Tabulation of the Function $\psi(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$

Introduction. The function $\phi(x)$ defined by

$$\phi(x) = \int_1^x \frac{dt}{t} \log|1 + t| \tag{1}$$

occurs quite frequently in problems in Quantum Electrodynamics, as for example problems on vacuum polarization [1], scattering of light by light [2], and similar fourth order processes [3].

For real values of x, the function $\phi(x)$ has been tabulated by Mitchell [4]. His tabulated function is $\phi(0) - \phi(x)$, rather than our $\phi(x)$ defined by (1). As in most literature he defines it as minus the integral from 0 to x. We shall however stick to the definition (1), which has already appeared in some of the literature [1].

The integrals appearing in the above mentioned problems cannot always be expressed in terms of $\phi(x)$ with the argument x real. In many cases there appear $\phi(x)$ with a complex x of absolute value 1. This happens when the integrals in question are expressed as combinations of $\phi\left(\frac{1-\delta}{1+\delta}\right)$ and $\phi\left(-\frac{1-\delta}{1+\delta}\right)$ where δ^2 is to take all values from $-\infty$ to $+\infty$. When δ^2 is negative, the complex argument $1 - \delta/1 + \delta$ will have absolute value 1, and the problem reduces to that of finding $\phi(e^{i\theta})$. On use of the expansion

$$\phi(x) = \sum_1^{\infty} \frac{(-)^{n+1}x^n}{n^2} - \frac{\pi^2}{12} \tag{2}$$

the latter function can be expressed as

$$\phi(e^{i\theta}) = -\theta^2/4 + i\psi(\theta) \tag{3}$$

where

$$\psi(\theta) = \sum_1^{\infty} \frac{\sin n\theta}{n^2}. \tag{4}$$

It was however shown by Kummer [5] that for complex x the imaginary part of $\phi(x)$ can be expressed in terms of $\psi(\theta)$ defined by equation (4).

The function $\psi(\theta)$, the Spence function, was tabulated by Clausen [6] for $\theta = 1, 2, \cdots, 180°$. We find that these given intervals are not enough for easy computation of the function for any argument.

A tabulation of the function $\psi(\theta)$ for enough intervals will be useful for physicists to carry out explicit numerical evaluations of their calculations for comparison with recent accurate experiment, which is necessary for the development of the theory. Therefore we found it worth while tabulating the function, given by Clausen for 180 intervals, for more intervals in such a way that the function can be easily computed for any value of the argument θ.

Some properties of the function $\psi(\theta)$. The series $\sum_1^\infty \frac{\sin n\theta}{n^2}$ is slowly convergent and the function is best computed by a numerical evaluation of the integral

$$\psi(\theta) = -\int_0^\theta \log\left(2\sin\frac{t}{2}\right)dt. \tag{5}$$

The function is zero at $\theta = 0$, $\theta = \pi$ and has a maximum between these two limits at $\theta = \pi/3$. For $\theta < \pi/2$ the function can be expressed by the expansion

$$\psi(\theta) = -\theta\log|\theta| + \theta + \sum_1^\infty \frac{(-)^{n-1}}{(2n)!}B_{2n}\frac{\theta^{2n+1}}{2n(2n+1)}. \tag{6}$$

For $\pi/2 < \theta < \pi$ the function can be expanded by use of the recurrence relation

$$\psi(\pi - \theta) = \psi(\theta) - \tfrac{1}{2}\psi(2\theta) \tag{7}$$

and is given by

$$\psi(\pi - \theta) = \theta\log 2 - \sum_1^\infty \frac{(-)^{n-1}}{(2n)!}B_{2n}(2^{2n} - 1)\frac{\theta^{2n+1}}{2n(2n+1)} \tag{8}$$

which shows that the function is linear in θ for θ near π.

Method of calculation. We start by evaluating the function from $\theta = \pi$ to $\theta = \pi/2$. The tabulated function $\log_{10}(\sin\frac{1}{2}t)$ to 7 figures is used to obtain the value of the integrand in (5), at 540 intervals: $\theta_0, \theta_1, \cdots$. We have then used the known expression

$$\psi(\theta_n) = w[\tfrac{1}{2}f(\theta_0) + f(\theta_1) + \cdots + f(\theta_{n-1}) + \tfrac{1}{2}f(\theta_n)] - \frac{w^2}{12}[f'(\theta_n) - f'(\theta_0)] \tag{9}$$

to compute $\psi(\theta_n)$. Here $f(\theta_i)$ is the integrand and w is the interval $\pi/1080$. The remainder containing w^4 and higher powers was found not to affect the result expressed by 7 figures.

