Coefficients and Roots of the Polynomials which Define the Derivatives of the Exponential of $(-\epsilon/T)$

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The n-th derivative of exp $(-\epsilon/T)$ can be shown to be of the form

$$d^n \exp [-\epsilon/T]/dT^n = T^{-n} \exp [-\epsilon/T] W_n[\epsilon/T] \tag{1}$$

$$W_n[\epsilon/T] = \sum_{l=1}^{n} b_{n,l}[\epsilon/T]^l.$$

The $b_{n,l}$ are constant coefficients in a homogeneous polynomial of the n-th order in $[\epsilon/T]$. These polynomials attain added importance from their relation to the Laguerre polynomials [1]

$$W_n[\epsilon/T] = (-1)^n\{L_n[\epsilon/T] - nL_{n-1}[\epsilon/T]\} \tag{2}$$

$L_n[\epsilon/T]$: the n-th order Laguerre polynomial.

The zeros of these polynomials are of interest since they locate the extreme values of the derivatives of exp $(-\epsilon/T)$. Furthermore, an accurate, simple computation of the W_n for values of (ϵ/T) which occur in physical problems sometimes requires use of the polynomial roots. This occurs whenever the direct evaluation of $W_n[\epsilon/T]$ by synthetic division introduces at intermediate steps of the calculation numbers which are larger than the value of the polynomial. In such cases the subtraction of these larger numbers can require the use of considerably more than the usual guard figures which allow for the effect of ordinary round-off error. Fortunately the occurrence of this disastrous subtraction in synthetic division does not imply the occurrence when the polynomial is calculated using its roots

$$W_n[\epsilon/T] = \prod_{l=1}^{n} [\epsilon/T - r_{n,l}] \tag{3}$$

$r_{n,l}$: the l-th root of W_n.

The polynomial coefficients, $b_{n,l}$, are integers. Table I lists all of their non-zero digits which were computed from the recursion relations

$$\begin{aligned} b_{n,1} &= (-1)^{n-1}n! \\ b_{n,l} &= -(n + l - 1)b_{n-1,l} + b_{n-1,l-1} (2 \leq l \leq n - 1) \\ b_{n,n} &= 1. \end{aligned} \tag{4}$$

The values in the Table were checked numerically by the distinct relations

$$(n - l)b_{n,l} = -n(n - 1)b_{n-1,l}. \tag{5}$$

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TABLE I. *Exact Values of the Coefficients, $b_{n,l}$*

n	l	$\beta_{n,l}$	$P_{n,l}$
1	1	+1	0
2	1	−2	0
	2	+1	0
3	1	+6	0
	2	−6	0
	3	+1	0
4	1	−2.4	1
	2	+3.6	1
	3	−1.2	1
	4	+1	0
5	1	+1.2	2
	2	−2.4	2
	3	+1.2	2
	4	−2	1
	5	+1	0
6	1	−7.2	2
	2	+1.8	3
	3	−1.2	3
	4	+3	2
	5	−3	1
	6	+1	0
7	1	+5.04	3
	2	−1.512	4
	3	+1.26	4
	4	−4.2	3
	5	+6.3	2
	6	−4.2	1
	7	+1	0
8	1	−4.032	4
	2	+1.4112	5
	3	−1.4112	5
	4	+5.88	4
	5	−1.176	4
	6	+1.176	3
	7	−5.6	1
	8	+1	0
9	1	+3.6288	5
	2	−1.4515 2	6
	3	+1.6934 4	6
	4	−8.4672	5
	5	+2.1168	5
	6	−2.8224	3
	7	+2.016	1
	8	−7.2	0
	9	+1	4
10	1	−3.6288	6
	2	+1.6329 6	7
	3	−2.1772 8	7
	4	+1.2700 8	7
	5	−3.8102 4	6
	6	+6.3504	5
	7	−6.048	4
	8	+3.24	3
	9	−9	1
	10	+1	0
11	1	+3.9916 8	7
	2	−1.9958 4	8
	3	+2.9937 6	8
	4	−1.9958 4	8
	5	+6.9854 4	7
	6	−1.3970 88	7
	7	+1.6632	6
	8	−1.188	5
	9	+4.95	3
	10	−1.1	2
	11	+1	0
12	1	−4.7900 16	8
	2	+2.6345 088	9
	3	−4.3908 48	9
	4	+3.2931 36	9
	5	−1.3172 544	9
	6	+3.0735 936	8
	7	−4.3908 48	7
	8	+3.9204	6
	9	−2.178	5
	10	+7.26	3
	11	−1.32	2
	12	+1	0
13	1	+6.2270 208	9
	2	−3.7362 1248	10
	3	+6.8497 2288	10
	4	−5.7081 024	10
	5	+2.5686 4608	10
	6	−6.8497 2288	9
	7	+1.1416 2048	9
	8	−1.2231 648	8
	9	+8.4942	6
	10	−3.7752	5
	11	+1.0296	4
	12	−1.56	2
	13	+1	0
14	1	−8.7178 2912	10
	2	+5.6665 88928	11
	3	−1.1333 17785 6	12
	4	+1.0388 74636 8	12
	5	−5.1943 73184	11
	6	+1.5583 11955 2	11
	7	−2.9682 13248	10
	8	+3.7102 6656	9
	9	−3.0918 888	8
	10	+1.7177 16	7
	11	−6.2462 4	5
	12	+1.4196	4
	13	−1.82	2
	14	+1	0
15	1	+1.3076 74368	12
	2	−9.1537 20576	12
	3	+1.9833 06124 8	13
	4	−1.9833 06124 8	13
	5	+1.0908 18368 64	13
	6	−3.6360 61228 8	12
	7	+7.7915 59776	11
	8	−1.1130 79968	11
	9	+1.0821 6108	10
	10	−7.2144 072	8
	11	+3.2792 76	7
	12	−9.9372	5
	13	+1.911	4
	14	−2.1	2
	15	+1	0

