_1 - x_2$). The functions $-p$ and q are tabulated side by side for $x_1 = 1(1)100$

and $x_2 = x_1(1)100$, where $-p = x_1^2 + x_2 + x_1x_2$ and $q = x_1x_2(x_1 + x_2)$. There is a brief introduction containing some notes on the early history of the subject and some obvious comments on how to obtain p and q from the table when x_i is either negative or non-integral, which is followed by some dozen references to other tables for solving cubics.

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87[H, I].—EDUARD L. STIEFEL, *Kernel polynomials in linear algebra and their numerical applications*, multilithed from typescript, 1955, 52 p., 26.7 × 20.3 cm., possibly available on loan from libraries or the author.

High-speed digital computing machines make it possible to deal with matrices of much larger orders than would have been considered fifteen years ago. As a consequence, in the last decade many older matrix methods have been re-examined, and a number of new methods—mostly iterative—for treating linear problems have been devised by FRAME, HESTENES, KANTOROVICH, LANCZOS, PURCELL, SOURIAU, STIEFEL, D. YOUNG, and others. The practical computer faces an almost impossible task of remembering all these methods for solving linear problems, not to mention understanding them or being able to predict which is best for a given task. The reviewer's outline [1] of methods of solving linear systems is terrifying by its length and lack of unity.

But now Professor Stiefel has unified several of the known iterative methods for matrix problems. The unifying principle is the theory of orthogonal polynomials over an arbitrary mass distribution on an interval of the real axis. The booklet reviewed here is a clear and careful exposition of this material.

In the first section the author reviews the theory of orthogonal polynomials $\{P_i(\lambda)\}$ over a mass distribution $\rho(\lambda)d\lambda$. He shows how one can generate them by orthogonalizing the sequence $\{\lambda P_{i-1}(\lambda)\}$, a far more efficient process than orthogonalizing the powers $\{\lambda^i\}$, as Lanczos has stressed. Although the author doesn't mention it here, this idea can be worked into a polynomial curve-fitting routine much more effective than the usual ones based on solving normal equations built from $\{\lambda^i\}$.

To solve a linear system $Ax = k$, the author considers simple iterative routines of the form $x_{i+1} = x_i + \Delta x_i$, where $\Delta x_i = (1/q_i)r_i$, $r_i = k - Ax_i$. Then $r_n = R_n(A)k$, where $R_n(\lambda) = \prod_{k=0}^{n-1} (1 - \lambda/q_k)$ is a polynomial with $R_n(0) = 1$. The author poses the following problem of choosing a best *strategy*: Suppose the eigenvalues of A are known to lie in an interval $[a, b]$. Then it is desired to find that polynomial $R_n(\lambda)$ of maximal degree n such that $R_n(0) = 1$ and such that $\int_a^b R_n(\lambda)^2\rho(\lambda)d\lambda$ is minimized. Since one doesn't know the eigenvalues λ_i, such a requirement is about all that one can make about the smallness of $R_n(\lambda_i)$. The author solves this minimization problem exactly in terms of orthogonal polynomials, and shows various ways in which it may be used to construct algorithms which yield small r_n. He points out, however, that the intermediate residuals r_i may be quite large; i.e., these processes may be unstable.

The author now shows how to modify the above algorithm so that *each inter-*

mediate r_i also corresponds to a best strategy, thus restoring stability. The change is to put $\Delta x_i = (1/q_i)(r_i + p_i \Delta x_{i-1})$, where p_i, q_i are scalars involved in the three-term recurrence relation for $\{P_i(\lambda)\}$. He shows that various known algorithms correspond to special choices of $\rho(\lambda)$—in particular, the conjugate gradient method of Lanczos, Hestenes, and Stiefel.

In another section the same ideas are applied to the calculation of eigenvalues of a matrix A with real eigenvalues. It is shown how to bring out any intermediate eigenvalue by iterations with an appropriate *kernel polynomial.* As a special case the author shows us the intriguing new *spectroscopic method* of Lanczos. The reviewer feels that the ideas of this booklet will prove more useful in eigenvalue problems than in solving linear systems, because of the greater number of other ways of dealing with linear systems.

The final section is devoted to an application of the *quotient-difference* algorithm of RUTISHAUSER and Stiefel to generating the orthogonal polynomials.

This booklet is an exceedingly valuable contribution, and it is hoped that the material will be made available to every mathematician concerned with numerical methods in linear algebra. A person studying the booklet carefully will see—probably for the first time—the unity behind the gradient methods, RICHARDSON's method, and the various conjugate gradient methods. And he will learn how to improve his techniques of curve fitting.

The author uses the notation of the Dirac delta function for atomic mass distributions, and also to motivate the derivation of the kernel polynomials. This is most unusual for a mathematician, and is probably done in an effort to reach non-mathematical readers. While the reviewer is neutral on the matter, he finds that several of his colleagues resent it emphatically. The reviewer's only criticism is that there is no direct reference to the work of L. F. Richardson, who in his 1910 paper [2] was considering similar questions of strategy. The reviewer also wonders whether the author couldn't as well assume at many places in the exposition that A has complex eigenvalues, and thus broaden the treatment substantially. After all, a limitation to real eigenvalues is almost a limitation to symmetric matrices.

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1. G. E. FORSYTHE, "Tentative classification of methods and bibliography on solving systems of linear equations," *National Bureau of Standards Applied Mathematics Series 29*, 1953, Superintendent of Documents, Washington, D. C., p. 1–28.

2. L. F. RICHARDSON, "The approximate arithmetical solution by finite differences of physical problems involving differential equations, with an application to the stresses in a masonry dam," Roy. Soc. London, *Phil. Trans.* (A), v. 210, 1910, p. 307–357.

Editorial note. These were notes of four lectures prepared under contract between the National Bureau of Standards and the American University under sponsorship of the Office of Naval Research. They will probably be published as part of a volume to appear within a few months as part of the NBS Applied Mathematics Series.

88[H, K, P, S, Z].—*Transactions of the Symposium on Computing, Mechanics, Statistics, and Partial Difference Equations* held at the University of Chicago, April 29–30, 1954. Vol. II. Editorial Committee: F. E. GRUBBS, F. J. MURRAY, & J. J. STOKER, Interscience Publishers, Inc., 1955, 216 p., $5.00.

This volume contains eleven papers based on invited addresses given by the separate authors at the second symposium on applied mathematics sponsored jointly by the American Mathematical Society and the Office of Ordnance Research, U. S. Army. This symposium was held on April 29 and 30, 1954 at the University of Chicago. A great variety of different fields of applied mathematics and mathematics is displayed in this book, ranging over operational research, statistics, and partial differential equations.

P. M. MORSE discusses Operations Research as an effort to discover regularities in the behavior of teams of men and equipment as they go about doing assigned jobs and to link these regularities with other knowledge so that the phenomena occurring can be modified and controlled. Examples of the application of operations research to various problems are given.

J. NEYMAN gives a consistent presentation of his fundamental contributions to the theory of inductive inference and answers objections raised by CARNAP. Neyman propounds the view that prior to any discussion based on a theory of inductive inference a theorizing step must be taken to create a mathematical model of the phenomenon being studied. He points out that the model may differ from the theory of probability based on the axioms of Kolmogoroff. The general discussion of the first part of the paper is applied to the problem of homogeneity of neutral V-particles.

H. O. HARTLEY reports progress on two topics in his paper entitled, "Some Recent Developments in Analysis of Variance." These topics are: (1) multiple decisions and comparisons and (2) short cut procedures based on substitute measures of dispersion.

The "Two Unsolved Problems of Statistical Mechanics" explained by J. E. MAYER are:

(1) The evaluation of the pressure P of a system with N molecules as a function of the activity Z and the temperature T as a sum over integrals of functions of the potential energy over configurations in 3 N dimensional space. The equivalence of this problem and the solution of a compact matrix equation is pointed out.

(2) The concise logical formulation of problems involving the rate of approach to equilibrium.

M. R. HESTENES in his paper, "Iterative Computational Methods," is concerned mainly with matrix problems and with problems that arise from variational principles. The main concern is with gradient methods and a discussion of the conjugate gradient method of solving linear systems of equations is included.

J. TODD's paper entitled "Motivation for Working in Numerical Analysis," contains a brief discussion of various separate topics in numerical analysis. These include evaluation of polynomials, increasing speed of convergence of sequences, modified differences, characteristic roots of finite matrices, quadrature, integral equations, and game theory. The author stresses the need for an experimental and empirical approach to many problems in numerical analysis.

In the paper, "Some Numerical Computations in Ordnance Problems," A. A. BENNETT lists and discusses briefly 40 problems considered by the Computation Laboratory of the Ballistic Research Laboratories, Aberdeen Proving Ground.

C. TRUESDELL in the paper, "The Simplest Rate Theory of Pure Elasticity," postulates a set of ten non-linear equations for the following ten quantities: The density, the three components of the velocity vector field, and the six components of the stress tensor. These equations embody the law that the rate of stress is a function of the rate of deformation. Simple special solutions of the equations are discussed.

J. J. STOKER reviews various definitions of stability in his paper, "On the Stability of Mechanical Systems," and discusses briefly the possibility of converting inherently unstable situations into stable ones by applying periodic external forces.

The longest paper in the collection is the one by F. J. BUREAU entitled "Divergent Integrals and Partial Differential Equations." In this paper the author reviews the definition of the finite part and the logarithmic part of divergent integrals and uses these concepts to obtain the solutions to the Cauchy problem for the wave equation, the equation for damped wave and the Euler-Poisson-Darboux equation. Linear partial differential equations with constant coefficients and systems of such equations are also discussed. HADAMARD's method of singularities for obtaining solutions to such equations is reviewed and new applications of this method are described. The usefulness of the notions of principle and logarithmic part of divergent integrals in the theory of elliptic equations is illustrated by a discussion of a solution of RADON's problem. Other related problems are also treated.