The last tabulated value at $\theta = \pi/2$ was found to agree with the known Catalan's number, $\sum_1^\infty \frac{(-)^n}{(2n+1)^2}$, which was a good check on this first part of the work.

The value of $\psi(\theta)$ from $\theta = \pi/2$ to $\theta = 0$ will then be computed from the already calculated values on use of the relation $\psi(\theta) = \psi(\pi - \theta) + \frac{1}{2}\psi(2\theta)$. On

reaching $\theta = \pi/4$ we had to start using newly calculated values of $\psi(\theta)$ to complete the table until $\theta = 0$. For example the value at $\theta = 44°$ is $\psi(136°) + \frac{1}{2}\psi(88°)$. The value at $\theta = 136°$ is contained in the first part of the work while the value at $\theta = 88°$ must be obtained from $\psi(92°) + \frac{1}{2}\psi(176°)$, which is contained in the beginning of the second part of the work.

The function was found to have the required maximum at $\theta = \pi/3$ and was further checked by directly evaluating it, using the expansion (6) for several values of θ.

The whole work has been carried out keeping the number of decimals 7, but it was found safer to tabulate it finally to 6 places of decimals which is enough for most purposes.

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x	$\psi(x)$	δ
0° 00′	0.000000	
		19897
10	0.019897	15864
20	0.035761	14342
30	0.050103	13354
40	0.063457	12618
50	0.076075	12033
1° 00	0.088108	11546
10	0.099654	11129
20	0.110783	10764
30	0.121547	10441
40	0.131988	10149
50	0.142137	9885
2° 00	0.152022	9641
10	0.161663	9418
20	0.171081	9210
30	0.180291	9016
40	0.189307	8834
50	0.198141	8662
3° 00	0.206803	8501
10	0.215304	8348
20	0.223652	8203
30	0.231855	8064
40	0.239919	7932
50	0.247851	7805
4° 00	0.255656	7684
10	0.263340	7568
20	0.270908	7456
30	0.278364	7348
40	0.285712	7244
50	0.292956	7144
5° 00	0.300100	7047
10	0.307147	6954
20	0.314101	6862
30	0.320963	6775
40	0.327738	6689
50	0.334427	6605
6° 00	0.341032	6526
10	0.347558	6446
20	0.354004	6370
30	0.360374	6296
40	0.366670	6223
50	0.372893	6152
7° 00	0.379045	6083
10	0.385128	6015
20	0.391143	5949
30	0.397092	5885
40	0.402977	5822
50	0.408799	5759
8° 00	0.414558	5700
10	0.420258	5640
20	0.425898	5582
30	0.431480	5525
40	0.437005	5469
50	0.442474	5414
9° 00	0.447888	5361
10	0.453249	5308
20	0.458557	5256
30	0.463813	5205
40	0.469018	5155
50	0.474173	5106
10° 00	0.479279	

x	$\psi(x)$	δ
10° 00′	0.479279	
		5057
10	0.484336	5010
20	0.489346	4963
30	0.494309	4918
40	0.499227	4871
50	0.504098	4827
11° 00	0.508925	4784
10	0.513709	4740
20	0.518449	4697
30	0.523146	4655
40	0.527801	4614
50	0.532415	4573
12° 00	0.536988	4533
10	0.541521	4493
20	0.546014	4454
30	0.550468	4415
40	0.554883	4378
50	0.559261	4339
13° 00	0.563600	4303
10	0.567903	4265
20	0.572168	4230
30	0.576398	4194
40	0.580592	4158
50	0.584750	4124
14° 00	0.588874	4089
10	0.592963	4055
20	0.597018	4022
30	0.601040	3988
40	0.605028	3955
50	0.608983	3923
15° 00	0.612906	3891
10	0.616797	3859
20	0.620656	3827
30	0.624483	3797
40	0.628280	3765
50	0.632045	3736
16° 00	0.635781	3705
10	0.639486	3675
20	0.643161	3646
30	0.646807	3617
40	0.650424	3588
50	0.654012	3559
17° 00	0.657571	3531
10	0.661102	3503
20	0.664605	3475
30	0.668080	3447
40	0.671527	3421
50	0.674948	3393
18° 00	0.678341	3367
10	0.681708	3340
20	0.685048	3314
30	0.688362	3288
40	0.691650	3262
50	0.694912	3237
19° 00	0.698149	3212
10	0.701361	3186
20	0.704547	3162
30	0.707709	3137
40	0.710846	3112
50	0.713958	3089
20° 00	0.717047	

