relaxation treatment can be programmed for machine, that part which requires judgment would be difficult to code.

The disadvantage of successive approximation lies in increasing the length of the calculation. However, as the numerical indeterminacy grows more stringent, the initial approximation improves and the number of iterations required decreases. Moreover, this is a comparatively less serious disadvantage for a method suitable for machine use. Thus the number of times the calculation is to be repeated for a different set of input data, the character of the equations with respect to cross-linking, and the comparative labor of constructing the machine programs must all be considered when deciding which method will prove most convenient for any particular study.

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1. EDWIN S. CAMPBELL, CM–847, University of Wisconsin Naval Research Laboratory, July 11, 1955.

A Smoothest Curve Approximation

A practical problem which often comes up in numerical work is the fitting of a curve to a finite set of known values in order to perform various operations such as integration. The usual method of approximation consists of fitting the points with one or more polynomials (independent of each other). By letting there be more points for each polynomial, with the polynomials being of comparable order, the error of approximation becomes asymptotic to a higher power of the interval length.

However, error analyses for such methods usually depend upon the boundedness of some derivative of a correspondingly high order [1]. But even if the function to be approximated is analytic, its correspondingly high order derivative may be of sufficient magnitude that for the given interval size, a simpler method would give better results. For instance, a fitting with an eighth order polynomial gives the following rule:

$$\int_0^8 f(t)dt = \frac{8}{28350}[989f(0) + 5888f(1) - 928f(2) + 10496f(3) - 4540f(4) + 10496f(5) - 928f(6) + 5888f(7) + 989f(8)].$$

An application of this formula to a positive function which was everywhere small over the range, apart from a sharp peak in the center, would lead to a negative result. Try to convince a prospector that there is a negative amount of mineral on his land because he finds a rich strike in the middle of it!

Frequently one contents oneself with a simpler rule which is repeated in blocks of so many intervals per block. However, this usually introduces discontinuities in the first derivative at the junctions of the blocks. If one were to integrate an interval broken up into 1000 equal subintervals, why should the 500th point be given twice as big a weighting as the 501st point (the middle point), as would be the case using Simpson's Rule? While this may not be a great objection for functions which do not undergo great changes of nature over several intervals, it may be quite unfortunate for those that do.

The main problem to be considered in this paper is the problem of obtaining and examining an integration rule based on integrating a function passing through a given set of points such that the function will have a small amount of twisting, and such that whatever twisting is necessary will be spread out. To be more specific, we will minimize

$$\int \left[\frac{d^2}{dt^2} f(t)\right]^2 dt,$$

where f ranges over an appropriate set of fitting functions, say those which are continuous and have continuous first and second derivatives.

It would be improper to consider an "angle-like" function, such as $f_0(t) = t$ $(0 \leq t \leq 1)$ and $f_0(t) = 1$ $(1 \leq t \leq 2)$, and argue that since $f_0''(t) = 0$ almost everywhere,

$$\int_0^2 [f_0''(t)]^2 dt = 0.$$

In fact if $\{f_i\}$ $(i = 1, 2, 3, \cdots)$ is a sequence of appropriate functions which converge to f_0, then

$$\lim_{i \to \infty} \int_0^2 [f_i''(t)]^2 dt = \infty.$$

Another application of this problem is that of bending a stick a little bit. A bent stick assumes a position which, subject to the bending constraints, will minimize its potential energy. Assuming the stick to be originally straight and uniform, Hooke's Law implies that its potential energy will be proportional to the integral along its arc length of the square of its curvature. If the problem is two dimensional, and if the bending is sufficiently slight that the arc length may be considered as being practically proportional to some coordinate axis, then we get the problem of minimizing

$$\int \left[\frac{d^2}{dt^2} f(t)\right]^2 dt.$$

In considering this stick application, one may also obtain the function given in Theorem 1 by mechanically balancing torques against curvatures.