A. H. T.

89[I].—E. W. DIJKSTRA & A. VAN WIJNGAARDEN, *Table of Everett's Interpolation Coefficients*, Computation Department of the Mathematical Centre, Amsterdam, Report R294, printed by Excelsior's Photo-Offset, The Hague, 1955. Introduction of 1 page plus 200 pages of tables (pages unnumbered), 24.7 cm.

The EVERETT interpolation formula for a function $f(x)$ that is tabulated at equal intervals of the argument $x_k = x_0 + kh$ is expressible as

$$f_p = (1 - p)f_0 + pf_1 + E_0^2(p)\delta^2 f_0 + E_1^2(p)\delta^2 f_1 + E_0^4(p)\delta^4 f_0 + E_1^4(p)\delta^4 f_1 + E_0^6(p)\delta^6 f_0 + E_1^6(p)\delta^6 f_1 + \cdots,$$

where $f_p = f(x_0 + ph)$, $f_k = f(x_k)$, $\delta^{2n} f_k$ are the $(2n)^{\text{th}}$ central differences of f_k, and the interpolation coefficients $E_k^{2n}(p)$, $k = 0, 1$, are given by

$$E_k^{2n}(p) = (-1)^{k+1}\binom{p + n + k - 1}{2n + 1}.$$

These present tables provide the six functions $E_k^{2n}(p)$; $k = 0, 1$; $n = 1, 2, 3$; for $p = 0(0.0001)1$; to 7D. Use could have been made of the symmetry relation $E_0^{2n}(p) = E_1^{2n}(1 - p)$ to reduce the size of the book by 50%, either by giving $E_0^{2n}(p)$ or $E_1^{2n}(p)$ alone for $p = 0(0.0001)1$ or by giving both functions only as far as $p = 0.5000$. But this present arrangement saves the reader the extra effort of looking up the coefficients in two different parts of the book, and it also reduces the chances of error. All six functions $E_k^{2n}(p)$ are tabulated alongside of each other. There are 51 arguments p on each page, in groups of five except for the last argument on the page which is repeated at the top of the next page. The fine interval of 0.0001 in p will, in many cases, obviate the need for interpolating in this table.

These present tables might be compared with two [1] earlier works each containing extensive tabulations of Everett coefficients. A. J. THOMPSON's excellent book includes, among other shorter basic tables of $E_k^{2n}(p)$, a main table of $E_k^{2n}(p)$ as far as $2n = 8$, for p at intervals of 0.001, with central differences to aid in the interpolation for $E_k^{2n}(p)$ when p occurs to more than 3D. The NATIONAL BUREAU OF STANDARDS volume includes $E_0^2(p)$ and $E_1^2(p)$ at intervals of 0.0001 and $E_0^4(p)$, $E_1^4(p)$, $E_0^6(p)$, $E_1^6(p)$ at intervals of 0.001, all to 10D. In the A-notation of this table, $E_0^{2n}(p) = A_{n+1}^{2n+2}(1 - p)$, $E_1^{2n}(p) = A_{n+1}^{2n+2}(p)$. However, the user of the Lagrangian table would have the extra work of looking up the desired values of $E_k^{2n}(p)$ in different parts of the book. Thus in comparison with earlier tables this present work has the advantages of finer intervals and more convenient arrangement, although it does not have as many decimal places.

The table was produced on the electronic computer ARRA of the Mathematisch Centrum in Amsterdam. The program included complete arithmetic checks which also covered the signals transmitted to the typewriter. The final sheets were reproduced by photo-offset.

The tables were computed for the "Empresa nacional bazan de construcciones navales militares' of Madrid."

Misprints in the Introduction: line 3, *for* $2k$, *read* $2n$; line 14, *for* $E_0^{2k}(p) = E_1^{2k}(1 - p)$, *read* $E_0^{2n}(p) = E_1^{2n}(1 - p)$.

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1. a) A. J. THOMPSON, *Tables of the Coefficients of Everett's Central-Difference Interpolation Formula. Tracts for Computers, No. V*, 2nd ed., Cambridge Univ. Press, 1943.
b) NATIONAL BUREAU OF STANDARDS, *Tables of Lagrangian Interpolation Coefficients*, Columbia Univ. Press, New York, 1944.

90[I].—OWEN R. MOCK, *Fifty Everett integration coefficients*. Three typewritten leaves (on vellum), deposited in UMT FILE.

The first fifty-one coefficients to use in EVERETT's integration formula

$$\int_{x_0}^{x_1} y(x)dx = \tfrac{1}{2} \sum_{\nu=0}^{\infty} M_{2\nu}\delta^{2\nu}[y(x_0) + y(x_1)],$$

listed to 50D and believed by the author to be accurate to 48D. There is no introduction or description; the formula is discussed by MILNE [1]. The list is believed to have been computed on a Model 701 International Business Machine.

C. B. T.

1. W. E. MILNE, *Numerical Calculus*, Princeton, 1949, p. 198.

91[I].—H. E. SALZER, "A simple method for summing certain slowly convergent series," *J. Math. Physics*, 33, 1955, p. 356–359.

This gives a table of (the integers)

$$B_{n,n-i}^{(m)}, \quad D_n^{(m)}, \quad \text{where} \quad A_{n,n-i}^{(m)} = \frac{(-1)^i(n-i)^{m-1}}{(m-1)_7}\binom{m-1}{i} = B_{n,n-i}^{(m)}/D_n^m$$

and where $D_n^{(m)}$ is the least common denominator of the $A_{n,n-i}^{(m)}$, $i = 0, 1, \cdots, m - 1$. The following values are covered: $n = 5, m = 4$; $n = 10, m = 4, m = 7$;

$n = 15$, $m = 4, 7, 11$; $n = 20$, $m = 4, 7, 11$. The motivation for this tabulation is the following: Consider the partial sums s_r of a convergent series as a function $s(x)$ of $x = r^{-1}$. The problem of the determination of $s = \lim s_n$ is regarded as that of interpolation (or rather extrapolation), for $s(0)$ given $s(p^{-1})$ for some integral values of p, in particular $p = n, n - 1, \cdots, n - m + 1$. This is solved by Lagrangian interpolation and

$$s \doteq \sum_{i=0}^{m-1} A_{n,\, n-i}^{(m)} s_{n-i}.$$

Various standard examples are discussed. The method appears effective when the sequence $\{s_r\}$ is monotone.

J. T.

92[I].—H. E. Salzer, "Equally weighted quadrature formulas over semi-infinite intervals," *J. Math. Physics*, 34, 1955, p. 54–63.

In general it is not possible to find real $x_i = x_i^{(n)}$ such that the quadrature formulae

$$(1) \qquad \int_0^\infty e^{-x} f(x)dx = 1/n \sum_{i=1}^{n} f(x_i)$$

or

$$(2) \qquad \int_{-\infty}^{+\infty} e^{-x^2} f(x)dx = \frac{\pi^{\frac{1}{2}}}{n} \sum_{i=1}^{n} f(x_i)$$

are exact when $f(x)$ is a polynomial of degree n. The polynomials having the $x_i^{(n)}$ for zeros are obtained formally for $n = 2(1)10$ in both cases [cf. L. M. Milne-Thomson, *Calculus of Finite Differences*, London, 1933, p. 177–180]. Among these only that for $n = 2$ in (1) and those for $n = 2, 3$ in (2) have all roots real. The roots are, however, given in all cases to 7D. There follows a detailed discussion of a few quadrature formulae of the above type which require exactitude only for polynomials of degree $m < n$, e.g., a seven-point quadrature which is exact for cubics in case (1) and a seven-point quadrature which is exact for quintics in case (2).

J. T.

93[I].—Herbert E. Salzer, *Formulas for calculating Fourier coefficients*, 5 typewritten pages (photocopy) placed in the UMT File.

These present formulas enable one to calculate the first twenty-four Fourier cosine and sine coefficients of a function $f(x)$ from its values (either given, or obtained through calculation or interpolation in a table of $f(x)$ at nine equally spaced points ranging from 0 to 2π). The coefficients a_i, $i = 1(1)5$, and b_i, $i = 1(1)4$, are tabulated for $n = 0(1)24$, to five decimal accuracy, for use in these formulas:

$$(1) \quad \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos nx \, dx = a_1[f(0) + f(2\pi)] + a_2\left[f\left(\frac{\pi}{4}\right) + f\left(\frac{7\pi}{4}\right)\right] + a_3\left[f\left(\frac{\pi}{2}\right) + f\left(\frac{3\pi}{2}\right)\right] + a_4\left[f\left(\frac{3\pi}{4}\right) + f\left(\frac{5\pi}{4}\right)\right] + a_5 f(\pi),$$

and

$$(2)\quad \frac{1}{2\pi}\int_0^{2\pi} f(x)\sin nx\,dx = b_1[f(0) - f(2\pi)] + b_2\left[f\left(\frac{\pi}{4}\right) - f\left(\frac{7\pi}{4}\right)\right] + b_3\left[f\left(\frac{\pi}{2}\right) - f\left(\frac{3\pi}{2}\right)\right] + b_4\left[f\left(\frac{3\pi}{4}\right) - f\left(\frac{5\pi}{4}\right)\right].$$

Apart from the error due to the approximate nature of a_i, b_i, and $f\left(\frac{m\pi}{4}\right)$, formulas (1) and (2) are exact whenever $f(x)$ is any polynomial of degree ≤ 8.

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94[I].—HERBERT E. SALZER, *Formulas for inverse osculatory interpolation*, 5 typewritten pages (ozalid) placed in the UMT FILE.

The author has previously given inverse interpolation formulas [1], [2] for finding $x = x_0 + ph$ from $f(x)$, in terms of $f_i \equiv f(x_i)$, where the $x_i \equiv x_0 + ih$ are equally spaced at intervals of h. Those formulas were obtained by the inversion of LAGRANGE's interpolation formula.