x	$\psi(x)$	δ
20° 00′	0.717047	
		3064
10	0.720111	3041
20	0.723152	3017
30	0.726169	2994
40	0.729163	2970
50	0.732133	2947
21° 00	0.735080	2925
10	0.738005	2902
20	0.740907	2879
30	0.743786	2858
40	0.746644	2835
50	0.749479	2813
22° 00	0.752292	2791
10	0.755083	2770
20	0.757853	2748
30	0.760601	2727
40	0.763328	2706
50	0.766034	2685
23° 00	0.768719	2665
10	0.771384	2643
20	0.774027	2623
30	0.776650	2603
40	0.779253	2582
50	0.781835	2563
24° 00	0.784398	2543
10	0.786941	2522
20	0.789463	2504
30	0.791967	2483
40	0.794450	2465
50	0.796915	2445
25° 00	0.799360	2426
10	0.801786	2407
20	0.804193	2388
30	0.806581	2370
40	0.808951	2351
50	0.811302	2333
26° 00	0.813635	2314
10	0.815949	2296
20	0.818245	2278
30	0.820523	2260
40	0.822783	2242
50	0.825025	2224
27° 00	0.827249	2207
10	0.829456	2190
20	0.831646	2171
30	0.833817	2155
40	0.835972	2137
50	0.838109	2121
28° 00	0.840230	2103
10	0.842333	2087
20	0.844420	2069
30	0.846489	2053
40	0.848542	2037
50	0.850579	2020
29° 00	0.852599	2004
10	0.854603	1987
20	0.856590	1971
30	0.858561	1956
40	0.860517	1939
50	0.862456	1923
30° 00	0.864379	

x	$\psi(x)$	δ
30° 00′	0.864379	
		1908
10	0.866287	
		1892
20	0.868179	
		1876
30	0.870055	
		1861
40	0.871916	
		1845
50	0.873761	
		1830
31° 00	0.875591	
		1814
10	0.877405	
		1800
20	0.879205	
		1784
30	0.880989	
		1770
40	0.882759	
		1755
50	0.884514	
		1739
32° 00	0.886253	
		1725
10	0.887978	
		1710
20	0.889688	
		1696
30	0.891384	
		1681
40	0.893065	
		1667
50	0.894732	
		1652
33° 00	0.896384	
		1638
10	0.898022	
		1624
20	0.899646	
		1610
30	0.901256	
		1596
40	0.902852	
		1581
50	0.904433	
		1568
34° 00	0.906001	
		1554
10	0.907555	
		1540
20	0.909095	
		1527
30	0.910622	
		1513
40	0.912135	
		1499
50	0.913634	
		1486
35° 00	0.915120	
		1472
10	0.916592	
		1459
20	0.918051	
		1446
30	0.919497	
		1432
40	0.920929	
		1420
50	0.922349	
		1406
36° 00	0.923755	
		1393
10	0.925148	
		1381
20	0.926529	
		1367
30	0.927896	
		1355
40	0.929251	
		1342
50	0.930593	
		1329
37° 00	0.931922	
		1316
10	0.933238	
		1304
20	0.934542	
		1292
30	0.935834	
		1278
40	0.937112	
		1267
50	0.938379	
		1254
38° 00	0.939633	
		1242
10	0.940875	
		1230
20	0.942105	
		1217
30	0.943322	
		1206
40	0.944528	
		1193
50	0.945721	
		1181
39° 00	0.946902	
		1170
10	0.948072	
		1157
20	0.949229	
		1146
30	0.950375	
		1133
40	0.951508	
		1123
50	0.952631	
		1110
40° 00	0.953741	

x	$\psi(x)$	δ
40° 00′	0.953741	
		1099
10	0.954840	
		1087
20	0.955927	
		1076
30	0.957003	
		1064
40	0.958067	
		1053
50	0.959120	
		1041
41° 00	0.960161	
		1031
10	0.961192	
		1018
20	0.962210	
		1008
30	0.963218	
		997
40	0.964215	
		985
50	0.965200	
		974
42° 00	0.966174	
		964
10	0.967138	
		952
20	0.968090	
		941
30	0.969031	
		931
40	0.969962	
		920
50	0.970882	
		908
43° 00	0.971790	
		899
10	0.972689	
		887
20	0.973576	
		877
30	0.974453	
		866
40	0.975319	
		856
50	0.976175	
		845
44° 00	0.977020	
		834
10	0.977854	
		825
20	0.978679	
		814
30	0.979493	
		803
40	0.980296	
		793
50	0.981089	
		783
45° 00	0.981872	
		773
10	0.982645	
		762
20	0.983407	
		753
30	0.984160	
		742
40	0.984902	
		733
50	0.985635	
		722
46° 00	0.986357	
		712
10	0.987069	
		703
20	0.987772	
		692
30	0.988464	
		683
40	0.989147	
		673
50	0.989820	
		663
47° 00	0.990483	
		653
10	0.991136	
		644
20	0.991780	
		634
30	0.992414	
		624
40	0.993038	
		615
50	0.993653	
		605
48° 00	0.994258	
		596
10	0.994854	
		586
20	0.995440	
		577
30	0.996017	
		567
40	0.996584	
		559
50	0.997143	
		548
49° 00	0.997691	
		540
10	0.998231	
		530
20	0.998761	
		521
30	0.999282	
		512
40	0.999794	
		503
50	1.000297	
		494
50° 00	1.000791	

