## Note on Predictor-Corrector Formulas

The various predictor-corrector formulas for solving differential equations have considerable appeal because of their local control of truncation error and machine error. Of course, there is an expense for this control in terms of additional computing, so that a different method might be chosen if the accuracy does not need to be examined at each step. However, predictor-corrector formulas are quite generally used [1], [2].

There seems to be a natural tendency to repeat the "correction" formula, just like you take two aspirin instead of one. In fact the convergence of the iterative procedure defined by a "correction" formula has been studied, but evidently without inquiring whether the successive steps in the iteration actually afford an improvement. The purpose of this note is to point out that repeating a "correction" formula may worsen the result rather than improve it.

Consider $y' = f(x, y)$, initial value $(x_0, y_0)$, predictor $y^p{}_{n+1} = y_{n-1} + 2hf(x_n, y_n)$, and corrector $y^c{}_{n+1} = y_n + (h/2)[f(x_n, y_n) + f(x_{n+1}, y^p{}_{n+1})]$.

This is a commonly used method with truncation errors $E^p = (h^3/3)y'''$, $E^c = -(h^3/12)y'''$; $h$ is the increment in $x$, and $y'''$ denotes an appropriate mean value of $y'''(x)$. If $y'''(x) > 0$ near the point in question and $y_{n+1}$ is the theoretical value we are seeking, then the signs of $E^p$ and $E^c$ show that $y^p{}_{n+1} - y_{n+1} < 0$ while $y^c{}_{n+1} - y_{n+1} > 0$; also $y^c{}_{n+1} > y^p{}_{n+1}$.

Now suppose $\partial f/\partial y > 0$ near the point in question: then applying the "corrector" to the "corrected" point gives

$$y_{n+1}^{cc} = y_n + (h/2)[f(x_n, y_n) + f(x_{n+1}, y^c{}_{n+1})] > y^c{}_{n+1} > y_{n+1},$$

so that $y_{n+1}^{cc}$ is farther than $y^c{}_{n+1}$ from $y_{n+1}$. It is also easy to give a geometric illustration for this result.

Criteria could be given indicating when repetition of the "corrector" formula will worsen and when it will improve $y^c{}_{n+1}$. The situation is similar with other predictor-corrector formulas, and improvement is only a 50–50 proposition in that it hinges on the sign of a derivative. However, it seems much more practical to never repeat a "corrector," and if $|y^c{}_{n+1} - y^p{}_{n+1}|$ is too large to accept, then a smaller value of $h$ should be used.

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1. F. B. HILDEBRAND, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1956.
2. W. E. MILNE, *Numerical Solution of Differential Equations*, John Wiley and Sons, New York, 1953.

## The Order of an Iteration Formula

The order of an iteration formula indicates the rate of convergence of the iteration [1]. For example, Newton's formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$, for solving $f(x) = 0$, is of order 2 since the number of correct decimal places approxi-
$x_{n+1} - \alpha = (x_n - \alpha)^2 f''(\alpha)/2f'(\alpha) + \theta$, where $\theta$ consists of terms of degree three and higher in $(x_m - \alpha)$ and when $f(\alpha) = 0$. More generally, for the iteration

formula $x_{n+1} = \phi(x_0, \cdots, x_n)$ which converges to $\alpha$, let $p_n = -\log|x_n - \alpha|$ so that $p_n$ is the number of correct decimal places in $x_n$; then we define the order $p$ of the iteration formula to be $p = \lim p_{n+1}/p_n$ as $n \to \infty$.

If $\phi$ depends on $x_n$ only, then the order is an integer. In fact the order in this case is the index $p$ of the smallest derivative of $\phi$ such that $\phi^{(p)}(\alpha) \neq 0$, as is seen by the power series expansion of $\phi(x_n)$ in powers of $(x_n - \alpha)$.

We now show that a particular iteration formula has order $(1 + \sqrt{5})/2 = 1.6^+$, which is interesting because this number is not an integer and also because it is large enough to indicate that the formula is quite useful. The formula in question is a common one for solving $y = f(x) = 0$ by repeated linear interpolation: $x_{n+1}$ is the point where the straight line through $(x_{n-1}, y_{n-1})$ and $(x_n, y_n)$ crosses the $x$ axis. Algebraically this gives $x_{n+1} = (x_{n-1}y_n - x_ny_{n-1})/(y_n - y_{n-1})$. The fact that the formula does not involve $f'(x)$ will often be a considerable advantage [2].

To investigate the order of the linear interpolation formula we expand $y = f(x)$ in powers of $(x - \alpha)$ and with a little algebraic simplification obtain $x_{n+1} - \alpha = (x_{n-1} - \alpha)(x_n - \alpha)f''(\alpha)/2f'(\alpha) + \theta$, where $\theta$ consists of terms of degree three and higher. From this equation it follows that $p_{n+1} = p_n + p_{n-1} + q_n$, where $q_n$ is bounded, so on dividing by $p_n$ and letting $n \to \infty$ we get $p = 1 + 1/p$, whose only positive root is $p = (1 + \sqrt{5})/2$. This is the desired result.

It may also be of interest to note that the interpolation formula can be written as $x_{n+1} = (x_n - y_n) \div [(y_n - y_{n-1})/(x_n - x_{n-1})]$, which shows a close relation to Newton's formula. Although $(y_n - y_{n-1})/(x_n - x_{n-1})$ approaches $f'(\alpha)$, the "one sidedness" of this difference quotient keeps the order of the interpolation formula less than the order of Newton's formula.

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1. D. R. Hartree, "Notes on iterative processes," Cambridge Phil. Soc., *Proc.*, v. 45, 1949, p. 230–236; see also E. Schröder, "Über unendlich viele Algorithmen zür Auflösung der Gleichungen," *Math. Ann.*, v. 2, 1870, p. 317–363.

2. A. S. Householder, *Principles of Numerical Analysis*, McGraw-Hill Book Co., Inc., New York, 1953, Sec. 3.3, p. 118–132.

## REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

**48[C, F].**—F. J. Duarte, "Tablas Logarítmicas de Factoriales Primarias, desde 2 hasta 1007, con 33 Decimales," Acad. Ci. Fís., Mat. y Nat., *Bol.*, Caracas, v. 15, 1953.

This booklet gives, for each prime $p = 2, 3, \cdots, 10007$, the value to 33D of $\log_{10}(p)$ where $(p) = 2\cdot3\cdot5\cdots p$; the last digit is asterisked when it has been rounded up. There is a short table of $\log_e(p)/y$, for $y = 50, 100(100)1000(1000)10000$, where $p$ is the greatest prime less than $y$, exhibiting the approach of this ratio to unity, a result which is equivalent to the prime number theorem.

This table is, presumably, based on the 36D table [1] by the same author