The formulas give p in terms of $f(x_0 + ph) \equiv f$, $f(x_i) \equiv f_i$, and $f'(x_i) \equiv f'$ for $x_i \equiv x_0 + ih$ at equally spaced intervals h, where the i ranges from $-[(n-1)/2]$ to $[n/2]$, for $n = 2(1)7$, where n is the number of points required in direct osculatory interpolation. Although the direct interpolation formula for $n = 6$ and $n = 7$ is of the 11th and 13th degree accuracy respectively, the inversion formula for p which is given does not go beyond the tenth degree terms (and in most practical problems it is very rarely that one will go that far; in fact, the first few terms will usually suffice).

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1. H. E. SALZER, "A new formula for inverse interpolation," Amer. Math. Soc., *Bull.*, v. 50, 1944, p. 513–516.
2. H. E. SALZER, "Inverse interpolation for eight-, nine-, ten-, and eleven-point direct interpolation," *J. Math. and Phys.*, v. 24, 1945, p. 106–108.

95[K].—E. S. PEARSON & H. O. HARTLEY, *Biometrika Tables for Statisticians*, v. 1, Cambridge University Press, 1954, xiv + 238 p. 26.0 cm. Price, $4.50.

One of the objectives of the originators of the journal *Biometrika* was to provide a place where mathematical tables of importance in statistics could be published. A number of such tables were collated to form the volume I of *Tables for Statisticians and Biometricians* which appeared for the first time in 1914. Four subsequent editions were issued and a second volume appeared in 1931. Due to the phenomenal growth of new developments in statistical methods since that time, it was felt that a modern version was warranted and thus the present *Biometrika* tables. This is the first volume of two and contains fifty-four of the more commonly used tables. The second volume will be concerned with the presentation of more specialized functions. The arrangement of the tables is such as to provide convenience to the user, the grouping being to collect the tabulations of individual or related functions. The sources from which the tables were drawn are given in an appendix. The older tables, having been in use for a considerable time, can be used with the satisfaction of knowing that they are error free. For the tables that have been newly compiled one can be certain that painstaking efforts, customary from English computers, have been made to guarantee their accuracy.

An outstanding feature of the volume is the comprehensive introduction prefacing the tables. Here the underlying mathematical theory is discussed, mathematical definitions are provided and a careful exposition is given of the method of interpolation to be used in each table, illustrated by example. In addition, the usefulness of each table is stressed by the presence of typical applications. Teachers of mathematical statistics should find this introduction an important supplement to their classroom texts. Because of the excellence of the introduction one may consider the volume as a book in mathematical statistics supplemented by tables rather than a book of tables preceded by an introduction.

The tables have been grouped under six main headings. The first group contains tables of the normal probability function:

1. The integral $P(X)$ and the ordinate $Z(X)$ in standard units to 7D for $X = 0(.01)4.5$ and to 10D for $X = 4.5(.01)6$ with first and second central differences. This is the fundamental table due to W. F. SHEPPARD.
2. Values of $-\log Q(X) = -\log [1 - P(X)]$ to 5D for $X = 5(1)50(10)100(50)500$.
3. Values of X for extreme values of Q and P to 4D for $Q = 0(.0001).02$.
4. Values of X in terms of Q and P to 4D for $Q = 0(.001).5$.
5. Values of Z in terms of Q and P to 5D for $Q = 0(.001).5$.
6. Table for probit analysis. Tabulates P, $Y + \frac{Q}{Z}$, $Y - \frac{Q}{Z}$, $\frac{1}{Z}$, and $\frac{Z^2}{PQ}$ for the expected probit $Y = 1(.01)9$. P is given to 3D for $Y = .5(.01)6.5$, to 4D for $Y = 6.5(.01)7.5$, to 5D for $Y = 7.5(.01)8.5$, and to 6D for $Y = 8.5(.01)9$. The other functions are given to 4D.

The second group is concerned with tables derived from the normal probability function:

7. The probability integral of the χ^2 distribution and the cumulative sum of the Poisson distribution. $Q(\chi^2/\nu)$ is given to 5D for $\chi^2 = .001(.001).01(.01).1(.1)2(.2)10(.5)20(1)40(2)134$ and $\nu = 1(1) \cdots$.
8. Values of χ^2 to 5 to 7S (mostly 6S) for $Q(\chi^2/\nu) = .995, .99, .975, .95, .9, .75, .5, .25, .1, .05, .025, .01, .005, .001$ and $\nu = 1(1)30(10)100$.
9. The probability integral of the t-distribution, $P(t|\nu)$ to 5D for $t = 0(.1)8$ and $\nu = 1(1)24, 30, 40, 60, 120, \infty$. Upper percentage points of t for $1 - P(t|\nu) = .001, .0001, .00001, .000005$ and $\nu = 1(1)10$ are given to 3 or more S.
10. Chart for determining the power function of the t-test. The probability β of establishing significance using the two-tail test at the $\alpha = 5\%$ or 1% level for values $\varphi = E(y)/\sqrt{2}\sigma_y$ is shown on a logarithmic scale. For $\alpha = .01$,

φ ranges from 2 to 5; for $\alpha = .05$ from 1.2 to 3.5 for the degrees of freedom $\nu = 6(1)10$, 12, 15, 20, 30, 60, ∞, β has the range .1 to .99.

11. Test for comparisons involving two variances which must be separately estimated. The table gives critical values to 2D of the ratio $\nu = (y - \eta)/(\lambda_1 s_1^2 + \lambda_2 s_2^2)^{\frac{1}{2}}$ for the significance level $\alpha = .05$ and degrees of freedom, ν_1, $\nu_2 = 6$, 8, 10, 15, 20, ∞; for $\alpha = .01$, ν_1, $\nu_2 = 10$, 12, 15, 20, 30, ∞ and for $c = \lambda_1 s_1^2/(\lambda_1 s_1^2 + \lambda_2 s_2^2) = 0(.1)1$.
12. Percentage points of the t-distribution. For values of $Q = 1 - P(t|\nu) = .4$, .25, .1, .05, .025, .01, .005, .0025, .001, .0005, and $\nu = 1(1)30$, 40, 60, 120 and ∞, values of the percentage points are given to 3D.
13. Percentage points for the distribution of the correlation coefficient, r, when $\rho = 0$. The values to 3D are given for $Q = .05, .025, .01, .005, .0025$, and .0005 for the degrees of freedom, $\nu = 1(1)20(5)50(10)100$.
14. Values of $z = \tanh^{-1} r$, with proportional parts are given to 4D for $z < 1$ and 3D for $z > 1$ for $r = .000(.002).998$.
15. Charts giving confidence limits for the population correlation coefficient ρ, given the sample coefficient, r with confidence coefficients .95 and .99. Values of r and s are given from $-1.(.05)1$ for sample sizes $= 3(1)8(2)12$, 20, 25, 50, 100, 200 and 400.
16. Percentage points of the B-distribution. Values of x are given to 5S for which

$$I_x(a, b) = \frac{\int_0^x u^{a-1}(1-u)^{b-1}du}{\int_0^1 u^{a-1}(1-u)^{b-1}du} = .5, .25, .1, .05, .025, .01 \text{ and } .005,$$

where $a = \frac{1}{2}\nu_2$, $b = \frac{1}{2}\nu_1$ for $\nu_1 = 1(1)10$, 12, 15, 20, 24, 30, 40, 60, 120, ∞ and $\nu_2 = 1(1)30$, 40, 60, 120, ∞.

17. Chart for determining the probability levels of the incomplete B-function, $I_x(a, b)$. Three separate charts are given: Chart A for $1 \le b \le 4$, Chart B for $4 \le b \le 15$ and Chart C for $15 \le b \le 60$. Each chart is further divided into upper and lower parts. The lower part contains a family of curves in Chart A corresponding to values of $a = 2(1)10(2.5)25(5)50(10)100(25)150$, 200. The lower parts of Charts B and C are given at the same intervals except that the starting values are $a = 5$ and 20 respectively. The top part contains a family of curves corresponding to selected values of $I_x(a, b)$. In charts A and B, $I_x(a, b) = .005$, .01, .02, .05, .1(.1).9, .95, .98, .99, .995, .999, .9995. In chart B, in addition, the curve corresponding to the value .9999 is also given. In chart C, $I_x(a, b) = .0005$, .001, .005, .01, .025, .05, .1(.1).9, .95, .975, .990, .995, .999, .9995.
18. Percentage points of the F-distribution (variance ratio). These are given for degrees of freedom, $\nu_1 = 1(1)10$, 12, 15, 20, 24, 30, 40, 60, 120, ∞, $\nu_2 = 1(1)30$, 40, 60, 120, ∞, for the upper 25, 10, 5, 2.5, 1, .5 and .1 percentage points to 2D if $F < 100$, 1D if $F < 1000$, to the nearest integer if $F < 30{,}000$ and to 4S for $F > 30{,}000$.
19. Percentage points of the largest variance ratio $s^2_{\max}/s_o^2$. Values of $s^2_{\max}/s_o^2$ for the upper 5 and 1% points are given to 2D for the degrees of freedom for

s_o^2, $v = 10, 12, 15, 20, 30, 60, \infty$, for $k = 1(1)10$. The quantity k here is the number of independent variance estimates each based on 1 degree of freedom of which $s^2_{\max}$ is the largest.

Group three comprises further tables of probability integrals, percentage points, etc., of distributions derived from the normal function.