x	$\psi(x)$	δ
50° 00′	1.000791	
		484
10	1.001275	
		476
20	1.001751	
		466
30	1.002217	
		458
40	1.002675	
		448
50	1.003123	
		440
51° 00	1.003563	
		431
10	1.003994	
		422
20	1.004416	
		413
30	1.004829	
		405
40	1.005234	
		395
50	1.005629	
		387
52° 00	1.006016	
		379
10	1.006395	
		369
20	1.006764	
		361
30	1.007125	
		353
40	1.007478	
		344
50	1.007822	
		335
53° 00	1.008157	
		327
10	1.008484	
		318
20	1.008802	
		311
30	1.009113	
		301
40	1.009414	
		294
50	1.009708	
		284
54° 00	1.009992	
		277
10	1.010269	
		268
20	1.010537	
		261
30	1.010798	
		252
40	1.011050	
		243
50	1.011293	
		236
55° 00	1.011529	
		227
10	1.011756	
		220
20	1.011976	
		211
30	1.012187	
		203
40	1.012390	
		195
50	1.012585	
		188
56° 00	1.012773	
		179
10	1.012952	
		171
20	1.013123	
		164
30	1.013287	
		155
40	1.013442	
		148
50	1.013590	
		140
57° 00	1.013730	
		132
10	1.013862	
		125
20	1.013987	
		116
30	1.014103	
		109
40	1.014212	
		101
50	1.014313	
		94
58° 00	1.014407	
		86
10	1.014493	
		78
20	1.014571	
		71
30	1.014642	
		63
40	1.014705	
		56
50	1.014761	
		48
59° 00	1.014809	
		40
10	1.014849	
		34
20	1.014883	
		26
30	1.014909	
		18
40	1.014927	
		11
50	1.014938	
		4
60° 00	1.014942	

x	$\psi(x)$	$-\delta$
60° 00′	1.014942	
10	1.014938	4
20	1.014927	11
30	1.014909	18
40	1.014883	26
50	1.014851	32
61° 00	1.014811	40
10	1.014763	48
20	1.014709	54
30	1.014648	61
40	1.014579	69
50	1.014504	75
62° 00	1.014421	83
10	1.014331	90
20	1.014235	96
30	1.014131	104
40	1.014020	111
50	1.013902	118
63° 00	1.013778	124
10	1.013646	132
20	1.013508	138
30	1.013363	145
40	1.013211	152
50	1.013052	159
64° 00	1.012886	166
10	1.012714	172
20	1.012535	179
30	1.012349	186
40	1.012156	193
50	1.011957	199
65° 00	1.011751	206
10	1.011538	213
20	1.011319	219
30	1.011093	226
40	1.010861	232
50	1.010622	239
66° 00	1.010376	246
10	1.010124	252
20	1.009866	258
30	1.009601	265
40	1.009329	272
50	1.009052	277
67° 00	1.008767	285
10	1.008477	290
20	1.008180	297
30	1.007876	304
40	1.007566	310
50	1.007250	316
68° 00	1.006928	322
10	1.006599	329
20	1.006265	334
30	1.005924	341
40	1.005576	348
50	1.005223	353
69° 00	1.004863	360
10	1.004497	366
20	1.004125	372
30	1.003747	378
40	1.003363	384
50	1.002973	390
70° 00	1.002576	397

x	$\psi(x)$	$-\delta$
70° 00′	1.002576	
10	1.002174	402
20	1.001766	408
30	1.001351	415
40	1.000931	420
50	1.000505	426
71° 00	1.000072	433
10	0.999634	438
20	0.999190	444
30	0.998740	450
40	0.998284	456
50	0.997822	462
72° 00	0.997355	467
10	0.996881	474
20	0.996402	479
30	0.995917	485
40	0.995426	491
50	0.994930	496
73° 00	0.994427	503
10	0.993919	508
20	0.993406	513
30	0.992886	520
40	0.992361	525
50	0.991831	530
74° 00	0.991294	537
10	0.990752	542
20	0.990205	547
30	0.989652	553
40	0.989093	559
50	0.988529	564
75° 00	0.987959	570
10	0.987384	575
20	0.986803	581
30	0.986217	586
40	0.985625	592
50	0.985028	597
76° 00	0.984425	603
10	0.983817	608
20	0.983204	613
30	0.982585	619
40	0.981961	624
50	0.981331	630
77° 00	0.980696	635
10	0.980056	640
20	0.979411	645
30	0.978760	651
40	0.978104	656
50	0.977443	661
78° 00	0.976776	667
10	0.976104	672
20	0.975427	677
30	0.974745	682
40	0.974058	687
50	0.973366	692
79° 00	0.972668	698
10	0.971965	703
20	0.971257	708
30	0.970544	713
40	0.969826	718
50	0.969103	723
80° 00	0.968375	728

