b) Since $-(1 - s)z < 0$, $sz > 0$ it follows from (5.1) that

$$\Phi_s(z) \leq \int_0^{sz} \frac{\tau}{1+\tau} d\tau \leq \int_0^{sz} \tau d\tau \leq \tfrac{1}{2}(sz)^2.$$

c) This result is an immediate consequence of the identity

$$\Phi_{\frac{1}{2}}(z) = -2\int_0^{z/s} \frac{\tau^2}{1-\tau^2} d\tau.$$

Mathematics Research Center,
United States Army,
University of Wisconsin,
Madison, Wisconsin

1. W. E. Milne, *Numerical Solutions of Differential Equations*, John Wiley and Sons, Inc., New York, 1953.
2. M. L. Juncosa & D. M. Young, "On the order of convergence of solutions of a difference equation to a solution of the diffusion equation," Soc. Indust. Appl. Math., *Jn.* v. 1, 1953, p. 111–135.
3. M. L. Juncosa & David Young, "On the convergence of a solution of a difference equation to a solution of the equation of diffusion," Amer. Math. Soc., *Proc.*, v. 5, 1954, p. 168–174.
4. M. L. Juncosa & D. M. Young, "On the Crank-Nicolson procedure for parabolic partial differential equations" Cambridge Phil. Soc., *Proc.*, v. 53, 1957, p.448–461.
5. P. D. Lax & R. D. Richtmyer, "Survey of the stability of linear finite difference equations," *Comm. Pure Appl. Math.*, v. 9, 1956, p. 267–293.

An Open Formula for the Numerical Integration of First Order Differential Equations (II)

By Herbert S. Wilf

In a previous paper [1], referred to below as *I*, a set of formulas was derived for the numerical integration of systems of first order differential equations. In what follows we will consider the questions of convergence and stability of the method and higher order formulas.

We quote here, for reference, the final results of *I*, namely the propagation formulas:

$$y_1 = y_0 + \frac{h}{12}[5f(x_0, y_0) + 8f(x_1, y_1) - f(x_2, y_2^*)] \tag{1}$$

$$y_2^* = 5y_0 - 4y_1 + 2h[f(x_0, y_0) + 2f(x_1, y_1)] \tag{2}$$

for the solution of

$$y' = f(x, y) \tag{3}$$

where y_0 is given.

1. Convergence. The solution of (1) and (2) for y_1 proceeds by taking as an initial guess the y_2^* from the preceding point. One then applies (2) and (1) successively until consecutive values of y_1 agree to sufficient accuracy.

Received 16 August 1957.

The convergence is governed by the following theorem.

THEOREM. *Suppose that on the interval* $[x_0, x_2]$,

$$H1: \quad \frac{\partial f}{\partial y} \text{ exists and is everywhere continuous.}$$

$$H2: \quad \left|\frac{\partial f}{\partial y}\right| \le M.$$

Further suppose that the mesh interval h has been chosen so that

$$H3: \quad 0 < h < \frac{\sqrt{21} - 3}{2M} = \frac{.7926}{M}.$$

Then the iterative process defined by (1) and (2) converges to a solution y_1 of the equations (1), (2).

Proof. Letting y_1 denote the solution of (1), (2) and $y_{1,r}$, $y^*_{2,r}$ denote the r-th iterated values of y_1 and y_2 respectively, we have

$$y_{1,r+1} = y_0 + \frac{h}{12}\,[5f(x_0, y_0) + 8f(x_1, y_{1,r}) - f(x_2, y^*_{2,r+1})]$$

and therefore,

$$(4) \quad y_1 - y_{1,r+1} = \frac{h}{12}\,\{8[f(x_1, y_1) - f(x_1, y_{1,r})] - [f(x_2, y_2^*) - f(x_2, y^*_{2,r+1})]\}.$$

Now,

$$(5) \qquad f(x_1, y_1) - f(x_1, y_{1,r}) = (y_1 - y_{1,r})\,\frac{\partial f}{\partial y}\,(x_1, \eta_1)$$

for some η_1, in $(y_1, y_{1,r})$ and

$$(6) \qquad f(x_2, y_2^*) - f(x_2, y^*_{2,r+1}) = (y_2^* - y^*_{2,r+1})\,\frac{\partial f}{\partial y}\,(x_2, \eta_2)$$

$$= \frac{\partial f}{\partial y}\,(x_2, \eta_2)\left[[-4(y_1 - y_{1,r}) + 4h(y_1 - y_{1,r})\,\frac{\partial f}{\partial y}\,(x_1, \eta_1)\right]$$

$$= 4(y_1 - y_{1,r})\,\frac{\partial f}{\partial y}\,(x_2, \eta_2)\left[h\,\frac{\partial f}{\partial y}\,(x_1, \eta_1) - 1\right].$$

Inserting (6), (5) in (4) we get

$$(7) \quad (y_1 - y_{1,r+1}) = (y_1 - y_{1,r})\,\frac{h}{12}\left\{8\,\frac{\partial f}{\partial y}\,(x_1, \eta_1) - 4\,\frac{\partial f}{\partial y}\,(x_2, \eta_2)\left[h\,\frac{\partial f}{\partial y}\,(x_1, \eta_1) - 1\right]\right\}$$

for some η_2 in $(y_2^*, y^*_{2,r+1})$.