1. THEOREM. *Let $t_0 < t_i < \cdots < t_n$. For each $i = 0, \cdots, n-1$, let f be a cubic on the interval $[t_i, t_{i+1}]$. Also let*

$$\frac{d^2}{dt^2} f(t_0) \equiv f''(t_0) = 0 = f''(t_n).$$

For $i = 1, \cdots, n-1$, let $f(t_i-) = f(t_i+)$, $f'(t_i-) = f'(t_i+)$ and $f''(t_i-) = f''(t_i+)$. Let $g(\lambda)$ be any admissible function on $[t_0, t_n]$ such that $g(t_i) = f(t_i)$ for

$i = 0, \cdots, n$. *Then*

$$\int_{t_0}^{t_n} [g''(t)]^2\, dt > \int_{t_0}^{t_n} [f''(t)]^2\, dt$$

if and only if $g \neq f$.

Proof: Define $\eta(t)$ as $g(t) - f(t)$. Then since f''' is constant in each interval

$$\int_{t_0}^{t_n} [g''(t)]^2 dt - \int_{t_0}^{t_n} [f''(t)]^2 dt = \int_{t_0}^{t_n} [\eta''(t)]^2 dt + 2 \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \eta''(t) f''(t)\, dt$$

$$= \int_{t_0}^{t_n} [\eta''(t)]^2 dt + 2 \sum_{i=0}^{n-1} [\eta'(t_{i+1}) f''(t_{i+1}) - \eta'(t_i) f''(t_i)]$$

$$- 2 \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f'''(t) \eta'(t)\, dt = \int_{t_0}^{t_n} [\eta''(t)]^2 dt + 2[\eta'(t_n) f''(t_n)$$

$$- \eta'(t_0) f''(t_0)] - 2 \sum_{i=0}^{n-1} f'''\ (i\text{-th interval})\ [\eta(t_{i+1}) - \eta(t_i)]$$

$$= \int_{t_0}^{t_n} [\eta''(t)]^2 dt + 2\,[0 - 0] - 2 \sum_{i=0}^{n-1} 0$$

which is positive for admissible non-zero η's.

Note that were t_0 and t_n interior to the end points of the interval on which f is defined, then for minimizing $\int [f''(t)]^2 dt$, f would be the same as before on $[t_0, t_n]$. Also, f would be linear at the left of t_0 and the right of t_n, and the first (as well as the second) derivatives would match at t_0 and t_n.

The function f of Theorem 1 consists of n cubics, and each cubic contains four coefficients. The restriction that f passes through $f(t_i)$ at t_i gives $2n$ determining equations. The conditions $f'(t_i-) = f'(t_i+)$ and

$$f''(t_i-) = f''(t_i+) \qquad (i = 1, \cdots, n-1)$$

give $n - 1$ equations each. The equations $f''(t_0) = 0$ and $f''(t_n) = 0$ give still two more, giving a total of $4n$. Therefore we have $4n$ linear equations to determine the $4n$ coefficients of f.

2. *Definition.* In Theorem 1, let $t_i = i$ $(0 \leq i \leq n)$. Then for m an integer $(0 \leq m \leq n)$, define $w(m, n)$ as $\int_0^n f(t)\, dt$ where $f(i) = \delta_{m,i}$ $(i = 0, \cdots, n)$.

The following is a table of w's for $n \leq 10$. In this table $w(m, n)$ is the $m + 1$-st entry of line n. Also the common denominator is factored out at the left.

$\frac{1}{2}$ [1; 1]

$\frac{1}{8}$ [3; 10; 3]

$\frac{1}{10}$ [4; 11; 11; 4]

$\frac{1}{28}$ [11; 32; 26; 32; 11]

$\frac{1}{38}$ [15; 43; 37; 37; 43; 15]

$\frac{1}{104}$ [41; 118; 100; 106; 100; 118; 41]

$\frac{1}{142}$ [56; 161; 137; 143; 143; 137; 161; 56]

$\frac{1}{388}$ [153; 440; 374; 392; 386; 392; 374; 440; 153]

$\frac{1}{530}$ [209; 601; 511; 535; 529; 529; 535; 511; 601; 209]

$\frac{1}{1448}$ [571; 1642; 1396; 1462; 1444; 1450; 1444; 1462; 1396; 1642; 571].

Since f (and therefore $\int f dt$) is linearly dependent on the $f(t_i)$, we get the integration rule $\int_0^n f(t)dt = \sum_{m=0}^{n} f(m)w(m, n)$. For instance, from the third line of the table,

$$\int_0^3 f(t)dt = \frac{1}{10}[4f(0) + 11f(1) + 11f(2) + 4f(3)].$$

Notice that as max $[m; n - m]$ gets large, $|w(m, n) - 1|$ gets small.