x	$\psi(x)$	$-\delta$
80° 00′	0.968375	
10	0.967642	733
20	0.966903	739
30	0.966160	743
40	0.965412	748
50	0.964658	754
81° 00	0.963900	758
10	0.963137	763
20	0.962369	768
30	0.961596	773
40	0.960818	778
50	0.960035	783
82° 00	0.959247	788
10	0.958455	792
20	0.957657	798
30	0.956855	802
40	0.956048	807
50	0.955236	812
83° 00	0.954419	817
10	0.953598	821
20	0.952771	827
30	0.951940	831
40	0.951105	835
50	0.950264	841
84° 00	0.949419	845
10	0.948569	850
20	0.947714	855
30	0.946855	859
40	0.945991	864
50	0.945123	868
85° 00	0.944249	874
10	0.943372	877
20	0.942489	883
30	0.941602	887
40	0.940711	891
50	0.939814	897
86° 00	0.938914	900
10	0.938008	906
20	0.937099	909
30	0.936184	915
40	0.935266	918
50	0.934342	924
87° 00	0.933415	927
10	0.932482	933
20	0.931546	936
30	0.930605	941
40	0.929659	946
50	0.928709	950
88° 00	0.927755	954
10	0.926796	959
20	0.925833	963
30	0.924866	967
40	0.923894	972
50	0.922918	976
89° 00	0.921938	980
10	0.920953	985
20	0.919964	989
30	0.918971	993
40	0.917973	998
50	0.916972	1001
90° 00	0.915966	1006

x	$\psi(x)$	$-\delta$
90° 00′	0.915966	
		1011
10	0.914955	
		1014
20	0.913941	
		1019
30	0.912922	
		1023
40	0.911899	
		1027
50	0.910872	
		1031
91° 00	0.909841	
		1035
10	0.908806	
		1040
20	0.907766	
		1044
30	0.906722	
		1047
40	0.905675	
		1052
50	0.904623	
		1056
92° 00	0.903567	
		1060
10	0.902507	
		1064
20	0.901443	
		1069
30	0.900374	
		1072
40	0.899302	
		1076
50	0.898226	
		1081
93° 00	0.897145	
		1084
10	0.896061	
		1088
20	0.894973	
		1093
30	0.893880	
		1096
40	0.892784	
		1100
50	0.891684	
		1104
94° 00	0.890580	
		1109
10	0.889471	
		1112
20	0.888359	
		1116
30	0.887243	
		1120
40	0.886123	
		1124
50	0.884999	
		1127
95° 00	0.883872	
		1132
10	0.882740	
		1135
20	0.881605	
		1140
30	0.880465	
		1143
40	0.879322	
		1147
50	0.878175	
		1151
96° 00	0.877024	
		1154
10	0.875870	
		1159
20	0.874711	
		1162
30	0.873549	
		1166
40	0.872383	
		1170
50	0.871213	
		1173
97° 00	0.870040	
		1178
10	0.868862	
		1181
20	0.867681	
		1184
30	0.866497	
		1189
40	0.865308	
		1192
50	0.864116	
		1196
98° 00	0.862920	
		1199
10	0.861721	
		1203
20	0.860518	
		1207
30	0.859311	
		1211
40	0.858100	
		1214
50	0.856886	
		1218
99° 00	0.855668	
		1221
10	0.854447	
		1225
20	0.853222	
		1228
30	0.851994	
		1232
40	0.850762	
		1236
50	0.849526	
		1239
100° 00	0.848287	