To prove convergence, i.e., that $(y_1 - y_{1,r}) \to 0$, we need only show that the coefficient of $y_1 - y_{1,r}$ in (7) is, in absolute value, < 1. We have

$$| y_1 - y_{1,r+1} | \leq | y_1 - y_{1,r} | \frac{h}{12} \{8M + 4M(hM + 1)$$

$$= | y_1 - y_{1,r} | \left(hM + \frac{h^2M^2}{3}\right).$$

Hypothesis $H3$ assures us that $hM\left(1 + \frac{hM}{3}\right) < 1$ which proves the theorem.

2. Stability. Let y denote the true solution of (3), z denote the calculated solution from (1), (2) and $\eta = y - z$ be the error.

Now y satisfies

$$y_1 = y_0 + \frac{h}{12} [5f(x_0, y_1) + 8f(x_1, y_1) - f(x_2, y_2^*)] + T_1 \tag{8}$$

$$y_2^* = 5y_0 - 4y_1 + 2h[f(x_0, y_0) + 2f(x_1, y_1)] \tag{9}$$

where T_1 is the truncation error given by (8.1) of I, and z satisfies

$$z_1 = z_0 + \frac{h}{12} [5f(x_0, z_0) + 8f(x_1, z_1) - f(x_2, z_2^*)] + h\rho_1 \tag{10}$$

$$z_2^* = 5z_0 - 4z_1 + 2h[f(x_0, z_0) + 2f(x_1, z_1)] + h\rho_2 \tag{11}$$

where ρ_1, ρ_2 are round off error in the evaluation of f.

By subtraction, we get

$$\eta_{n+1}\left\{1 - hg + \frac{h^2g^2}{3}\right\} = \eta_n\left\{1 - \frac{h^2g^2}{6}\right\} + T \tag{12}$$

where we have assumed

a) $\frac{\partial f}{\partial y}$ is constant $(=g)$ over the interval in question.

b) Round off error is negligible.

c) Truncation error is constant $(=T)$.

The solution of (12) is

$$\eta_n = \lambda^n\eta_0 + (\lambda^n - 1) \frac{T}{hg(1 - hg/2)} \tag{13}$$

where

$$\lambda = \frac{1 - (hg)^2/6}{1 - hg + \frac{1}{3}(hg)^2}\,. \tag{14}$$

The initial error η_0 will therefore decrease in magnitude when $| \lambda | < 1$ which occurs for $-\infty < hg < 0$ and for $hg > 2$.

We may then summarize the discussion of this and the preceding sections by noting that convergence and stability occur simultaneously only when

$$-.7926 < hg < 0. \tag{15}$$

If $\partial f/\partial y$ is positive, errors introduced at any stage will grow in magnitude, and the integration had better be performed in the reverse direction. This behaviour is characteristic of numerical integration techniques.

3. A Higher Order Formula. The formulas corresponding to (1) and (2) for an error of order h^5 are as follows:

$$(16)\quad y_1 = y_0 + \frac{h}{24}[9f(x_0, y_0) + 19f(x_1, y_1) - 5f(x_2, y_2^*) + f(x_3, y_3^*)] + 0(h^5)$$

$$(17)\quad y_2^* = y_0 + \frac{h}{3}[f(x_0, y_0) + 4f(x_1, y_1) + f(x_2, y_2^*)]$$

$$(18)\quad y_3^* = 9y_1 - 8y_0 - 3h[f(x_0, y_0) + 2f(x_1, y_1) - f(x_2, y_2^*)].$$

These formulas are used to find y_1 as follows:

a) Guess $y_1 = y_2^*(x - h)$, $y_2^* = y_3^*(x - h)$.

b) Calculate improved values of y_3^*, y_2^*, y_1 in that order.

c) Repeat from b) until y_1 has converged before proceeding to the next point.

Note that no starter formulas are required since initially we may guess $y_0 = y_1 = y_2^* = y_3^*$ and proceed from b) above.

Nuclear Development Corporation of America,
White Plains, New York

1. HERBERT S. WILF, "An open formula for the numerical integration of first order differential equations," *MTAC*, v. 11, 1957, p. 201–203.

TECHNICAL NOTES AND SHORT PAPERS

Note on the Computation of the Zeros of Functions Satisfying a Second Order Differential Equation

By D. J. Hofsommer

It has been pointed out by P. Wynn [1] that, if a function satisfies a second order differential equation, this fact may be used with advantage in the computation of its zeros. In his note he only pays attention to Richmonds formula which, incidentally, was already known to Schröder [2]. We will elaborate his idea to construct another iteration formula.

Let $f(x)$ be the function, the roots of which are to be computed. Let α be such a root and let x be a first approximation. If the approximation is sufficiently close,

$$(1)\quad \alpha = x - f/f' - \tfrac{1}{2}(f''/f')(f/f')^2 - \tfrac{1}{6}[3(f''/f')^2 - f'''/f'](f/f')^3 + 0[(f/f')^4].$$

This series may be used either for direct computation in taking enough terms, or for obtaining an iterative process if only few terms are retained. If $f(x)$ satisfies the homogeneous differential equation

$$(2)\qquad f'' = 2Pf' + Qf + 2S,$$

substitution in the series (1) yields

$$(3)\quad \alpha = x - f/f' + (P + S/f')(f/f')^2 - \tfrac{1}{3}(4P^2 - P' + Q + 10PS/f' + S'/f' + 6S^2/f'^2)(f/f')^3 + 0[(f/f')^4]$$

Received 3 May 1957.