20. Moment constants of the mean deviation and of the range. For the mean deviation this table gives for $n = 2(1)20, 30, 60$ the expectation to 6D, the standard deviation to 4D, variance to 5D and first and second moment ratios to 3 or 4D. For the range and the same values of n, values of the expectation to 5D, standard deviation to 4D, and variance of the range to 5D, are given with the first and second moment ratios to 4 and 3D respectively. The factors d_n/V_n and d_n^2/V_n where d_n and V_n are the expectation and variance are given to 3S.
21. Percentage points of the distribution of the mean deviation. The lower and upper .1, .5, 1, 2.5, 5 and 10 percentage points of the standardized mean deviation as obtained from the probability integral table are given for sample size $n = 2(1)10$ to 3D.
22. Percentage points of the distribution of the range. The same lower and upper percentage points as in Table 21 of the distribution of the range are given to 2D for $n = 2(1)20$.
23. Probability integral of the range W, in normal samples of size n. The values of $P(W/n)$ are tabulated to 4D for $n = 2(1)20$ and $W = 0.00(.05)7.25$.
24. Percentage points of the extreme standardized deviate from population mean $(x_n - \mu)/\sigma$ or $(\mu - x_1)/\sigma$. The lower and upper .1, .5, 1, 2.5, 5, 10% points of $X_n = (x_n - \mu)/\sigma$ are given for $n = 1(1)30$ to 3D.
25. Percentage points of the extreme standardized deviate from sample mean $(x_n - \bar{x})/\sigma$ or $(\bar{x} - x_1)/\sigma$. The lower and upper .1, .5, 1, 2.5, 5, 10% points of $u = (x_n - \bar{x})/\sigma$ are given for $n = 1(1)9$ to 2D.
26. Percentage points of the extreme studentized deviate from sample mean $(x_n - \bar{x})/s_\nu$ or $(\bar{x} - x_1)s_\nu$. Lower and upper 1 and 5% points of the studentized extreme deviate $t_n = (x_n - \bar{x})s_\nu$ where s_ν is an estimate of σ based on ν degrees of freedom are given for $n = 3(1)9$ and $v = 10, 15, 30, \infty$ for the lower percentage points and $= 10(1)20, 24, 30, 40, 60, 120, \infty$ for the upper ones.
27. Mean range in normal samples of size n. Values of the expected value, W_n, in samples from a normal population are given to 5D for $n = 2(1)500(10)1000$.
28. Mean positions of ranked normal deviates (normal order statistics). Values of $\xi(i|n)$, the expectation of the i-th ranked normal deviate for $n = 2(1)26(2)50$, $i = 1(1)25$ to 3D for $n \leq 20$; to 2D for $n > 20$.
29. Percentage points of the studentized range $q = (x_n - x_1)/s_\nu$. Lower and upper 1 and 5% points of q for $n = 2(1)20$ and $\nu = 10(1)20, 24, 30, 40, 60, 120, \infty$ for lower percentage points and $= 1(1)20, 24, 30, 40, 60, 120, \infty$ for the upper ones.
30. Tables for analysis of variance based on range. In part A the scale factor, c, to 2D and the equivalent degrees of freedom, ν, to 1D appropriate to a simple classification into K groups of n observations are given for $n = 2(1)10$ and

$K = 1(1)5, 10, \infty$, together with constant differences. In part B the like values for analysis of double classification with K blocks and n treatments for $n = 2(1)9$ and $K = 2(1)10, 20, \infty$.

31. Percentage points of the ratio s^2_{max}/s^2_{min} in a set of K independent mean squares. For the upper 5% points, values are given to 3S; for the upper 1% points to either 2, 3 or 4S for $\nu = 2(1)10, 12, 15, 20, 30, 60, \infty$ and $k = 2(1)12$.
32. Test for heterogeneity of variance: percentage points of Bartlett's test, M. The 5% and 1% significant points of M are given to 2D for $k = 3(1)15$ and $c_1 = 0(.5)5(1)10(2)14$. Corresponding to each combination of k and c_1, two percentage points denoted by (a) and (b) are given, representing the maximum and minimum values of the true percentage point.
33. Test for heterogeneity of variance: table to facilitate interpolation in Table 32. This table required only for accurate interpolation in Table 32, gives $C = c_1^3/k^2$ and $\Delta C = c_1 - c^3/k^2$ to 3D for all the marginal entries k and c_1 of Table 32.
34. Test for departure from normality.
    A. Percentage points of the distribution of $a =$ (mean deviation)/(standard deviation)
    B. Percentage points of the distribution of $\sqrt{b_1} = m_3/m_2^{\frac{3}{2}}$
    C. Percentage points of the distribution of $b_2 = m_4/m_2^2$

    Part A gives the upper and lower 10, 5 and 1% points for a, to 4D, the expectation and standard deviation to 5D for size of sample $n = 11(5)51(10)101(100)1001$. Part B gives the upper and lower 5 and 1% points for $\sqrt{b_1}$ to 3D and the standard deviation to 4D for size of sample $n = 25(5)50(10)100(25)200(50)1000(200)2000(500)5000$. Part C gives the upper and lower 5 and 1% points for b_2 to 2D for $n = 200(50)1000(200)2000(500)5000$.
35. Moments of $s/\sigma = \chi/\sqrt{\nu}$ and factors for determining confidence limits for σ. Values of the expectation are given to 6D of the standard deviation to 5D and of the standard deviation multiplied by $2\sqrt{\nu}$ to 4D. Values of the moment ratios $\beta_1 = \mu_3^2/\mu_2^3$ and $\beta_2 = \mu_4/\mu_2^2$ of s/σ are given to either 4 or 5D for ν (degrees of freedom) $= 1(1)20(5)100$. In addition, the factors $\sqrt{\nu}/\chi_\alpha$ and $\sqrt{\nu}/\chi_{1-\alpha}$ for confidence limits of σ are given for $\alpha = .025$ and $.005$ to 3 and 2D respectively.

The fourth section gives tables related to various discrete distributions.

36. Test for the significance of the difference between Poisson variables. In Part A the values of b are given for which

$$\sum_{i=0}^{b} \binom{r}{i}\left(\tfrac{1}{2}\right)^r \leqq a \quad \text{and} \quad \sum_{i=0}^{b+1} \binom{r}{i}\left(\tfrac{1}{2}\right)^r > a$$

for α (the nominal significance level) $= .1, .05, .025, .01, .005$ and $r = a + b$ in the range $1(1)80$. Part B for the single-tail test shows for values of $m = 2.5(2.5)25.0$ and the same nominal significance levels as in Part A, the probability of rejecting the hypothesis $m_1 = m_2$ when it is true. The probabilities are given to 3D for $\alpha = .1, .05, .025$ and to 4D for $\alpha = .01, .005$.

37. Individual terms of certain binomial distributions:

$$f(i|n, p) = \binom{n}{i} p^i (1 - p)^{n-i}$$

Values of $f(i|n, p)$ are given to 5D for $n = 5, i = 0(1)5$; $n = 10, i = 0(1)10$; $n = 20$, $i = 0(1)20$; $n = 25$, $i = 0(1)23$; $n = 30$, $i = 0(1)26$ and $p = .01$, .02(.02).1(.1).5.

38. Significance tests in a 2 × 2 contingency table. The table shows (1) in bold type for given a, A and B, the value of $b(<a)$ which is just significant at the probability levels of 5, 2.5, 1 and ½% for a single-tail test and (2) in small type for the given A, B and $r = a + b$, the exact probability to 3D that b is equal to or less than the bold type integer. A covers the range from 3(1)15; B from $2(1)A$, and a from $A(-1)$ until $b = 0$.

39. Individual terms, $e^{-m}m^i/i!$, of the Poisson distribution. Values are given to 6D for $m = .1(.1)15.0$ and i ranging from 0(1) until the contribution becomes negligible in the 6th decimal place.

40. Confidence limits for the expectation of a Poisson variable. Tabulates the Poisson variable m for the lower and upper .001, .005, .01, .025, .05 confidence limits to 3 or 4S for frequencies $c = 0(1)50$.

41. Charts providing confidence limits for p in binomial sampling given a sample fraction c/n. Confidence coefficients, .95 and .99. These are charts of the confidence belts with the confidence limit P as ordinate and the ratio c/n as abscissa for n ranging from 8 to 1000.

Miscellaneous tables such as the PEARSON type curves, rank correlation, and orthogonal polynomials are given in the fifth section.

42. Percentage points of Pearson curves, for given β_1, β_2, expressed in standardized measure. Tabulates the standardized deviate $X = \dfrac{x - \mu}{\sigma}$ to 3S for $\beta_1 = 0$, .01, .03, .05, .1, .15, .2(.1)1, and $\beta_2 = 1.8(.2)5$ for $P = .005$, .01, .025, .05, .95, .975, .99, .995.

43. Chart relating the type of Pearson frequency curve to the values of β_1, β_2. This chart drawn with β_1 and β_2 as coordinates in a rectangular coordinate system associates various types of Pearson curves with points in the $\beta_1\beta_2$-plane and indicates the significantly important regions of application.

44. Distribution of SPEARMAN's rank correlation coefficients, r_s, in random rankings. This table gives the probability Q to 3 or 4D of exceeding or equalling a given score S_r. Results are given for $n = 4(1)10$, $S_r = 12(2)308$.

45. Distribution of KENDALL's rank correlation coefficient t_k, in random rankings. This table gives the probability Q to 3 or 4D of exceeding or equalling a given score for $n = 4(1)9$, $S = 0(2)30$; $n = 6, 7, 10$, $S = 1(1)35$.

46. Distribution of the concordance coefficient W, in random rankings. Exact values of the probability Q that a given score will be attained or exceeded are given for $n = 3$, $m = 3(1)10$; $n = 4$, $m = 3(1)6$; $n = 5$, $m = 3$.

47. Orthogonal polynomials. This table gives values of orthogonal polynomials $\phi_i(x)$ for $i = 1(1)6$ for sample sizes $n = 3(1)52$. The observational range covered is $t = 1(1)n$ for $n = 3(1)12$ and the half range $t = 1(1)[(n + 1/2)]$ for $n > 12$.

The final section gives various auxiliary tables which are of fundamental importance.