The following rules which follow from Theorems 6 and 7 are probably the most useful for writing down the w's. Define $D(n)$ as the common denominator for the $w(m, n)$'s. Since

$$D(2n) = 2\left[\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{2n} + \left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^{2n}\right]$$

and

$$D(2n+1) = \sqrt{2}\left[\left(\frac{1+\sqrt{3}}{\sqrt{2}}\right)^{2n+1} + \left(\frac{1-\sqrt{3}}{\sqrt{2}}\right)^{2n+1}\right],$$

it follows that $D(2n + 2) = D(2n) + 2D(2n + 1)$ and

$$D(2n + 3) = D(2n + 1) + D(2n + 2).$$

A similar rule also holds for the numerators. For

$$m < 2n,\ D(2n + 2)w(m, 2n + 2) = D(2n)w(m, 2n) + 2D(2n + 1)w(m, 2n + 1),$$

and for $m < 2n + 1$, $D(2n + 3)w(m, 2n + 3) = D(2n + 1)w(m, 2n + 1) + D(2n + 2)w(m, 2n + 2)$. For

$$0 < m < n,\ w(m, n) = 1 + (-1)^{m+1}\frac{D(|2m - n|)}{2D(n)}$$

where $D(0) = 4$. Also note the obvious fact that $w(m, n) = w(n - m, n)$.

3. *Definition.* For n a positive integer, consider a function f on $[0, n]$ such that f is a cubic on $[i, i + 1]$ $(i = 0, \cdots, n - 1)$; $f(i) = 0$ $(i = 0, \cdots, n)$;

$$f'(0) \neq 0; f''(n) = 0;$$

and for $i = 1, \cdots, n - 1$, $f'(i-) = f'(i+)$ and $f''(i-) = f''(i+)$. Then define $R(n)$ as $f''(0)/f'(0)$. Define $I(n)$ as $\int_0^n f(t)dt/f'(0)$. Also define $R(0) = I(0) = 0$.

Define x as the number $2 + \sqrt{3}$. Define y as $\dfrac{1+\sqrt{3}}{\sqrt{2}} = \sqrt{x}$.

4. Lemma. *For n a non-negative integer,* $R(n) = -2\sqrt{3}\dfrac{x^n - x^{-n}}{x^n + x^{-n}}$.

Proof: $R(0) = 0 = -2\sqrt{3}\dfrac{x^0 - x^{-0}}{x^0 + x^{-0}}$. To use mathematical induction, let us show that the validity of this formula for $R(n)$ implies its validity for $R(n + 1)$.

Let f be the function of Definition 3 defined on $[0, n+1]$. $f(0) = f(1) = 0$ and $f'(0) \neq 0$ imply that there exist real numbers a and $\lambda \neq 0$ such that $f(t) = \lambda[t - at^2 + (a-1)t^3]$ for $0 \leq t \leq 1$. Then

$$R(n) = \frac{f''(1)}{f'(1)} = \frac{4a-6}{a-2}$$

implies that $a = \dfrac{6 - 2R(n)}{4 - R(n)}$. Therefore

$$R(n+1) = \frac{f''(0)}{f'(0)} = -2a = -4\left[\frac{3 - R(n)}{4 - R(n)}\right]$$

which, substituting for $R(n)$, equals

$$-2\sqrt{3}\,\frac{x^{n+1} - x^{-(n+1)}}{x^{n+1} + x^{-(n+1)}}.$$

5. Lemma. *For n a non-negative integer,* $I(n) = \dfrac{1}{12} - \dfrac{(-1)^n}{6(x^n + x^{-n})}$;

Proof. $I(0) = 0 = \dfrac{1}{12} - \dfrac{(-1)^0}{6(x^0 + x^{-0})}$. To use mathematical induction, let us show that the validity of this formula for $I(n)$ implies its validity for $I(n+1)$. Let f, a, and λ be the same as in the proof of Lemma 4. Then

$$\begin{aligned} I(n+1) &= \frac{1}{f'(0)}\int_0^{n+1} f(t)dt = \frac{1}{\lambda}\left[\int_0^1 f(t)dt + f'(1)I(n)\right] \\ &= \frac{1}{4} - \frac{a}{12} + (a-2)I(n) = \frac{1}{4} - \frac{R(n+1)}{2}\left[I(n) - \frac{1}{12}\right] - 2I(n) \\ &= \frac{1}{12} - \frac{(-1)^{n+1}}{6[x^{n+1} + x^{-(n+1)}]} \end{aligned}$$

on substituting for $R(n+1)$ and $I(n)$.

