Equation (8) becomes (omitting brackets)

$$(8') \quad M_2^{-1} = \begin{pmatrix} 0+0P_11P_10+0P_11P_22P_21P_10, & 0P_11+0P_11P_22P_21, & 0P_11P_22 \\ 1P_10+ \quad 1P_22P_21P_10, & 1+ \quad 1P_22P_21, & 1P_22 \\ 2P_21P_10, & 2P_21, & 2 \end{pmatrix}.$$

Recurrence formulas similar to (9) may be induced from this array.

4. Computation Procedure. We have inverted matrices of both described types using a Datatron computer without magnetic tape storage. It was found convenient to use interpretive matrix algebra commands of a three-address form. Storage locations were reserved both for square and for diagonal matrices. These locations were given pseudo-addresses. Typical single word commands would perform the following operations:

(i) Multiply the matrix in pseudo-address A by that in B and store the product in C
(ii) Invert the matrix in A and store the inverse in B
(iii) Punch onto cards the matrix (or its transpose) stored in A.

A standard punch card format is used.

The sub-matrices [0] to [n] are first computed and punched. These results are used by a second program to compute the inverse. Using the interpretive three-address commands, a large inversion may be programmed in a few minutes. Our programs limit us to 20×20 matrices (α_i and P_i) and to a 200×200 matrix M. It required about $2\frac{1}{2}$ hours to invert an 80×80 matrix by this method on our computer.

H. I. Meyer
B. J. Hollingsworth

United Gas Corporation
Shreveport, Louisiana

1. L. Fox, "Practical solution of linear equations and inversion of matrices," NBS Applied Mathematics Series No. 39, U. S. Gov. Printing Office, Washington, D. C., 1954.
2. R. A. Frazer, W. J. Duncan, & A. R. Collar, *Elementary Matrices* (Sect. 4.9), Cambridge University Press, New York, 1950.
3. P. Stein & E. L. Peck, "On the numerical solution of Poisson's equation over a rectangle," *Pac. J. of Math.*, v. 5, 1955, p. 999–1011.
4. H. M. Wagner, *Matrix Inversion on an Automatic Calculator by Row and Column Partitioning*, RAND Corp. Report P-417, 1953.

Note on the General Solution of the Confluent Hypergeometric Equation

The note [1] on a Webb & Airey-Adams-Bateman-Olsson error by Murlan S. Corrington, which is concerned with the confluent hypergeometric differential equation

$$(1) \qquad xy'' + (\gamma - x)y' - \alpha y = 0,$$

does not quite cover all the possibilities for its general solution. A statement of the complete situation does not seem to have appeared in print before now, though in the Fletcher, Miller, and Rosenhead *Index of Mathematical Tables* [2], the statement is deliberately cautious and correct. It seems worth while to give a full statement here.

If α and γ are constants, which may be complex, the complete solution of (1) is

$$y = AM(\alpha, \gamma, x) + Bx^{1-\gamma}M(\alpha - \gamma + 1, 2 - \gamma, x)$$

where A and B are arbitrary constants, and

$$M(\alpha, \gamma, x) = 1 + \frac{\alpha}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha + 1)}{\gamma(\gamma + 1)}\frac{x^2}{2!} + \cdots$$

so long as both solutions exist and differ.

The first solution ceases to have a meaning when γ is zero or a negative integer, *unless α is also a non-positive integer such that* $|\alpha| \leqslant |\gamma|$. In the latter case the numerator vanishes with or before the denominator, and both *solutions still hold.* In other cases, as γ approaches zero or a negative integer, $M(\alpha, \gamma, x)/\Gamma(\gamma)$ remains finite, but becomes a multiple of $x^{1-\gamma}M(\alpha - \gamma + 1, 2 - \gamma, x)$.

The case $\alpha = \gamma = -p$, p zero or a positive integer, as indicated in the *Index*, is of peculiar interest. The complete solution then becomes

$$\begin{aligned} y &= A\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^p}{p!}\right) + B\left(\frac{x^{p+1}}{(p+1)!} + \frac{x^{p+2}}{(p+2)!} + \cdots\right) \\ &= (A - B)\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^p}{p!}\right) + Be^x. \end{aligned}$$

Since the substitution $y = x^{1-\gamma}z$ interchanges the roles of the "first" and "second" solutions, leaving a differential equation that is still confluent hypergeometric, and since one of γ and $2 - \gamma$, here assumed real, must be positive, it is clear that there is no loss of generality if γ is taken positive; the logarithmic form of solution thus need not be considered for the *first* solution.