48. Powers of integers. This table gives the first k powers of the integer n, for $n = 1(1)100$, $k = 2(1)7$.
49. Sums of powers of integers. This table gives the sums of the powers of the integers, $S(n^k)$ for $n = 1(1)100$, $k = 1(1)7$.
50. Squares of integers. Values of n^2 are given for $n = 0(1)999$.
51. Factorials of integers, their logarithms; square roots; and their reciprocals. For $n = 1(1)100$, $n!$ and $1/n!$ are given to 6S, while $\log_{10} n!$, $\sqrt{n}$, $1/\sqrt{n}$, and $1/n$ are given to 7D. For $n = 101(1)250$, $n!$ is given to 6S and $\log_{10} n!$ to 7D. For $n = 251(1)1000$, 7D are given of only $\log_{10} n!$
52. Miscellaneous functions of p and $q = 1 - p$ over the unit range. For $p = .00(.01)1.00$, $\Gamma(1 + p)$ and $\log_{10} \{\Gamma(1 + p)\}$ are tabulated to 7D. The quantities $1 - p^2$, pq and $p^2 + q^2$ are tabulated to 4D while $\sqrt{1 - p^2 1}$, $\sqrt{1 - p^2}$ and $\sqrt{pq}$ are tabulated to 5D. For $p = .910(.002).990(.001)1.000$, the quantities $\sqrt{1 - p^2 1}/\sqrt{1 - p^2}$ and $\sqrt{pq}$ are given to 5D.
53. Natural logarithms, $\log_e x$. $\text{Log}_e\, x$ is given to 5D together with proportional parts for $x = 1.00(.05)2.00(.01)9.99$.
54. Useful constants. Mathematical constants, their reciprocals and logarithms are given to 10D. Among these constants are multiples and powers of π, e, e^2, $\log_{10} e$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{10}$ and 1 radian. The binomial coefficients $\binom{n}{i}$ are given for $n = 4(1)12$ and $i = 0(1)12$. Conversion tables from metric to British and British to metric are given for measurements of length, area, volume, weight and capacity.

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96[K].—F. W. DAVID & N. L. JOHNSON, "Statistical treatment of censored data. Part I. Fundamental formulae," *Biometrika*, v. 41, 1954, p. 228–240.

This paper is concerned with censored samples in which the smallest k of n observations are fully measured, while of the $(n - k)$ largest observations, it is known only that $x > x_k$. The authors consider a population described by a continuous random variable with p.d.f., $f(t)$, and with distribution function, $F(X) = \int_{-\infty}^{X} f(t)dt$. Arranged in ascending order of magnitude, observations of a (complete) random sample from a population of this type correspond to x_1, $x_2, \cdots, x_n$. Variables $X_1, X_2, \cdots, X_n$, defined by the equations $F(X_r) = r/(n + 1)$, $(r = 1, 2, \cdots, n)$, are introduced and approximations to the moments of the x_r's are found by expanding x_r about X_r in an inverse TAYLOR series. Central moments and cumulants are then obtained from these noncentral moments. With the first four moments or cumulants known, functions of the PEARSON system are available for approximating distributions of the x_r's and functions of them.

The first four orders of cumulants of $F(x_r)$ are listed in Table 1 which is appended to this paper. The first four orders of cumulants of the ordered variables $x_1, x_2, \cdots, x_n$ are listed in Table 2 with an accuracy of $0[(n + 2)^{-3}]$. Sums of powers and products of the first k natural numbers which enter into calculation of the cumulants of Tables 1, and 2, are given in Table 3.

The application of results obtained in the paper is illustrated by determining upper and lower five percentage points of $\bar{x}_k$, which is the mean of the first k observations (in ascending order of magnitude) of a censored sample from a normal population. Calculations are given to 3D for $n = 10$, $k = 2(2)8$ and for $n = 20$, $k = 4(4)16$.

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97[K].—F. G. FOSTER & A. STUART, "Distribution-free tests in time-series based on the breaking of records," Roy. Stat. Soc., *Jn.*, s.B., v. 16, 1954, p. 1–22.

Two distribution free tests making use of the statistics involving the upper records and the lower records for the randomness of a series are introduced in this paper. An observation in a time series is called a lower (upper) record if it is smaller (greater) than all previous observations in the series. The statistics used are s, the sum of the number of upper and lower records in a series and d, the difference between the number of upper and lower records. The hypothesis of randomness of a series can be tested by the critical region $s \geq s_0$ or $|d| \geq d_0$. The first test provides a test against trend in dispersion while the second tests against a trend in location. Comparable round-trip tests (S and D) are also introduced involving counting the records in the series in both directions.

The joint distribution of s and d is obtained under the null hypothesis and the two variables are shown to be independently distributed as well as being asymptotically normal. In the round trip case the corresponding statistics are established to be asymptotically normally distributed and the variances have been tabulated.

The powers of the tests against trend in the mean have been computed using empirical sampling techniques on a high speed computer.

The following tables are given:

Table 1. $P(s \leq s_0)$ to 3 or 4D for $n = 3(1)15$ and the normal approximation for $n = 15$.

Table 2. $P(d \leq d_0)$ to 3 or 4D for $n = 3(1)6$ with normal approximation for $n = 6$.

Table 3. Mean and standard error of s and standard error of d to 3D for $n = 10(5)100$ with approximation for $n = 100$.

Table 4. The standard error of D to 2D based on 1000 samples of size n for $n = 10, 25, (25), 125$.

Table 5. The correlation between d and d' to 2D for $n = 10, 25(25)125$.

Table 6. The power of the d test against normal regression at the 5 percent level for trends $\Delta = .01(.01).07$ and sample sizes $n = 25(25)125$ based on 1000 samples in each case.

Table 7. The power of the D test against normal regression at the 5 percent level for trends $\Delta = .01(.01).06, .08$, and sample size $n = 25(25)125$, based on 1000 samples in each case.

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98[K].—H. E. DANIELS, "A distribution-free test for regression parameters," *Ann. Math. Stat.*, v. 25, 1954, p. 499–513.

Consider the usual regression model with $y_i = \alpha_0 + \beta_0 x_i + \epsilon_i$, where we assume only that the x_i are distinct and that the errors ϵ_i are independent and satisfy Pr $(\epsilon_i > 0) =$ Pr $(\epsilon_i < 0) = \frac{1}{2}$. The lines $\alpha = -x_i\beta + y_i$ divide the (β, α) plane into polygonal regions, some of them open. Let m denote the minimum number of lines which must be crossed to escape from (β_0, α_0) into an open polygon. The distribution of m is given to 3D for $n = 3(1)30$, providing confidence regions for (β_0, α_0) and tests for an hypothesis specifying both parameters. The power of the m-test is compared with certain competitors, and the complications arising when not all x_i are distinct are examined.

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99[K].—BENJAMIN EPSTEIN, "Tables for the distribution of the number of exceedances," *Ann. Math. Stat.*, v. 25, 1954, p. 762–768.

Two random samples, each of size n, are taken from the same continuous distribution. U_r^n, the number of exceedances, is the number of values in the second sample exceeding the rth smallest in the first sample, r fixed. Taking advantage of certain symmetries, it is possible to give the complete distribution function, Pr $(U_r^n \leq x)$, in terms of r up to $[n/2]$ and $r - 1 \leq x \leq n - r - 1$. Table 1 gives these to not less than 4D and 3S for $n = 2(1)15(5)20$. [WAYNE UNIVERSITY Technical Report No. 6, by the same author, gives values for $n = 2(1)20(5)50$.] Table 2 indicates, for the case $n = 5$, how to complete the array of probabilities from the portion given in Table 1.

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100[K].—C. K. TSAO, "An extension of MASSEY's distribution of the maximum deviation between two-sample cumulative step functions," *Ann. Math. Stat.*, v. 25, 1955, p. 587–592.

Let $S_n(x)$ and $S_m'(x)$ be the cumulative distributions observed in two random samples, of sizes n and m respectively, from the same continuous population. Denote by x_r and y_r the rth smallest in the respective samples, and for $r \leq \min(m, n)$ let

$$d_r = \max_{x \leq x_r} |S_n(x) - S_m'(x)|$$

and

$$d_r' = \max_{x \leq \max(y_r, x_r)} |S_n(x) - S_m'(x)|.$$

Under the above assumption the distributions of d_r and d_r' are independent of the distribution of the population. Table I gives $P(d_r \leq c/m)$ to 5D for $n = m = 3(1)10, 15, 20, 30, 40$; $c = 1(1)12$; $r = 2(1)10$. Table II gives $P(d_n' \leq c/m)$ to 5D for the same range of arguments.

These tables may be used in testing the hypothesis that two populations have identical distribution functions.

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101[K].—YOSHIHIKO HIRAGO, HIDENORI MORIMURA, & HISAO WATANABE, "Tables for three-sample test," Inst. Stat. Math., *Annals*, v. 5, 1954, p. 97–102.

The authors provide tables for a generalization, to three samples, of a two-sample test for randomness originated by WALD and WOLFOWITZ [1] and studied in greater detail by MOOD [2].

Let $x_1, x_2, \cdots, x_{n_1}, y_1, y_2, \cdots, y_{n_2}$, and $z_1, z_2, \cdots, z_{n_3}, n_1 \leq n_2 \leq n_3$, be three random samples drawn from the respective populations having continuous cdf's $F(x)$, $G(y)$, $H(z)$. To test whether the $n = n_1 + n_2 + n_3$ observations form a single random sequence from the same population, i.e., whether $F \equiv G \equiv H$, the values are arranged in a single ascending order and the runs of x, y, z counted, regardless of length. Let r_1, r_2, r_3 be the number of runs of x's, y's, z's, respectively and put $r = r_1 + r_2 + r_3$. For given values of n_1, n_2, n_3 the critical value of r at the α-level of significance is defined as the largest integer, r_0, for which Prob $[r \leq r_0] \leq \alpha$. Table 1, for $\alpha = .05$, is for all n_i for which $1 \leq n_1 \leq n_2 \leq n_3 \leq 10$. It was obtained from the exact distribution of r discussed by Mood in the above paper. Table 2, for $\alpha = .05$, is for $10 \leq n_1 \leq n_2 \leq n_3 \leq 30$, the n_i selected from the set 10(2)20(5)30, but excluding $n_3 = n_1$. Table 3 gives the $\alpha = .01$ and .05 critical values for all n_i for which $11 \leq n_1 = n_2 = n_3 \leq 30$. Tables 2 and 3 were obtained by using the normal approximation given by Mood with a correction based on his formulas for the first two moments of r_i.