6. Theorem. *For n a positive integer*

$$w(0, n) = \frac{y^{n+1} - (-y)^{-(n+1)}}{2\sqrt{6}[y^n + (-y)^{-n}]}.$$

Proof. Let f be the f of Definition 2. Then $f(0) = 1, f(1) = 0$ and $f''(0) = 0$ imply that there exists a real number a such that

$f(t) = 1 - at + (a-1)t^3$ for $0 \leq t \leq 1$.

$R(n-1) = \dfrac{f''(1)}{f'(1)} = \dfrac{6(a-1)}{2a-3}$. Therefore $a = \dfrac{3R(n-1) - 6}{2R(n-1) - 6}$. Using

$$R(n) = -4\,\frac{3 - R(n-1)}{4 - R(n-1)}$$

from Lemma 4, we see that $R(n-1) = 4\dfrac{3+R(n)}{4+R(n)}$ and so $a = 3 + \dfrac{6}{R(n)}$.

$$\begin{aligned} w(0,n) &= \int_0^n f(t)dt = \int_0^1 f(t)dt + y'(1)I(n-1) \\ &= \frac{3}{4} - \frac{a}{4} + [2a-3]\left[\frac{1}{12} - \frac{(-1)^{n-1}}{6(x^{n-1}+x^{1-n})}\right] \\ &= \frac{1}{2} - \frac{a}{12} + \frac{2a-3}{6}\,\frac{(-1)^n}{x^{n-1}+x^{1-n}} \\ &= \frac{1}{4} - \frac{1}{2R(n)} + \left[\frac{1}{2} + \frac{2}{R(n)}\right]\frac{(-1)^n}{x^{n-1}+x^{1-n}}. \end{aligned}$$

On substituting for $R(n)$ from lemma 4 and after some manipulation, we obtain

$$w(0,n) = \frac{y^{n+1} - (-y)^{-(n+1)}}{2\sqrt{6}[y^n + (-y)^{-n}]}.$$

7. THEOREM. *For*

$$0 < m < n,\ w(m,n) = 1 - (-1)^m \frac{y^{n-2m} + (-y)^{2m-n}}{2[y^n + (-y)^{-n}]}.$$

Proof. Let f be the function of Theorem 1 where $t_i = i - m$ $(i = 0, \cdots, n)$ and $f(t_i) = \delta_{m,i}$. Then $f(0) = 1, f(-1) = f(+1) = 0, f'(0-) = f'(0+)$ and $f''(0-) = f''(0+)$ imply there exist real numbers a and b so that $f(t) = 1 + at + bt^2 - (1+a+b)t^3$ for $0 \le t \le 1$ and $f(t) = 1 + at + bt^2 + (1-a+b)t^3$ for $-1 \le t \le 0$. Define $\alpha = n - m - 1$ and $\beta = m - 1$. Then

$$R(\alpha) = \frac{f''(1)}{f'(1)} = \frac{6+6a+4b}{3+2a+b}$$

and

$$\mathrm{R}(\beta) = -\frac{f''(-1)}{f'(-1)} = \frac{6-6a+4b}{3-2a+b}.$$

Solving these last two equations, we obtain

$$a = \frac{3R(\alpha) - 3R(\beta)}{2R(\alpha)R(\beta) - 7R(\alpha) - 7R(\beta) + 24}$$

and

$$b + 3 = \frac{-6R(\alpha) - 6R(\beta) + 36}{2R(\alpha)R(\beta) - 7R(\alpha) - 7R(\beta) + 24}.$$

Now,

$$\begin{aligned} w(m,n) - 1 &= \int_{-m}^{n-m} f(t)dt - 1 \\ &= I(\alpha)f'(1) - I(\beta)f'(-1) + \int_{-1}^{1} f(t)dt - 1 \\ &= I(\alpha)[-2a - (b+3)] + I(\beta)[2a - (b+3)] + \frac{1}{6}(b+3). \end{aligned}$$