x	$\psi(x)$	$-\delta$
100° 00′	0.848287	
		1243
10	0.847044	
		1247
20	0.845797	
		1249
30	0.844548	
		1254
40	0.843294	
		1257
50	0.842037	
		1260
101° 00	0.840777	
		1264
10	0.839513	
		1267
20	0.838246	
		1271
30	0.836975	
		1274
40	0.835701	
		1278
50	0.834423	
		1281
102° 00	0.833142	
		1285
10	0.831857	
		1288
20	0.830569	
		1291
30	0.829278	
		1295
40	0.827983	
		1298
50	0.826685	
		1302
103° 00	0.825383	
		1305
10	0.824078	
		1308
20	0.822770	
		1312
30	0.821458	
		1315
40	0.820143	
		1318
50	0.818825	
		1322
104° 00	0.817503	
		1324
10	0.816179	
		1329
20	0.814850	
		1331
30	0.813519	
		1335
40	0.812184	
		1338
50	0.810846	
		1341
105° 00	0.809505	
		1345
10	0.808160	
		1347
20	0.806813	
		1351
30	0.805462	
		1355
40	0.804107	
		1357
50	0.802750	
		1361
106° 00	0.801389	
		1364
10	0.800025	
		1367
20	0.798658	
		1370
30	0.797288	
		1373
40	0.795915	
		1377
50	0.794538	
		1379
107° 00	0.793159	
		1383
10	0.791776	
		1386
20	0.790390	
		1389
30	0.789001	
		1392
40	0.787609	
		1395
50	0.786214	
		1398
108° 00	0.784816	
		1402
10	0.783414	
		1404
20	0.782010	
		1407
30	0.780603	
		1411
40	0.779192	
		1413
50	0.777779	
		1417
109° 00	0.776362	
		1419
10	0.774943	
		1423
20	0.773520	
		1426
30	0.772094	
		1428
40	0.770666	
		1432
50	0.769234	
		1434
110° 00	0.767800	

x	$\psi(x)$	$-\delta$
110° 00′	0.767800	
		1438
10	0.766362	
		1440
20	0.764922	
		1444
30	0.763478	
		1446
40	0.762032	
		1449
50	0.760583	
		1452
111° 00	0.759131	
		1455
10	0.757676	
		1458
20	0.756218	
		1461
30	0.754757	
		1464
40	0.753293	
		1467
50	0.751826	
		1469
112° 00	0.750357	
		1472
10	0.748885	
		1475
20	0.747410	
		1478
30	0.745932	
		1481
40	0.744451	
		1484
50	0.742967	
		1486
113° 00	0.741481	
		1490
10	0.739991	
		1492
20	0.738499	
		1495
30	0.737004	
		1497
40	0.735507	
		1501
50	0.734006	
		1503
114° 00	0.732503	
		1506
10	0.730997	
		1508
20	0.729489	
		1511
30	0.727978	
		1515
40	0.726463	
		1516
50	0.724947	
		1520
115° 00	0.723427	
		1522
10	0.721905	
		1525
20	0.720380	
		1527
30	0.718853	
		1531
40	0.717322	
		1532
50	0.715790	
		1536
116° 00	0.714254	
		1538
10	0.712716	
		1541
20	0.711175	
		1543
30	0.709632	
		1546
40	0.708086	
		1549
50	0.706537	
		1551
117° 00	0.704986	
		1554
10	0.703432	
		1557
20	0.701875	
		1559
30	0.700316	
		1561
40	0.698755	
		1564
50	0.697191	
		1567
118° 00	0.695624	
		1569
10	0.694055	
		1572
20	0.692483	
		1574
30	0.690909	
		1577
40	0.689332	
		1579
50	0.687753	
		1582
119° 00	0.686171	
		1585
10	0.684586	
		1586
20	0.683000	
		1590
30	0.681410	
		1591
40	0.679819	
		1595
50	0.678224	
		1596
120° 00	0.676628	

x	$\psi(x)$	$-\delta$
120° 00′	0.676628	
		1599
10	0.675029	
		1602
20	0.673427	
		1604
30	0.671823	
		1606
40	0.670217	
		1609
50	0.668608	
		1611
121° 00	0.666997	
		1614
10	0.665383	
		1616
20	0.663767	
		1618
30	0.662149	
		1621
40	0.660528	
		1623
50	0.658905	
		1625
122° 00	0.657280	
		1628
10	0.655652	
		1630
20	0.654022	
		1632
30	0.652390	
		1635
40	0.650755	
		1637
50	0.649118	
		1640
123° 00	0.647478	
		1641
10	0.645837	
		1644
20	0.644193	
		1646
30	0.642547	
		1649
40	0.640898	
		1651
50	0.639247	
		1653
124° 00	0.637594	
		1655
10	0.635939	
		1658
20	0.634281	
		1659
30	0.632622	
		1662
40	0.630960	
		1665
50	0.629295	
		1666
125° 00	0.627629	
		1669
10	0.625960	
		1670
20	0.624290	
		1673
30	0.622617	
		1676
40	0.620941	
		1677
50	0.619264	
		1679
126° 00	0.617585	
		1682
10	0.615903	
		1684
20	0.614219	
		1686
30	0.612533	
		1688
40	0.610845	
		1690
50	0.609155	
		1692
127° 00	0.607463	
		1695
10	0.605768	
		1696
20	0.604072	
		1699
30	0.602373	
		1701
40	0.600672	
		1703
50	0.598969	
		1704
128° 00	0.597265	
		1707
10	0.595558	
		1709
20	0.593849	
		1711
30	0.592138	
		1714
40	0.590424	
		1715
50	0.588709	
		1717
129° 00	0.586992	
		1719
10	0.585273	
		1721
20	0.583552	
		1723
30	0.581829	
		1725
40	0.580104	
		1728
50	0.578376	
		1729
130° 00	0.576647	