TABLE I. *Exact Values of the Coefficients, $b_{n,l}$*—Continued

n	l	$\beta_{n,l}$	$P_{n,l}$
16	1	−2.0922 78988 8	13
	2	+1.5692 09241 6	14
	3	−3.6614 88230 4	14
	4	+3.9666 12249 6	14
	5	−2.3799 67349 76	14
	6	+8.7265 46949 12	13
	7	−2.0777 49273 6	13
	8	+3.3392 39904	12
	9	−3.7102 6656	11
	10	+2.8857 6288	10
	11	−1.5740 5248	9
	12	+5.9623 2	7
	13	−1.5288	6
	14	+2.52	4
	15	−2.4	2
	16	+1	0
17	1	+3.5568 74280 96	14
	2	−2.8454 99424 768	15
	3	+7.1137 48561 92	15
	4	−8.2993 73322 24	15
	5	+5.3945 92659 456	15
	6	−2.1578 37063 7824	15
	7	+5.6514 78024 192	14
	8	−1.0091 92504 32	14
	9	+1.2614 90630 4	13
	10	−1.1213 25004 8	12
	11	+7.1357 04576	10
	12	−3.2435 0208	9
	13	+1.0395 84	8
	14	−2.2848	6
	15	+3.264	4
	16	−2.72	2
	17	+1	0
18	1	−6.4023 73705 728	15
	2	+5.4420 17649 8688	16
	3	−1.4512 04706 63168	17
	4	+1.8140 05883 2896	17
	5	−1.2698 04118 30272	17
	6	+5.5024 84512 64512	16
	7	−1.5721 38432 18432	16
	8	+3.0881 29063 2192	15
	9	−4.2890 68143 36	14
	10	+4.2890 68143 36	13
	11	−3.1193 22286 08	12
	12	+1.6541 86060 8	11
	13	−6.3622 5408	9
	14	+1.7478 72	8
	15	−3.3292 8	6
	16	+4.1616	4
	17	−3.06	2
	18	+1	0
19	1	+1.2164 51004 08832	17
	2	−1.0948 05903 67948 8	18
	3	+3.1019 50060 42521 6	18
	4	−4.1359 33413 90028 8	18
	5	+3.1019 50060 42521 6	18
	6	−1.4475 76694 86510 08	18
	7	+4.4805 94531 72531 2	17
	8	−9.6012 73996 55424	16
	9	+1.4668 61305 02912	16
	10	−1.6298 45894 4768	15
	11	+1.3335 10277 2992	14
	12	−8.0818 80468 48	12
	13	+3.6264 84825 6	11
	14	−1.1955 44448	10
	15	+2.8465 344	8
	16	−4.7442 24	6
	17	+5.2326	4
	18	−3.42	2
	19	+1	0
20	1	−2.4329 02008 17664	18
	2	+2.3112 56907 76780 8	19
	3	−6.9337 70723 30342 4	19
	4	+9.8228 41858 01318 4	19
	5	−7.8582 73486 41054 72	19
	6	+3.9291 36743 20527 36	19
	7	−1.3097 12247 73509 12	19
	8	+3.0404 03432 24217 6	18
	9	−5.0673 39053 73696	17
	10	+6.1934 14399 01184	16
	11	−5.6303 76726 3744	15
	12	+3.8388 93222 528	14
	13	−1.9686 63191 04	13
	14	+7.5717 81504	11
	15	−2.1633 66144	10
	16	+4.5070 128	8
	17	−6.6279 6	6
	18	+6.498	4
	19	−3.8	2
	20	+1	0
21	1	+5.1090 94217 17094 4	19
	2	−5.1090 94217 17094 4	20
	3	+1.6178 79835 43746 56	21
	4	−2.4268 19753 15619 84	21
	5	+2.0627 96790 18276 864	21
	6	−1.1001 58288 09747 6608	21
	7	+3.9291 36743 20527 36	20
	8	−9.8228 41858 01318 4	19
	9	+1.7735 68668 80793 6	19
	10	−2.3647 58225 07724 8	18
	11	+2.3647 58225 07724 8	17
	12	−1.7914 83503 8464	16
	13	+1.0335 48175 296	15
	14	−4.5430 68902 4	13
	15	+1.5143 56300 8	12
	16	−3.7858 90752	10
	17	+6.9593 58	8
	18	−9.0972	6
	19	+7.98	4
	20	−4.2	2
	21	+1	0