Substituting the values for a and $b+3$ into the last equation and multiplying both sides by the common denominator of a and $b+3$, we get

$$\begin{aligned} &[2R(\alpha)R(\beta) - 7R(\alpha) - 7R(\beta) + 24][w(m,n) - 1] \\ &\qquad = 12I(\alpha)[R(\beta) - 3] + 12I(\beta)[R(\alpha) - 3] - R(\alpha) - R(\beta) + 6. \end{aligned}$$

Now substitute for $R(\alpha)$, $R(\beta)$, $I(\alpha)$ and $I(\beta)$ the expressions of lemmas 4 and 5 and multiply both sides of the equation by $(x^\alpha + x^{-\alpha})\ (x^\beta + x^{-\beta})$.

Then collect like terms to get

$$[(48 + 28\sqrt{3})x^{\alpha+\beta} + (48 - 28\sqrt{3})x^{-(\alpha+\beta)}]\ [w(m, n) - 1]$$
$$= (6 + 4\sqrt{3})\ (-1)^\beta x^\alpha + (6 + 4\sqrt{3})\ (-1)^\alpha x^\beta$$
$$+ (6 - 4\sqrt{3})\ (-1)^\beta x^\alpha + (6 - 4\sqrt{3})\ (-1)^\beta x^{-\alpha}.$$

Divide both sides of this equation by $2\sqrt{3}$ and substitute $x = 2 + \sqrt{3}$ to get

$$2[x^{\alpha+\beta+2} - x^{-(\alpha+\beta+2)}]\ [w(m, n) - 1]$$
$$= (-1)^\beta x^{\alpha+1} + (-1)^\alpha x^{\beta+1} - (-1)^\beta x^{-(\alpha+1)} - (-1)^\alpha x^{-(\beta+1)}.$$

Since $x = y^2$, we may divide both sides of this equation by

$$[y^{\alpha+\beta+2} - (-y)^{-(\alpha+\beta+2)}]\ [y^{\alpha+\beta+2} + (-y)^{-(\alpha+\beta+2)}]$$

which using the binomial expansion, equals

$$\sum_{j=2}^{k+1} \frac{2_j k! B_j}{j!(k - j + 1)!n^j} + \frac{y}{n\sqrt{6}} \frac{1 - (-x)^{-(n+1)}}{1 + (-x)^{-n}}$$
$$- \frac{1}{n} \sum_{i=0}^{k} \sum_{m=0}^{n} \frac{n^{-i}(-2m)^i k!}{i!(k - i)!} \frac{(-x)^{-m}}{1 + (-x)^{-n}}$$
$$= \sum_{i=1}^{k} \frac{2^{i+1}k!}{(i + 1)!(k - i)!n^{i+1}} \left[B_{i+1} - \frac{i + 1}{2[1 + (-x)^{-n}]} \sum_{m=0}^{n} (-m)^i(-x)^{-m}\right]$$
$$= \frac{1}{2} \sum_{i\geq 1} \frac{h^{i+1}}{(i + 1)!} \left[\frac{d^i}{dt^i} P(1) + (-1)^i \frac{d^i}{dt^i} P(-1)\right]$$
$$\left[B_{i+1} - \frac{i + 1}{2[1 + (-x)^{-n}]} \sum_{m=0}^{n} (-m)^i(-x)^{-m}\right].$$

8. *Definition.* For $i - 1$ a positive integer, define B_i as i factorial times the coefficient of t^i in the power series expansion of $t/(e^t - 1)$. In other words, let $0 = B_3 = B_5 = B_7 = \cdots$, and let $B_2, B_4, B_6, B_8, \cdots$ be the Bernoulli's Numbers

$$\frac{1}{6}, -\frac{1}{30}, \frac{1}{42}, -\frac{1}{30}, \frac{5}{66}, -\frac{691}{2730}, \frac{7}{6}, \cdots.$$

9. THEOREM. *Let $P(t)$ be a finite polynomial on the interval (a, b). Then*

$$\int_a^b P(t)dt = \sum_{m=0}^{n} P(a + mh)w(m, n)\ h$$
$$- \sum_{i\geq 1} \frac{h^{i+1}}{(i + 1)!} \left[\frac{d^i}{dt^i} P(b) + (-1)^i \frac{d^i}{dt^i} P(a)\right]$$
$$\left[B_{i+1} - \frac{i + 1}{2[1 + (-x)^{-n}]} \sum_{m=0}^{n} (-m)^i(-x)^{-m}\right]$$

where h is the sub-interval length $= \dfrac{b - a}{n}$.