The second solution is not distinct from the first if $\gamma = 1$ and ceases to have a meaning if $\gamma \geqslant 1$ is an integer, *unless α is also an integer such that* $1 \leqslant \alpha < \gamma$. In the latter case, as before, the numerator vanishes with or before the denominator and *both solutions are valid.*

The cases emphasized by use of italics, often overlooked (as by Corrington) lead to polynomial M-functions.

When $\gamma \geqslant 1$ is an integer, whilst α is *not* a positive integer less than γ, the second solution of (1) is replaced by a logarithmic solution, given by equation (2) of Corrington's note; but this is *not* the complete solution, as stated, a further term $AM(\alpha, \gamma, x)$ being needed. This addition, with A replaced by a new arbitrary constant $D = A + C\{\psi(1 - \alpha) - \psi(\gamma) - \psi(1)\}$, and with the Beta-function terms written out, replaces Corrington's solution by the full solution

$$y = (C \ln x + D)M(\alpha, \gamma, x) + CN(\alpha, \gamma, x) + CS(\alpha, \gamma, x)$$

in which

$$\begin{aligned} N(\alpha, \gamma, x) = &\left(\frac{1}{\alpha} - \frac{1}{\gamma} - 1\right)\frac{\alpha}{\gamma}\frac{x}{1!} \\ &+ \left(\frac{1}{\alpha} + \frac{1}{\alpha + 1} - \frac{1}{\gamma} - \frac{1}{\gamma + 1} - 1 - \tfrac{1}{2}\right)\frac{\alpha(\alpha + 1)}{\gamma(\gamma + 1)}\frac{x^2}{2!} + \cdots \end{aligned}$$

and

$$S(\alpha, \gamma, x) = (-1)^{\gamma}\Gamma(\alpha - \gamma + 1)\Gamma(\gamma - 1)\frac{\Gamma(\gamma)}{\Gamma(\alpha)}x^{1-\gamma}$$

$$\times\left(1 + \frac{\alpha - \gamma + 1}{2 - \gamma}\frac{x}{1!} + \frac{(\alpha - \gamma + 1)(\alpha - \gamma + 2)}{(2 - \gamma)(3 - \gamma)}\frac{x^2}{2!} + \cdots \text{ to } \gamma - 1 \text{ terms}\right)$$

$$= \frac{\gamma - 1}{\alpha - 1}\frac{1}{x} - \frac{(\gamma - 1)(\gamma - 2)}{(\alpha - 1)(\alpha - 2)}\frac{1!}{x^2} + \frac{(\gamma - 1)(\gamma - 2)(\gamma - 3)}{(\alpha - 1)(\alpha - 2)(\alpha - 3)}\frac{2!}{x^3} - \cdots$$

$$+ (-1)^{\gamma}\frac{(\gamma - 1)(\gamma - 2)\cdots 2.1}{(\alpha - 1)(\alpha - 2)\cdots(\alpha - \gamma + 1)}\frac{(\gamma - 2)!}{x^{\gamma-1}}.$$

This function $S(\alpha, \gamma, x)$ is the part omitted by Webb and Airey, and others. Of course, if $\gamma = 1$, then $S = 0$ and the solution given by Webb and Airey is correct.

J. C. P. Miller

The University Mathematical Laboratory
Cambridge, England

1. *A Webb & Airey-Adams-Bateman-Olsson Error*, M.T.E. **117**, *MTAC*, v. 2, 1947, p. 352–353.
2. A. Fletcher, J. C. P. Miller, & L. Rosenhead, *An Index of Mathematical Tables*, Scientific Computing Service Ltd., London, 1946, article 22.56, p. 336.

Approximation and Table of the Weierstrass $\wp$ Function in the Equianharmonic Case for Real Argument

The most readily accessible table of the Weierstrass $\wp$ function for the equianharmonic case appears to be that in Jahnke-Emde [2], which is mainly to 4D.

The equianharmonic case for the function $\wp(u; g_2, g_3)$ is that in which $g_2 = 0$, $g_3 = 1$. The function behaves like $1/u^2$ near the origin. This leads us to consider tabulating

$$f(u; 0, 1) \equiv \wp(u; 0, 1) - 1/u^2. \tag{1}$$

Because of symmetries, we may restrict attention to the interval $0 < u \leq \omega_2$, where $\omega_2 \approx 1.52995$ and is the real half-period of $\wp$; see F. Tricomi [1].

It is possible to represent $\wp$ on the above interval to 7S by a relatively simple approximant. Write

$$f(u; 0, 1) \approx c_3u^4 + c_6u^{10} + c_9u^{16} + c_{12}u^{22} + c_{15}u^{28}, \tag{2}$$

the polynomial being obtained by truncation of Maclaurin's series. Thus we have

$$\wp(u; 0, 1) \approx g(u)/u^2 = h(y)/u^2, \tag{3}$$