TABLE I. *Exact Values of the Coefficients, $b_{n,l}$*—Continued

n	l	$\beta_{n,l}$	$P_{n,l}$
22	1	−1.1240 00727 77760 768	21
	2	+1.1802 00764 16648 8064	22
	3	−3.9340 02547 22162 688	22
	4	+6.2288 37366 43424 256	22
	5	−5.6059 53629 79081 8304	22
	6	+3.1767 07056 88146 37056	22
	7	−1.2101 74116 90722 42688	22
	8	+3.2415 37813 14435 072	21
	9	−6.3029 90192 22512 64	20
	10	+9.1043 19166 54740 48	19
	11	−9.9319 84545 32444 16	18
	12	+8.2766 53787 77036 8	17
	13	−5.3055 47299 8528	16
	14	+2.6236 22291 136	15
	15	−9.9947 51585 28	13
	16	+2.9151 35879 04	12
	17	−6.4304 46792	10
	18	+1.0507 266	9
	19	−1.2289 2	7
	20	+9.702	4
	21	−4.62	2
	22	+1	0
23	1	+2.5852 01673 88849 7664	22
	2	−2.8437 21841 27734 74304	23
	3	+9.9530 26444 47071 60064	23
	4	−1.6588 37740 74511 93344	24
	5	+1.5758 95853 70786 33676 8	24
	6	−9.4553 75122 24718 02060 8	23
	7	+3.8271 75644 71909 67500 8	23
	8	−1.0934 78755 63402 76428 8	23
	9	+2.2780 80740 90422 4256	22
	10	−3.5436 81152 51768 2176	21
	11	+4.1879 86816 61180 6208	20
	12	−3.8072 60742 37436 928	19
	13	+2.6846 06933 72551 68	18
	14	−1.4750 58754 79424	17
	15	+6.3216 80377 6896	15
	16	−2.1072 26792 5632	14
	17	+5.4230 10127 92	12
	18	−1.0633 35319 2	11
	19	+1.5545 838	9
	20	−1.6364 04	7
	21	+1.1688 6	5
	22	−5.06	2
	23	+1	0
24	1	−6.2044 84017 33239 43936	23
	2	+7.1351 56619 93225 35526 4	24
	3	−2.6162 24093 97515 96359 68	25
	4	+4.5783 92164 45652 93629 44	25
	5	−4.5783 92164 45652 93629 44	25
	6	+2.8996 48370 82246 85965 312	25
	7	−1.2427 06444 63820 08270 848	25
	8	+3.7725 01706 93739 53679 36	24
	9	−8.3833 37126 52754 52620 8	23
	10	+1.3972 22854 42125 75436 8	23
	11	−1.7782 83632 89978 23283 2	22
	12	+1.7513 39941 49220 98688	21
	13	−1.3471 84570 37862 2976	20
	14	+8.1423 24326 46420 48	18
	15	−3.8772 97298 31628 8	17
	16	+1.4539 86486 86860 8	16
	17	−4.2764 30843 7312	14
	18	+9.7826 84936 64	12
	19	−1.7162 60515 2	11
	20	+2.2582 3752	9