Proof: If this theorem holds for a given $a < b$, then for λ a positive real number, this theorem also holds for end points λa and λb, since new $h = \lambda$ old h and

$\frac{d^i}{dt^i} P(t)$ at $t = a$ (or b) $= \lambda^{-i} \frac{d^i}{dt^i} P\left(\frac{t}{\lambda}\right)$ at $t = \lambda a$ (or λb). Also, the three terms of

this equation are invariant under translations. Therefore, it is sufficient to prove this theorem for $-a = b = 1$. Also, since this equation is linear in P, it is sufficient to verify it for $P(t) = t^k$ where k is a non-negative integer.

If k is odd, then $P(t) = -P(-t)$ implies that all three terms of the equation of this theorem are zero. Therefore, we may let k be even.

$$\frac{1}{2}\left[\sum_{m=0}^{n} P(-1 + mh)\, w(m, n)\, h - \int_{-1}^{1} P(t)dt\right]$$

$$= \frac{1}{n}\left[\sum_{m=0}^{n}\left(\frac{2m - n}{n}\right)^k w(m, n)\right] - \frac{1}{k + 1}$$

$$= \frac{1}{n}\left\{\sum_{m=1}^{n-1}\left(\frac{n - 2m}{n}\right)^k\left[1 - (-1)^m \frac{y^{n-2m} + (-y)^{2m-n}}{2[y^n + (-y)^{-n}]}\right] + 2w(0, n)\right\} - \frac{1}{k + 1}$$

$$= \frac{1}{n}\left\{\sum_{m=0}^{n}\left(\frac{n - 2m}{n}\right)^k\left[1 - (-1)^m \frac{x^{-m}}{1 + (-x)^{-n}}\right] - 1 + 2w(0, n)\right\} - \frac{1}{k + 1}.$$

If n is even, $\frac{1}{n}\sum_{m=0}^{n}\left(\frac{n - 2m}{n}\right)^k = \frac{2}{n}\sum_{i=0}^{n/2}\left(\frac{2i}{n}\right)^k$.

If n is odd, $\frac{1}{n}\sum_{m=0}^{n}\left(\frac{n - 2m}{n}\right)^k = \frac{2}{n}\sum_{i=0}^{(n-1)/2}\left(\frac{2i + 1}{n}\right)^k$.

Using the Euler-Maclaurin sum formula [2], either of these last two expressions equals

$$\frac{1}{k + 1} + \frac{1}{n} + \sum_{j=2}^{k+1} \frac{2^j k! B_j}{j!(k - j + 1)!\, n^j}.$$

Substituting, we get

$$\frac{1}{2}\left[\sum Pwh - \int Pdt\right] = \sum_{j=2}^{k+1} \frac{2^j k! B_j}{j!(k - j + 1)!\, n^j} + \frac{2w(0, n)}{n} - \frac{1}{n}\sum_{m=0}^{n}\left(\frac{n - 2m}{n}\right)^k \frac{(-x)^{-m}}{1 + (-x)^{-n}}$$

and

$$D(2n + 1) = \sqrt{2}[y^{2n+1} + (-y)^{-(2n+1)}],$$

we see that

$$\phi_1\left(\frac{1}{x}\right) = -\frac{1}{6}.$$

$$\phi_2\left(\frac{1}{x}\right) = \frac{\sqrt{3}}{36}[-D(1)] = -\frac{\sqrt{3}}{18}.$$

$$\phi_3\left(\frac{1}{x}\right) = \frac{1}{36}\left[4 - \frac{1}{2}D(2)\right] = 0.$$

$$\phi_4\left(\frac{1}{x}\right) = \frac{\sqrt{3}}{216}[11D(1) - D(3)] = \frac{\sqrt{3}}{18}.$$

$$\phi_5\left(\frac{1}{x}\right) = \frac{1}{216}\left\{-66 + \frac{1}{2}[26D(2) - D(4)]\right\} = \frac{1}{9}.$$