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1. A. WALD & J. WOLFOWITZ, "On a test whether two samples are from the same population," *Ann. Math. Stat.*, v. 11, 1940, p. 147–162.
2. A. M. MOOD, "The distribution theory of runs," *Ann. Math. Stat.*, v. 11, 1940, p. 367–392.

102[K].—W. J. DIXON, "Power under normality of several nonparametric tests," *Ann. Math. Stat.*, v. 25, 1954, p. 610–614.

Consider a sample of size N_1 from a normal population with mean μ_1 and variance σ^2 and an independent sample of size N_2 from a normal population with mean μ_2 and variance σ^2. This paper presents tabulations of the power and power efficiency [1] of the rank-sum, maximum deviation, median, and total number of runs tests for the null hypothesis $\mu_1 = \mu_2$ with significance level α and alternative hypotheses $\delta = |\mu_1 - \mu_2|/\sigma$. Table I contains power function values to 3D and power efficiencies to 2S for the cases $\alpha = 1/10, 1/35, 1/126$, $N_1 = N_2 = 3(1)5$, $\delta = 0(.25)2(.5)5$ for all tests; also for the cases $\alpha = 2/126, 4/126$, $N_1 = N_2 = 5$, $\delta = 0(.5)4.5$ for the rank sum test; for the cases $\alpha = 10/126$, $N_1 = N_2 = 5$, $\delta = .5(.5)4.5$ for the maximum absolute deviation test; and for the cases $\alpha = 26/126$, $N_1 = N_2 = 5$, $\delta = .5(.5)3$ for the median test. Table II contains power function values to 3D and power efficiencies to 2S for all tests except the total number of runs test for a randomized significance level of $\alpha = .025$, $N_1 = N_2 = 5$, $\delta = .5(.5)4.5$. Table III gives limiting power efficiencies as $\delta \to 0$ to 3S for the rank sum test for the cases: $\alpha = 1/3$ and $(N_1, N_2) = (2, 2)$; $\alpha = 1/5$ and $(N_1, N_2) = (2, 3)$; $\alpha = 2/15, 4/15$ and $(N_1, N_2) = (2, 4)$; $\alpha = 1/10$, $1/5$ and $(N_1, N_2) = (3, 3)$; $\alpha = 1/70$ and $(N_1, N_2) = (4, 4)$; $\alpha = 1/126, 2/126$, $4/126$ and $(N_1, N_2) = (5, 5)$; $0 < \alpha < 1$ and $(N_1, N_2) = (\infty, \infty)$.

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1. W. J. Dixon, "Power functions of the sign test and power efficiency for normal alternatives," *Ann. Math. Stat.*, v. 24, 1953, p. 467–473.

103[K].—E. S. Page, "Control charts for the mean of a normal population," Roy. Stat. Soc., *Jn.* s. B., v. 16, 1954, p. 131–135.

Consider the well known statistical quality control procedure in which the method is to examine a fixed number of items at regular intervals, to measure a certain dimension, x, and to plot the sample mean, $\bar{x}$, on a control chart. If the point falls outside the control limits drawn at $\mu \pm \frac{B\sigma}{\sqrt{N}}$ (where μ is the required process mean, and σ is the process standard deviation), the process mean is said to be "out of control" and some appropriate action is taken. However, it is the premise of this paper that unnecessary restriction exists when the width of these control limits is fixed, and the author proceeds to present a method by which B and N are chosen to obtain a "best" scheme.

For an estimated process mean, m (perhaps different from μ), and a known constant standard deviation, σ, it is desired to detect any change in the mean. Consider only the case in which we are interested in a single control limit at $\mu + B\sigma/\sqrt{N}$ and a constant process mean, m. Let $P(m)$ denote the probability that a given sample yields a point outside the control limit. Define the "average run length," L, as the average number of articles inspected between two successive occasions when action is taken. The average run length for a constant process mean, m, is $L = N/P(m)$.

The average number of articles produced between two successive occasions when action is taken is L/f where f is the fraction of output that is inspected. It is easily seen that the average run length is a measure of the amount of scrap or defective articles produced when quality is bad, and is also a measure of the amount of acceptable articles produced between interferences caused by process inspection when the quality is satisfactory. If μ is a satisfactory value for the mean, m, then for $m = \mu$ there is no need for any corrective action to be taken. It is desirable that the production should only rarely be suspended. Define L_0 as the average run length when $m = \mu$. Let k (a positive number) be such that a shift in the mean of $m \geq \mu + k\sigma$ should be detected as soon as possible after it has occurred. Define L_1 as the average run length when $m = \mu + k\sigma$. A "best" inspection scheme might be defined as one that minimizes L_1 for some given large value of L_0, or, alternately, as one which maximizes L_0 for some given small value of L_1.

If only deviations from the mean in one direction are considered, the control limit is $\mu + \frac{B\sigma}{\sqrt{N}}$ and $L_0 = \frac{N}{\Phi(B)}$, $L_1 = N/\Phi(B - k\sqrt{N})$, where

$$\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_x^\infty \exp(-x^2/2)dx.$$

In accord with the above principles for the one-sided scheme, Tables 1a–c give N and B, the latter to 2D, which minimize L_1 for which $L_0 = 2{,}000$, $5{,}000(5{,}000)20{,}000$, $40{,}000$, $60{,}000$ is specified for a given $k = .2(.1)1.8$. Also Tables 2a–c give N to 2 or 3S and B to 2D which maximize L_0 for which $L_1 = 10(5)25(25)100$ is specified for a given $k = .1(.1)1$.

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104[K].—T. E. Sterne, "Some remarks on confidence or fiducial limits," *Biometrika*, v. 41, 1954, p. 275–278.

Define $P_{n,r}(\pi) = \binom{n}{r}\pi^r(1 - \pi)^{n-r}$ and $P(\pi, n, s) = \sum_r P_{n,r}(\pi)$ for all r for which $P_{n,r}(\pi) \leq P_{n,s}(\pi)$. From these it is possible to find confidence limits corresponding to any n, s, and any desired value ϵ, where n is the number of trials and s the number of successes. It is sufficient to consider the shortest interval of π that contains all values of π for which $P(\pi, n, s) \geq \epsilon$ for a given n and s. The lower limit is the smallest π satisfying the inequality and the upper limit is the largest π. Tables for confidence limits for π to 2 and 3D with confidence coefficients 0.5 and 0.9 are given for all possible values of s when $n = 1(1)10$.

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105[K].—H. A. David, H. O. Hartley, & E. S. Pearson, "The distribution of the ratio, in a single normal sample, of range to standard deviation," *Biometrika*, v. 41, 1954, p. 482–493.

Percentage points of the distribution of w/s, where w is the range and s the sample standard deviation of a sample of n from a normal population, are obtained partly by the use of an exact distribution applicable in the case of small samples, and partly by fitting a curve of the Pearson system to the first four moments for selected values of n and interpolating between these results. The final table gives upper α percentage points for $n = 3(1)20(10)60(20)100$, 150, 200, 500, 1000 and $\alpha = 0.10, 0.05, 0.025, 0.01, 0.005$ to 3D for $n \leq 9$ and 2D for $n > 9$, and lower percentage points for the same values of α and $n \geq 10$. The use of this ratio as a test of homogeneity in both large and small samples is considered, and it is compared to the ratio of the mean deviation to the sample standard deviation and the sample fourth standard moment as a test of normality.

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106[K].—MARTIN WEIBULL, "The distribution of t- and F-statistics and of correlation and regression coefficients in stratified samples from normal populations with different means," *Skandinavisk Akuarietidskrift*, v. 36, 1953, supplement to haft 1–2.

This long paper is principally concerned with the application of the standard statistics used in exact significant tests in samples from normal to samples from stratified universes in which the distributions within each stratum are normal with variances constant in all strata but with means different in different strata. It begins with a discussion of non-central χ^2, t- and F-statistics and includes an illustrative table of the non-central F distribution in which both the χ^2's appearing in the numerator and denominator are non-central. 1% and 5% points of this F are given to 2D for the degrees of freedom for the numerator, $b_1 = 1, 2, 4, 8$, the degrees of freedom for the denominator, $b_2 = 1, 2, 4, 8, 16, 32, \infty$, but only for $b_1 < b_2$, and the non-centrality parameters β_1 and $\beta_2 = 0, \frac{1}{2}, 1$.

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107[K].—A. J. DUNCAN, "The use of ranges in comparing variabilities," *Industrial Quality Control*, v. 11, no. 5, 1955, p. 18–22.

The author tabulates the quantities d_2^* and ν such that $E(\bar{R}/d_2^*)^2 = \sigma^2$ and the mean and variance of $\sqrt{\nu}\bar{R}/d_2^*\sigma$ are those of a χ^2-distribution with ν degrees of freedom, where $\bar{R}$ is the mean of the ranges of k samples of n from a normal distribution with variance σ^2. d_2^* is given to 2D and ν to 1D for $k = 1(1)20, 25, 30, 50$ and $n = 2(1)7$. This is an extension of Table I of DAVID [1] where $c = d_2^*$ is given to 2D and ν to 1D for $k = 1(1)5, 10$ and $n = 2(1)10$, which in turn is an extension, with minor corrections, of the original table given by PATNAIK [2]. The methods used for the extension are not indicated.

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1. H. A. DAVID, "Further applications of range to the analysis of variance," *Biometrika*, v. 38, 1951, p. 393–407.
2. P. B. PATNAIK, "The use of mean range as an estimator of variance in statistical tests," *Biometrika*, v. 37, 1950, p. 78–87.

108[K].—T. LEWIS, "99.9 and 0.1% points of the χ^2 distribution," *Biometrika*, v. 40, 1953, p. 421–426.