TABLE I. *Exact Values of the Coefficients, $b_{n,l}$*—Continued

n	l	$\beta_{n,l}$	$P_{n,l}$
24	21	−2.1507 024	7
	22	+1.3965 6	5
	23	−5.52	2
	24	+1	0
25	1	+1.5511 21004 33309 85984	25
	2	−1.8613 45205 19971 83180 8	26
	3	+7.1351 56619 93225 35526 4	26
	4	−1.3081 12046 98757 98179 84	27
	5	+1.3735 17649 33695 88088 832	27
	6	−9.1567 84328 91305 87258 88	26
	7	+4.1423 54815 46066 94236 16	26
	8	−1.3314 71190 68378 66004 48	26
	9	+3.1437 51422 44782 94732 8	25
	10	−5.5888 91417 68503 01747 2	24
	11	+7.6212 15569 57049 56928	23
	12	−8.0831 07422 27173 7856	22
	13	+6.7359 22851 89311 488	21
	14	−4.4412 67814 43502 08	20
	15	+2.3263 78378 98977 28	19
	16	−9.6932 43245 79072	17
	17	+3.2073 23132 7984	16
	18	−8.3851 58517 12	14
	19	+1.7162 60515 2	13
	20	−2.7098 85024	11
	21	+3.2260 536	9
	22	−2.7931 2	7
	23	+1.656	5
	24	−6	2
	25	+1	0
26	1	−4.0329 14611 26605 63558 4	26
	2	+5.0411 43264 08257 04448	27
	3	−2.0164 57305 63302 81779 2	28
	4	+3.8648 76502 46330 40076 8	28
	5	−4.2513 64152 70963 44084 48	28
	6	+2.9759 54906 89674 40859 136	28
	7	−1.4171 21384 23654 48028 16	28
	8	+4.8080 90410 80256 27238 4	27
	9	−1.2020 22602 70064 06809 6	27
	10	+2.2704 87138 43454 35084 8	26
	11	−3.3025 26746 81388 14668 8	25
	12	+3.7528 71303 19759 2576	24
	13	−3.3679 61425 94655 744	23
	14	+2.4056 86732 81896 96	22
	15	−1.3746 78133 03941 12	21
	16	+6.3006 08109 76396 8	19
	17	−2.3164 00040 3544	18
	18	+6.8129 41295 16	16
	19	−1.5936 70478 4	15
	20	+2.9357 08776	13
	21	−4.1938 6968	11
	22	+4.5388 2	9
	23	−3.588	7
	24	+1.95	5
	25	−6.5	2
	26	+1	0
27	1	+1.0888 86945 04183 52160 768	28
	2	−1.4155 53028 55438 57808 9984	29
	3	+5.8981 37618 97660 74204 16	29
	4	−1.1796 27523 79532 14840 832	30
	5	+1.3565 71652 36461 97066 9568	30
	6	−9.9481 92117 34054 45157 6832	29
	7	+4.9740 96058 67027 22578 8416	29
	8	−1.7764 62878 09652 58063 872	29
	9	+4.6878 88150 53249 86557 44	28