$$\phi_6\left(\frac{1}{x}\right) = \frac{\sqrt{3}}{1296}[-302D(1) + 57D(3) - D(5)] = -\frac{\sqrt{3}}{18}.$$

$$\phi_7\left(\frac{1}{x}\right) = \frac{1}{1296}\left\{2416 + \frac{1}{2}[-1191D(2) + 120D(4) - D(6)]\right\} = -\frac{5}{9}.$$

$$\phi_8\left(\frac{1}{x}\right) = \frac{\sqrt{3}}{7776}[15{,}619D(1) - 4293D(3) + 247D(5) - D(7)] = -\frac{17\sqrt{3}}{54}.$$

$$\phi_9\left(\frac{1}{x}\right) = \frac{1}{7776}\left\{-156{,}190 + \frac{1}{2}[88{,}234D(2) - 14{,}608D(4) + 502D(6) - D(8)]\right\} = \frac{7}{3}.$$

Therefore, the equation

$$\int_a^b P(t)dt = T(n) + \sum_{m=0}^{n} P(a+mh)w(m,n)h - \sum_{i\geq 1} \frac{h^{i+1}}{(i+1)!} \left[\frac{d^i}{dt^i}P(b) + (-1)^i \frac{d^i}{dt^i}P(a)\right]\left[B_{i+1} - (-1)^i \frac{i+1}{2}\phi_i\left(\frac{1}{x}\right)\right].$$

gives us the series of this theorem.

10. Corollary. $\sum_{m=0}^{n} w(m,n) = n.$

11. Theorem. *Let $P(t)$ be a finite polynomial on the interval (a, b). Then*

$$\begin{aligned}\int_a^b P(t)dt &= T(n) + \sum_{m=0}^{n} P(a+mh)w(m,n)h \\ &- \frac{\sqrt{3}}{72}h^3[P^{ii}(b) + P^{ii}(a)] + \frac{1}{720}h^4[P^{iii}(b) - P^{iii}(a)] \\ &+ \frac{\sqrt{3}}{864}h^5[P^{iv}(b) + P^{iv}(a)] - \frac{1}{2016}h^6[P^{v}(b) - P^{v}(a)] \\ &- \frac{\sqrt{3}}{25{,}920}h^7[P^{vi}(b) + P^{vi}(a)] + \frac{29}{518{,}400}h^8[P^{vii}(b) - P^{vii}(a)] \\ &- \frac{17\sqrt{3}}{4{,}354{,}560}h^9[P^{viii}(b) + P^{viii}(a)] - \frac{31}{9{,}580{,}032}h^{10}[P^{ix}(b) - P^{ix}(a)] \\ &+ \cdots, \text{ where } h = \frac{b-a}{n} \text{ and } \lim_{n\to\infty}(2+\sqrt{3})^n T(n) = 0.\end{aligned}$$

Proof: For each i, $\sum_{m=0}^{\infty} (-m)^i (-x)^{-m}$ is an alternating series. Therefore, for sufficiently large n,

$$\left| \sum_{m=0}^{\infty} (-m)^i (-x)^{-m} - \sum_{m=0}^{n} (-m)^i (-x)^{-m} \right| < (n+1)^i x^{-(n+1)}.$$

Since h^{i+1} is proportional to $1/n^{i+1}$,

$$\lim_{n\to\infty} x^n \left\{ \int_a^b P(t)dt - \sum_{m=0}^{n} P(a+mh)w(m,n)h \right.$$
$$+ \sum_{i\geq 1} \frac{h^{i+1}}{(i+1)!} \left[\frac{d^i}{dt^i} P(b) + (-1)^i \frac{d^i}{dt^i} P(a) \right]$$
$$\left. \left[B_{i+1} - \frac{i+1}{2} \sum_{m=0}^{\infty} (-m)^i (-x)^{-m} \right] \right\} = 0.$$

If we define $\phi_i(t$ as) $\sum_{m=0}^{\infty} m^i (-t)^m$, then $\phi_{i+1}(t) = \frac{d}{dt} \phi_i(t)$.