x	$\psi(x)$	$-\delta$
130° 00′	0.576647	
		1731
10	0.574916	
		1733
20	0.573183	
		1735
30	0.571448	
		1737
40	0.569711	
		1739
50	0.567972	
		1741
131° 00	0.566231	
		1742
10	0.564489	
		1745
20	0.562744	
		1747
30	0.560997	
		1748
40	0.559249	
		1751
50	0.557498	
		1752
132° 00	0.555746	
		1754
10	0.553992	
		1756
20	0.552236	
		1758
30	0.550478	
		1760
40	0.548718	
		1762
50	0.546956	
		1763
133° 00	0.545193	
		1766
10	0.543427	
		1767
20	0.541660	
		1769
30	0.539891	
		1771
40	0.538120	
		1772
50	0.536348	
		1775
134° 00	0.534573	
		1776
10	0.532797	
		1778
20	0.531019	
		1780
30	0.529239	
		1781
40	0.527458	
		1784
50	0.525674	
		1785
135° 00	0.523889	
		1787
10	0.522102	
		1788
20	0.520314	
		1790
30	0.518524	
		1793
40	0.516731	
		1793
50	0.514938	
		1796
136° 00	0.513142	
		1797
10	0.511345	
		1799
20	0.509546	
		1801
30	0.507745	
		1802
40	0.505943	
		1804
50	0.504139	
		1805
137° 00	0.502334	
		1808
10	0.500526	
		1809
20	0.498717	
		1810
30	0.496907	
		1813
40	0.495094	
		1814
50	0.493280	
		1815
138° 00	0.491465	
		1817
10	0.489648	
		1819
20	0.487829	
		1821
30	0.486008	
		1822
40	0.484186	
		1823
50	0.482363	
		1825
139° 00	0.480538	
		1827
10	0.478711	
		1828
20	0.476883	
		1830
30	0.475053	
		1832
40	0.473221	
		1833
50	0.471388	
		1834
140° 00	0.469554	

x	$\psi(x)$	$-\delta$
140° 00′	0.469554	
		1837
10	0.467717	
		1837
20	0.465880	
		1839
30	0.464041	
		1841
40	0.462200	
		1842
50	0.460358	
		1844
141° 00	0.458514	
		1845
10	0.456669	
		1847
20	0.454822	
		1848
30	0.452974	
		1850
40	0.451124	
		1851
50	0.449273	
		1852
142° 00	0.44742[1]	
		1854
10	0.445567	
		1856
20	0.443711	
		1857
30	0.441854	
		1858
40	0.439996	
		1860
50	0.438136	
		1861
143° 00	0.436275	
		1863
10	0.434412	
		1864
20	0.432548	
		1865
30	0.430683	
		1867
40	0.428816	
		1869
50	0.426947	
		1869
144° 00	0.425078	
		1871
10	0.423207	
		1873
20	0.421334	
		1873
30	0.419461	
		1875
40	0.417586	
		1877
50	0.415709	
		1878
145° 00	0.413831	
		1879
10	0.411952	
		1880
20	0.410072	
		1882
30	0.408190	
		1883
40	0.406307	
		1884
50	0.404423	
		1886
146° 00	0.402537	
		1887
10	0.400650	
		1888
20	0.398762	
		1890
30	0.396872	
		1890
40	0.394982	
		1892
50	0.393090	
		1894
147° 00	0.391196	
		1894
10	0.389302	
		1896
20	0.387406	
		1897
30	0.385509	
		1899
40	0.383610	
		1899
50	0.381711	
		1901
148° 00	0.379810	
		1902
10	0.377908	
		1903
20	0.376005	
		1904
30	0.374101	
		1906
40	0.372195	
		1907
50	0.370288	
		1908
149° 00	0.368380	
		1909
10	0.366471	
		1910
20	0.364561	
		1912
30	0.362649	
		1912
40	0.360737	
		1914
50	0.358823	
		1915
150° 00	0.356908	