TABLE I. *Exact Values of the Coefficients,* $b_{n,l}$—Continued

n	l	$\beta_{n,l}$	$P_{n,l}$
27	10	−9.3757 76301 06499 73114 88	27
	11	+1.4489 83610 16459 04935 936	27
	12	−1.7563 43769 89647 33255 68	26
	13	+1.6887 92086 43891 66592	25
	14	−1.2990 70835 72224 3584	24
	15	+8.0418 67078 28055 552	22
	16	−4.0209 33539 14027 776	21
	17	+1.6261 12828 32878 88	20
	18	−5.3140 94210 2248	18
	19	+1.3984 45844 796	17
	20	−2.9440 96515 36	15
	21	+4.9068 27525 6	13
	22	−6.3725 0328	11
	23	+6.2969 4	9
	24	−4.563	7
	25	+2.2815	5
	26	−7.02	2
	27	+1	0
28	1	−3.0488 83446 11713 86050 1504	29
	2	+4.1159 92652 25813 71167 70304	30
	3	−1.7835 96815 97852 60839 33798 4	31
	4	+3.7158 26699 95526 26748 6208	31
	5	−4.4589 92039 94631 52098 34496	31
	6	+3.4185 60563 95884 16608 73113 6	31
	7	−1.7906 74581 12129 80128 38297 6	31
	8	+6.7150 29679 20486 75481 43616	30
	9	−1.8652 86022 00135 20967 0656	30
	10	+3.9378 26046 44729 88708 2496	29
	11	−6.4437 15348 73194 36068 0448	28
	12	+8.2987 24312 76083 64633 088	27
	13	−8.5115 12115 65213 99623 68	26
	14	+7.0149 82512 90011 53536	25
	15	−4.6766 55008 60007 69024	24
	16	+2.5331 88129 65837 49888	23
	17	−1.1175 82998 37869 4848	22
	18	+4.0174 55222 92994 88	20
	19	−1.1746 94509 62864	19
	20	+2.7821 71207 0152	17
	21	−5.2993 73727 648	15
	22	+8.0293 54132 8	13
	23	−9.5209 7328	11
	24	+8.6240 7	9
	25	−5.7493 8	7
	26	+2.6535 6	5
	27	−7.56	2
	28	+1	0
29	1	+8.8417 61993 73970 19545 43616	30
	2	−1.2378 46679 12355 82736 36106 24	32
	3	+5.5703 10056 05601 22313 62478 08	32
	4	−1.2069 00512 14546 93167 95203 584	33
	5	+1.5086 25640 18183 66459 94004 48	33
	6	−1.2069 00512 14546 93167 95203 584	33
	7	+6.6092 17090 32042 72110 21352 96	32
	8	−2.5964 78142 62588 21186 15531 52	32
	9	+7.5730 61249 32548 95126 28633 6	31
	10	−1.6829 02499 85010 87805 84140 8	31
	11	+2.9068 31590 65018 78937 36243 2	30
	12	−3.9638 61259 97752 89460 03968	29
	13	+4.3195 92398 69346 10309 0176	28
	14	−3.7974 43866 98326 24447 488	27
	15	+2.7124 59904 98804 46033 92	26
	16	−1.5822 68277 90969 26853 12	25
	17	+7.5623 11622 36250 18048	23
	18	−2.9656 12400 92647 1296	22
	19	+9.5385 19418 18455 68	20

TABLE I. *Exact Values of the Coefficients, $b_{n,l}$*—Continued

n	l	$\beta_{n,l}$	$P_{n,l}$
29	20	−2.5101 36688 99593 6	19
	21	+5.3788 64333 56272	17
	22	−9.3140 50794 048	15
	23	+1.2885 05050 56	14
	24	−1.4005 48968	12
	25	+1.1671 2414	10
	26	−7.1823 024	7
	27	+3.0693 6	5
	28	−8.12	2
	29	+1	0
30	1	−2.6525 28598 12191 05863 63084 8	32
	2	+3.8461 66467 27677 03502 26472 96	33
	3	−1.7948 77684 72915 94967 72354 048	34
	4	+4.0384 74790 64060 88677 37796 608	34
	5	−5.2500 17227 83279 15280 59135 5904	34
	6	+4.3750 14356 52732 62733 82612 992	34
	7	−2.5000 08203 72990 07276 47207 424	34
	8	+1.0267 89083 67478 06559 97960 192	34
	9	−3.1374 11089 00627 42266 60433 92	33
	10	+7.3206 25874 34797 31955 41012 48	32
	11	−1.3310 22886 24508 60355 52911 36	32
	12	+1.9158 66275 65580 56572 35251 2	31
	13	−2.2106 14933 44900 65275 79136	30
	14	+2.0648 60102 67214 89543 3216	29
	15	−1.5732 26744 89306 58699 6736	28
	16	+9.8326 67155 58166 16872 96	26
	17	−5.0609 31624 19644 35155 2	25
	18	+2.1500 68990 67169 16896	24
	19	−7.5441 01721 65505 856	22
	20	+2.1838 18919 42646 432	21
	21	−5.1995 68855 77729 6	19
	22	+1.0129 03023 85272	18
	23	−1.6014 27705 696	16
	24	+2.0307 96003 6	14
	25	−2.0307 96003 6	12
	26	+1.5621 50772	10
	27	−8.9011 44	7
	28	+3.5322	5
	29	−8.7	2
	30	+1	0