$$\text{Since } \phi_0(t) = \frac{1}{1+t}, \ \phi_1(t) = \frac{-t}{(1+t)^2}; \ \phi_2(t) = \frac{-t+t^2}{(1+t)^3};$$

$$\phi_3(t) = \frac{-t+4t^2-t^3}{(1+t)^4}; \ \phi_4(t) = \frac{-t+11t^2-11t^3+t^4}{(1+t)^5};$$

$$\phi_5(t) = \frac{-t+26t^2-66t^3+26t^4-t^5}{(1+t)^6};$$

$$\phi_6(t) = \frac{-t+57t^2-302t^3+302t^4-57t^5+t^6}{(1+t)^7}$$

etc., and keeping in mind that

$$1 + \frac{1}{x} = \frac{\sqrt{6}}{y} = \sqrt{\frac{6}{x}}, \ D(2n) = 2\left[y^{2n} + (-y)^{-2n}\right]$$

to get

$$w(m,n) - 1 = -\frac{y^{\alpha+1}(-y)^{-(\beta+1)} + y^{\beta+1}(-y)^{-(\alpha+1)}}{2[y^{\alpha+\beta+2} + (-y)^{-(\alpha+\beta+2)}]}$$
$$= (-1)^m \frac{y^{n-2m} + (-y)^{2m-n}}{2[y^n + (-y)^{-n}]}.$$

12. *Examples*. Let $g(t)$ be a function on $[-1, 1]$ and let us consider the error $\sum_{m=0}^{n} g\left(-1 + \frac{2m}{n}\right) \frac{2\,w(m,n)}{n} - \int_{-1}^{1} g(t)\,dt$. If g is constant or $g(t) = -g(-t)$, then the error is zero, as is true of most rules.

If we let $n = 10$ and consider $g(t)$ as t^2, t^4 and t^6 respectively, then the error is 7.7×10^{-4}, 4.5×10^{-3}, and 1.1×10^{-2} respectively. If we now apply the end point terms of Theorem 11, the remaining errors are -2.9×10^{-9}, -1.8×10^{-8} and -4.2×10^{-8} respectively. If we let $n = 14$, then the above quoted errors become 2.8×10^{-4}, 1.6×10^{-3}, and 4.0×10^{-3} and also -5.5×10^{-12}, -3.2×10^{-11} and -8.1×10^{-11}. The corresponding errors using Simpson's Rule are 0, 4.3×10^{-4} and 2.1×10^{-3} for 10 intervals and 0, 1.1×10^{-4} and 5.5×10^{-4} for 14 intervals.

If we let $g(t) = \cos(\pi t/4)$ and $n = 12$, then the error is -1.7×10^{-5}. If we apply the $\frac{1}{720} h^4 [g'''(1) - g'''(-1)]$ correction term, the remaining error is -7.1×10^{-7}. The corresponding error using Simpson's Rule is 6.7×10^{-5}.

13. *Semi-Infinite Interval.* Let us define $w(m, \infty)$ as $\lim_{n\to\infty} w(m, n)$. Then

$$w(0, \infty) = \frac{1}{4} + \frac{\sqrt{3}}{12}$$

and

$$w(m, \infty) = 1 - \frac{1}{2}(-x)^{-m} \text{ for } m > 0.$$

Given a function $g(t)$ for which the integral $\int_0^\infty g(t)\,dt$ is convergent, one may approximate this integral by the expression $h \sum_{m=0}^{N} w(m, \infty) g(mh)$ where N is sufficiently large that $h \sum_{m=N+1}^{\infty} w(m, \infty) g(mh)$ is insignificant. If one also knows the first few derivatives of $g(t)$ at $t = 0$, one may also add the terms obtained from Theorem 11.

If $g(t) = e^{-\lambda t}$, then the series

$$h \sum_{m=0}^{\infty} w(m, \infty) g(m,h) - \frac{\sqrt{3}}{72} h^3 g''(0) - \frac{1}{720} h^4 g'''(0) + \frac{\sqrt{3}}{864} h^5 g''''(0) + \frac{1}{2016} h^6 g'''''(0) + \cdots$$

will converge to $\int_0^\infty g(t)\,dt$ whenever $h\lambda < 2\pi$.

For some functions such as $g(t) = e^{-t^2}$, this sequence of derivatives is divergent for any $h > 0$. However, this fact does not imply that using this formula with the first few derivative terms would not do a good job of approximation.

Notice that the weight terms $w(m, \infty)$ are easy to compute on a machine, requiring only one multiplication per term.

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