Points of the χ^2 distribution have been computed correct to 8D for the lower 0.1% points for degrees of freedom $n = 1(1)30(10)100, 120$, and for the upper 0.1% points for $n = 40(10)100, 120$.

Using $m = \frac{1}{2}n$, $v = \frac{1}{2}\chi^2$, and $P = \dfrac{1}{\Gamma(m)}\displaystyle\int_v^\infty t^{m-1}e^{-t}dt$, so that $2v$ is the 100P% point, the author had $f(v) = 0$ where for n even,

$$f(v) = \left\{1 + v + \frac{v^2}{2!} + \cdots + \frac{v^{m-1}}{(m-1)!}\right\} - \mathrm{P}e^v$$

and for n odd,

$$f(v) = \left\{\sqrt{\frac{2}{\pi}}\, e^{v} \int_{\sqrt{2v}}^{\infty} e^{-\frac{1}{2}t^2}dt + \frac{v^{\frac{1}{2}}}{(1/2)!} + \frac{v^{\frac{3}{2}}}{(3/2)!} + \cdots + \frac{v^{m-1}}{(m-1)!}\right\} - \mathrm{P}e^{v}.$$

Then each value of v was obtained by computing $f(v)$ approximately for an approximate value v_0 of v and then improving by iteration. The first v_0 was obtained by a formula due to WILSON and HILFERTY [1],

$$\sqrt[3]{\frac{v}{m}} \sim 1 - \frac{1}{9m} + x\sqrt{\frac{1}{9m}},$$

the x being the normal deviate corresponding to the 100P% value of v.

Tables of these computations are exhibited.

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1. E. B. WILSON & M. M. HILFERTY, "The distribution of chi-square," Nat. Acad. Sci., *Proc.*, v. 17, 1931, p. 684–688.

109[K].—S. H. ABDEL-ATY, "Approximate formulae for the percentage points of the non-central χ^2 distribution," *Biometrika*, v. 41, 1954, p. 538–540.

The main point of this note is to derive the two approximation formulas of the title and compare them numerically with the exact values. The cube root transformation of WILSON and HILFERTY turns out to be the appropriate one for transforming the non-central χ^2 variate, χ'^2, to an approximately normal variate y. The CORNISH-FISHER expansion [1] is then used to obtain a formula for the 100α-percentage point of Y (the variate y after standardization) in ascending powers of $r^{-\frac{1}{2}}$, where $r = f + \lambda$ and f is the number of degrees of freedom and λ is the parameter. This formula is written out to the term in $r^{-\frac{3}{2}}$, with the coefficients expressed in terms of quantities whose values are given for $\alpha = .5(.1).1$, .05, .01, .005, .001. The numerical comparison with a few exact values indicates that even for small degrees of freedom and moderate values of λ the expansion agrees very well for both the upper and lower 5-percent points. The probability integral approximation for Y is obtained by straightforward application of the EDGEWORTH expansion. Numerical comparison with exact values for a few values of f and λ show agreement to 3D or better.

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1. E. A. CORNISH & R. A. FISHER, "Moments and cumulants in the specification of distributions," *Extrait de la Revue de l'Institut International de Statistique*, v. 4, 1937, p. 1–14.

110[K].—A. M. FREUDENTHAL & E. J. GUMBEL, "Minimum life in fatigue," Amer. Stat. Assn., *Jn.*, v. 49, 1954, p. 575–597.

The authors fit fatigue data by the "survivorship function"

$$l(N)_S = \exp\left[-\left(\frac{N - N_{0,S}}{V_S - N_{0,S}}\right)^{\alpha_S}\right].$$

In this formula, $l(N)_S$ is the probability of surviving N cycles under stress S; V_S and $N_{0,S}$ are location parameters having dimensions of N and the relation $V_S > N_{0,S} \geq 0$; $N_{0,S}$ stands for the minimum life measured in cycles to failure at the stress level S; $1/\alpha_S$ is a dimensionless scale parameter. The problem of estimating the three unknown parameters α_S, $N_{0,S}$, and V_S from a set of fatigue data taken at a series of stress levels S is a formidable one. The authors use the method of moments to carry out the estimation. This procedure is facilitated by the authors' Table 1. This table gives for $1/\alpha_S = 0(.01)1(.2)2(1)5$ the values to 4D of the standardized distance from the "characteristic number," $\sqrt{S}$ to the minimum life, of the standardized distance from $\sqrt{S}$ to the mean, and of the third central moment in standard units. The method is applied to published data of RAVILLY on the fatigue strength under tension of aluminum and nickel wire. The question of the sampling errors of estimate obtained by the authors' procedure is left open.

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111[K].—MAX HALPERIN, SAMUEL W. GREENHOUSE, JEROME CORNFIELD, & JULIA ZALOKAR, "Tables of percentage points for the Studentized maximum absolute deviate in normal samples," Am. Stat. Assn., *Jn.*, v. 50, 1955, p. 185–195.

Let x_i, $i = 1$ to k, be independent normally distributed variates each with mean μ and variance σ^2. The Studentized maximum absolute deviation is defined by $d = \max\limits_{i=1,2,\cdots,k} \dfrac{|x_i - \bar{x}|}{s}$, where ms^2/σ^2 is distributed as χ^2 with m degrees of freedom and independent of x_i. Tables of the upper and lower limit for the upper 5% and 1% points of d are given to 3S for $k = 3(1)10(5)20(10)40$, 60 and $m = 3(1)10(5)20(10)40$, 60, 120, ∞. Examples illustrate the use of the tables.

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112[K].—J. H. CADWELL, "The probability integral of range for samples from a symmetrical unimodal population," *Ann. Math. Stat.*, v. 25, 1954, p. 803–806.

An asymptotic expression is given for the probability integral of range for samples from a symmetrical unimodal population. Its accuracy is investigated for the case of a normal parent population for sample sizes of 20 to 100. Over this range errors are small and by use of a given table of corrections the probability integral can be found with a maximum error of 0.0001. The .1, .5, 1, 5, 10, 25, 50, 75, 90, 95, 99, 99.5, 99.9 percentage points are given to 3S, for sample sizes of 20(20)100 of the range w from a normal population with unit standard deviation and also the mean value of w for the same sample sizes.

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113[K].—S. R. BROADBENT, "The quotient of a rectangular or triangular and a general variate," *Biometrika*, v. 41, 1954, p. 330–337.

The paper discusses the distribution of the quotient of the value of a random variable from a rectangular or a triangular distribution and the value of a random variable from an independent distribution. The denominator distribution is restricted to one for which a frequency function exists almost everywhere and for which, if the cumulative distribution function is $F(x) = z$, the inverse function, $G(z) = x$, is defined, non-zero, and finite for almost all z.

The tables of the paper are for the cases in which the denominator distribution is a "normal" distribution. Table 1 is entitled "Percentage points of the quotient of a rectangular and an independent normal variate. If y is rectangular with range $(1 - \alpha, 1 + \alpha)$ and x is normal with mean 1 and variance β^2, the table gives percentage points of (y/x)." The values given are solutions for Q to 3D of [1]

$$Hh_0\left(-\frac{1}{\beta}\right) + \frac{\beta Q}{2\alpha\sqrt{2\pi}}\left\{Hh_1\left(\frac{1-\alpha}{\beta Q} - \frac{1}{\beta}\right) - Hh_1\left(\frac{1+\alpha}{\beta Q} - \frac{1}{\beta}\right)\right\} = k,$$

$k = .01, .05, .95, .99$; $\alpha = 0(.02).10$; $\beta = 0(.01).05$. Table 2 is entitled "Percentage points of the quotient of a triangular and an independent normal variate. If y is triangular with range $(1 - 2\alpha, 1 + 2\alpha)$, and x is normal with mean 1 and variance β^2, the table gives percentage points of (y/x)." The values given are solutions for Q to 3D of

$$Hh_0\left(-\frac{1}{\beta}\right) + \frac{\beta^2 d^2}{(2\alpha)^2\sqrt{2\pi}}\left\{Hh_2\left(\frac{1-2\alpha}{\beta Q} - \frac{1}{\beta}\right) - 2Hh_2\left(\frac{1}{\beta Q} - \frac{1}{\beta}\right) + Hh_2\left(\frac{1+2\alpha}{\beta Q} - \frac{1}{\beta}\right)\right\} = k,$$

$k = .01, .05, .95, .99$; $\alpha = .00(.01).05$; $\beta = .00(.01).05$.

In all cases tabled, $Hh_0(-1/\beta)$ is negligible and could be omitted from the above formulas without affecting the solution.

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1. For the definition of $Hh_n(x)$ see *BAAS Math. Tables*, v. I, London, 1941, p. x.

114[K].—S. B. CHOWDHURY, "The most powerful unbiased critical regions and the shortest unbiased confidence intervals associated with the distribution of the classical D^2-statistic," *Sankhyā*, v. 14, 1954, p. 71–80.

This continuation of an earlier paper by P. K. BOSE [1] includes two short tables for determining the shortest unbiased upper and lower 95% confidence intervals for a parameter Δ^2 used in testing the equality of means of two p-variate normal populations. Let the (known) variance matrices of the two populations be $A = (\alpha_{ij})$, $A' = (\alpha'_{ij})$, and their unknown vector means be $\{\alpha_i\}$, $\{\alpha_i'\}$. In terms of these quantities,

$$\Delta^2 = p^{-1}\sum\sum\beta^{ij}(\alpha_i - \alpha_i')(\alpha_j - \alpha_j'),$$

where β^{ij} are the elements of the inverse matrix $(\beta_{ij})^{-1}$, $\beta_{ij}=(n'\alpha_{ij}+n\alpha'_{ij})/(n+n')$. To test the equality of means a random sample is drawn from each population, of, say, respective sizes n, n', and sample means $\{a_i\}$, $\{a_i'\}$. One then computes a sample statistic D_i^2 (to which the "classical D^2" statistic introduced by MAHALANOBIS is related by $D^2 = D_1^2 - 2/\bar{n}$), corresponding to Δ^2, by means of $D_1^2 = p^{-1}\sum\sum\beta^{ij}(a_i - a_i')(a_j - a_j')$, and from this, the quantities $l^2 = kD_1^2/2$, $L = (\bar{n})^{\frac{1}{2}}l$, where $\bar{n}$ is the harmonic mean of n, n'.