x	$\psi(x)$	$-\delta$
150° 00′	0.356908	
10	0.354992	1916
20	0.353075	1917
30	0.351157	1918
40	0.349238	1919
50	0.347317	1921
151° 00	0.345396	1921
10	0.343473	1923
20	0.341549	1924
30	0.339624	1925
40	0.337698	1926
50	0.335771	1927
152° 00	0.333843	1928
10	0.331914	1929
20	0.329984	1930
30	0.328053	1931
40	0.326121	1932
50	0.324187	1934
153° 00	0.322253	1934
10	0.320318	1935
20	0.318382	1936
30	0.316444	1938
40	0.314506	1938
50	0.312567	1939
154° 00	0.310626	1941
10	0.308685	1941
20	0.306743	1942
30	0.304800	1943
40	0.302856	1944
50	0.300911	1945
155° 00	0.298965	1946
10	0.297018	1947
20	0.295070	1948
30	0.293121	1949
40	0.291171	1950
50	0.289220	1951
156° 00	0.287269	1951
10	0.285316	1953
20	0.283363	1953
30	0.281409	1954
40	0.279454	1955
50	0.277498	1956
157° 00	0.275541	1957
10	0.273583	1958
20	0.271625	1958
30	0.269665	1960
40	0.267705	1960
50	0.265744	1961
158° 00	0.263782	1962
10	0.261819	1963
20	0.259856	1963
30	0.257891	1965
40	0.255926	1965
50	0.253960	1966
159° 00	0.251993	1967
10	0.250026	1967
20	0.248057	1969
30	0.246088	1969
40	0.244118	1970
50	0.242148	1970
160° 00	0.240176	1972

x	$\psi(x)$	$-\delta$
160° 00′	0.240176	
10	0.238204	1972
20	0.236231	1973
30	0.234258	1973
40	0.232283	1975
50	0.230308	1975
161° 00	0.228333	1975
10	0.226356	1977
20	0.224379	1977
30	0.222401	1978
40	0.220423	1978
50	0.218443	1980
162° 00	0.216463	1980
10	0.214483	1980
20	0.212502	1981
30	0.210520	1982
40	0.208537	1983
50	0.206554	1983
163° 00	0.204570	1984
10	0.202586	1984
20	0.200601	1985
30	0.198615	1986
40	0.196629	1986
50	0.194642	1987
164° 00	0.192654	1988
10	0.190666	1988
20	0.188677	1989
30	0.186688	1989
40	0.184698	1990
50	0.182708	1990
165° 00	0.180717	1991
10	0.178725	1992
20	0.176733	1992
30	0.174740	1993
40	0.172747	1993
50	0.170753	1994
166° 00	0.168759	1994
10	0.166764	1995
20	0.164769	1995
30	0.162773	1996
40	0.160777	1996
50	0.158780	1997
167° 00	0.156783	1997
10	0.154785	1998
20	0.152787	1998
30	0.150788	1999
40	0.148789	1999
50	0.146789	2000
168° 00	0.144789	2000
10	0.142789	2000
20	0.140788	2001
30	0.138786	2002
40	0.136784	2002
50	0.134782	2002
169° 00	0.132780	2002
10	0.130777	2003
20	0.128773	2004
30	0.126769	2004
40	0.124765	2004
50	0.122760	2005
170° 00	0.120755	2005

x	$\psi(x)$	$-\delta$
170° 00′	0.120755	
10	0.118750	2005
20	0.116744	2006
30	0.114738	2006
40	0.112732	2006
50	0.110725	2007
171° 00	0.108718	2007
10	0.106710	2008
20	0.104702	2008
30	0.102694	2008
40	0.100686	2008
50	0.098677	2009
172° 00	0.096668	2009
10	0.094659	2009
20	0.092649	2010
30	0.090639	2010
40	0.088629	2010
50	0.086619	2010
173° 00	0.084608	2011
10	0.082597	2011
20	0.080586	2011
30	0.078574	2012
40	0.076562	2012
50	0.074551	2011
174° 00	0.072538	2013
10	0.070526	2012
20	0.068513	2013
30	0.066500	2013
40	0.064487	2013
50	0.062474	2013
175° 00	0.060461	2013
10	0.058447	2014
20	0.056433	2014
30	0.054419	2014
40	0.052405	2014
50	0.050391	2014
176° 00	0.048377	2014
10	0.046362	2015
20	0.044347	2015
30	0.042332	2015
40	0.040317	2015
50	0.038302	2015
177° 00	0.036287	2015
10	0.034272	2015
20	0.032256	2016
30	0.030241	2015
40	0.028225	2016
50	0.026209	2016
178° 00	0.024194	2015
10	0.022178	2016
20	0.020162	2016
30	0.018146	2016
40	0.016130	2016
50	0.014114	2016
179° 00	0.012097	2017
10	0.010081	2016
20	0.008065	2016
30	0.006049	2016
40	0.004033	2016
50	0.002016	2016
180° 00	0.000000	2016