Table 1 gives, for $p = 1(1)10$ and for 14 values of L ranging from $L = 2$ to $L = 400$, the lower 95% confidence limit λ for the parameter λ related to Δ^2 in the same way as l is related to D_1^2, for all alternatives $\lambda > \lambda_0$. Here λ_0 represents the hypothesis to be tested, that is, λ_0 represents the bound for λ which would assure that Δ^2 is sufficiently small to make the means equal within desirable tolerances. The values of $\lambda = \lambda(L)$ are the solutions, for the given values of L, of

$$\int_0^L \psi(x, \lambda(L))dx = 1 - \alpha,$$

where $\alpha = .95$, and

$$\psi(L, \lambda) = L^{p/2}\lambda^{-(p-2)/2}e^{-(L^2+\lambda^2)/2}I_{(p-1)/2}(L\lambda)$$

(where I is the imaginary Bessel function) is the probability density function of L.

Table 2 gives, for the same values of p and L, the upper 95% confidence limit $\bar{\lambda}$, for all alternatives $\lambda < \lambda_0$, in precisely analogous manner to Table 1, with the limits of integration 0 to L in the equation for $\bar{\lambda}(L)$ being replaced by L to ∞.

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1. P. K. BOSE, "On recursion formulae, tables, and Bessel function populations, associated with the distribution of classical D^2-statistic," *Sankhyā*, v. 8, 1947, p. 235–248.

115[K].—D. R. COX, "The mean and coefficient of variation of range in small samples from non-normal populations," *Biometrika*, v. 41, 1954, p. 469–481.

The author is interested in the mean range and the coefficient of variation of the range in small samples of 2, 3, 4, and 5, from non-normal populations. Different types of populations covering a wide range of values of β_1 and β_2, the usual measures of skewness and kurtosis, are considered: symmetrical and unsymmetrical mixed normal distributions, the normal distribution, the rectangular distribution, exponential type distributions, the PEARSON system, and SHONE's [1] numerical results for 5 populations of discrete values. Based on these results he provides a table for the normalized mean range and the coefficient of variation of the range to 3D for sample sizes of 2, 3, 4 and 5, for $\beta_2 = 1(.2)2(.5)5(1)9$; β_1 is not a determining factor. Comparisons are made between the distribution function of the range and the ratio of two ranges from the exponential e^{-x} and the unit normal population. The theory is applied to the point estimation of dispersion by use of the range, the control chart for the range, the comparison of several ranges by use of the range ratio test and the use of the range in the t-test for non-normal as well as normal distributions. Many short tables give numerical illustrations. The author concludes that if β_2 is known, it is possible to make a rough correction for non-normality in methods that use ranges of small samples, and that on the whole these methods are less affected by non-normality than corresponding methods using variances of small samples.

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1. K. J. Shone, "Relations between the standard deviation and the distribution of range in non-normal populations," Roy. Stat. Soc., *Jn.*, s.B., v. 11, 1949, p. 85–88.

116[K].—H. A. David, "The distribution of the range in certain non-normal populations," *Biometrika*, v. 41, 1954, p. 463–468.

Explicit expressions for the probability integral $P(w|n)$ and the expectation $E(w|n)$ of the range w of a sample of n are obtained for several non-normal distributions with varying degrees of skewness and/or kurtosis. Values of $Q(w|n) = 1 - P(w|n)$ for $w = k(n, \alpha)\sigma_x$, where $k(n, \alpha)$ is the $100\alpha\%$ point of w/σ for the normal distribution, are given to 3D in Table 1 for $\alpha = .01, .05$, $n = 2(2)12$, and various degrees of skewness in the populations considered. Part of these values were obtained from the explicit expressions and part by direct quadrature using related tables. Table 2 gives the ratio $E(w|n)/d_n\sigma_x$ to 3D, where d_n is the expected range, in standard units, of a sample of n from the normal distribution, for $n = 2(2)12$ and those distributions for which $E(w|n)$ was obtained explicitly. Table 2 indicates that appreciable non-normality has little effect on the use of the range to estimate the standard deviation, but Table 1 further substantiates the difficulty in distinguishing between changes in variability and departures from normality.

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117[K, L].—H. J. Gawlik, *A table of a function related to the error function*, Armament Research and Development Establishment, BR. Memo. B4/1/55, Fort Halstead, Sevenoaks, Kent. 3 typescript pages mimeographed, 10 leaves of tables (photo-offset) stapled, 33 cm. Deposited in the UMT File.

The table lists $F(x)$ to 10D for $x = 0(.01)10.00$, where

$$F(x) = e^{-x^2/2} \int_0^x e^{t^2/2} dt.$$

There is a brief discussion of the computational techniques used and the author's estimate that all digits are accurate except for possible round off inaccuracy in the last digit. There is no discussion of use of the table or interpolation. The calculation was on the AMOS machine.

C. B. T.

118[L].—OWEN R. MOCK, *Table of Bessel functions at multiples of* $\pi/2$. 8 leaves (mimeographed both sides), deposited in UMT FILE.

A table of $J_n(x)$ for $n = 0(1)59$ and $x = k\pi/2$ to 20D for all non-negative integral values of k giving significant values at this precision and not exceeding 25. There is no introduction or description. The tables are believed to have been computed on a Model 701 International Business Machine, and the author believes that all digits are correct.

C. B. T.

119[L].—ALBERT D. WHEELON & JOHN T. ROBACKER, *A Table of Integrals involving Bessel Functions*, The Ramo-Wooldridge Corporation, Los Angeles, California, 1954, 91 p., 28 cm. Multilithed from manuscript and typescript. Deposited in UMT FILE.

The authors note that the preparation of this table of 400 integrals was halted when they learned of the BATEMAN Manuscript Project which was doing similar work. They hope the tables contain the most frequently encountered integrals. This set is much less comprehensive than those in the Bateman Manuscript Project, "Tables of Integral Transforms," v. 2, New York, 1954, p. 5–177. There are, however, a few formulas in the present work which are not found in "Tables of Integral Transforms."

The authors' careful indication of sources of their formulas is valuable.

Sometimes the statement of the formula is careless. In particular, the reviewer noted the following:

p. 40, 11.306. The important condition $b \leq a$ is not stressed.

p. 41, 11.311. The interpretation of the formula for $a \geq 1$ might be noted.

p. 46, 11.201. Many similar integrals published by W. N. BAILEY, "Some infinite integrals involving Bessel functions," London Math. Soc., *Proc.*, v. 40, 1936, p. 37–48, are not listed. However, this particular integral seems new.

p. 15, 1.207. The integral on the left is divergent. The authors probably intend to have the principal value used.

The indexing scheme is clearly described, and integrals are easily found in the list.

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120[M, S].—A. J. W. DUIJVESTIJN & J. BERGHUIS, *The computation and the expansion of some triple integrals originating from the theory of cosmic rays.* R 261, *Math. Centrum*, Amsterdam. Seventeen leaves mimeographed from typescript deposited in the UMT FILE.

This work contains tables of K and $J(r)$ for $a = 0.005, 0.01, 0.05, 0.1, 0.5, 1$ and $J^*(B)$ for $a = 0.1$ and 1. Two cases are considered: (1) $\alpha = 0.0245/r^3$, $\beta = 0.0245/r_1^3$, and $\gamma = 0.0245/r_2^3$; (2) $\ln \alpha = \ln 0.454 - \ln r + \ln (1 + 4r) - 4r^{2/3}$, $\ln \beta = \ln 0.454 - \ln r_1 + \ln (1 + 4r_1) - 4r_1^{2/3}$, $\ln \gamma = \ln 0.454 - \ln r_1 + \ln (1 + 4r_2) - 4r_2^{2/3}$.

Here $r_{1,2} = (a^2 + r^2 \pm 2ar\cos\theta)^{1/2}$, and

$$I(r, B) = r\int_{\theta=0}^{\pi/2} (1 - e^{-B\alpha})(1 - e^{-B\beta})(1 - e^{-B\gamma})d\theta$$

$$J^*(B) = B^{-5/2}\int_{r=0}^{\infty} I(r, B)dr$$

$$J(r) = \int_{B=0}^{\infty} B^{-5/2}I(r, B)dB$$

$$K = \int_{r=0}^{\infty} J(r)dr = \int_{B=0}^{\infty} J^*(B)dB.$$

Behavior of $J^*(B)$ for small and large positive values of B is discussed, and the method of computation is described.

The integrals are of interest in studies of cosmic rays.

C. B. T.

TABLE ERRATA

In this issue references have been made to errata in Review **80**, Review **81**, and Review **89**.

In *MTAC*, 9, July 1955, reference was made to errata in Review **55** and Review **59**.

245.—E. Jahnke & F. Emde, *Tables of Functions*. Dover edition, 1945.

On p. 211, section 9, chapter VIII, *the left side of the last formula should read* $-\Omega_p(z) + N_p(z) \cdots$ *instead of* $\Omega_p(z) + N_p(z) \cdots$.

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246.—Robert E. Greenwood, "Coupon Collector's Test for Random Digits," *MTAC*, 9, 1955, p. 3.

Entry for $n = 13$ *should read* .00800 8315 2 *instead of* .0080 9315 2.

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NOTES

R. C. Archibald

Professor Raymond C. Archibald, who founded *Mathematical Tables and Other Aids to Computation* and who was always one of its principal contributors and the most valued adviser to the Editorial Committee, died on July 26, 1955 at Sackville, New Brunswick, Canada.

Professor Archibald was the Chairman of the Editorial Committee from the first issue of the journal in 1943 through 1949.

A short history of some of Professor Archibald's professional and personal accomplishments will appear in *MTAC* shortly.