Legend: $W_n(x) = \sum_{l=1}^{n} b_{n,l}\, x^l$

$b_{n,l} = \beta_{n,l} \times 10^{P_{n,l}}, \qquad 1 \leqq \beta_{n,l} < 10$

TABLE II. *Roots of W_n (ϵ/T)*

n	l	$r_{n,l}$	C	$e_{n,l}$
2	1	2.00000 00000 0000	0	0
3	1	1.26794 91924 3112	0	5
3	2	4.73205 08075 6888	0	5
4	1	0.93582 22275 2409	0	5
4	2	3.30540 72893 3227	0	5
4	3	7.75877 04831 4363	0	5

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
5	1	0.74329 19279 8143	0	5
5	2	2.57163 50076 4627	0	5
5	3	5.73117 87516 8905	0	50
5	4	10.95389 43126 8321	1	6
6	1	0.61703 08532 7825	0	50
6	2	2.11296 59585 7838	0	50
6	3	4.61083 31510 1760	0	50
6	4	8.39906 69712 0486	0	50
6	5	14.26010 30659 2081	1	10
7	1	0.52766 81217 1117	0	50
7	2	1.79629 98096 4345	0	50
7	3	3.87664 15204 7699	0	50
7	4	6.91881 65667 0471	0	50
7	5	11.23461 04290 8311	1	6
7	6	17.64596 35523 8068	1	16
8	1	0.46102 42198 0496	0	50
8	2	1.56358 61896 5431	0	50
8	3	3.35205 05025 3674	0	50
8	4	5.91629 72490 2042	0	50
8	5	9.42069 93830 2156	0	50
8	6	14.19416 55480 0748	1	10
8	7	21.09217 69079 5447	1	22
9	1	0.40938 35732 0319	0	5
9	2	1.38496 31848 0312	0	5
9	3	2.95625 45561 6887	0	5
9	4	5.18194 31010 4007	0	5
9	5	8.16170 96881 4582	0	5
9	6	12.07005 51268 3715	1	7
9	7	17.24973 55261 4898	1	15
9	8	24.58595 52436 5281	1	30
10	1	0.36817 84529 4174	0	5
10	2	1.24335 79621 4047	0	5
10	3	2.64603 38413 8420	0	5
10	4	4.61688 25146 3485	0	50
10	5	7.22178 65393 9663	0	50
10	6	10.56732 08077 4184	1	56
10	7	14.83591 45152 6107	1	110
10	8	20.38218 19854 4899	1	207
10	9	28.11834 33810 4993	1	39
11	1	0.33452 86763 2476	0	5
11	2	1.12825 33558 7666	0	5
11	3	2.39586 99247 4731	0	5
11	4	4.16684 09879 2878	0	5
11	5	6.48735 30313 8081	0	5
11	6	9.42835 48133 3561	0	5
11	7	13.10172 35803 6780	1	9
11	8	17.69648 75668 4621	1	16
11	9	23.57778 70883 6019	1	28
11	10	31.68280 09748 3192	1	50

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
12	1	0.30652 67021 3005	0	5
12	2	1.03279 73987 7972	0	5
12	3	2.18961 19419 6840	0	5
12	4	3.79904 76060 5497	0	5
12	5	5.89491 11715 0275	0	5
12	6	8.52729 20080 7027	0	5
12	7	11.77100 65654 9068	1	7
12	8	15.74226 02870 2615	1	12
12	9	20.63580 56686 4129	1	213
12	10	26.82634 99493 7082	1	36
12	11	35.27439 07009 6510	1	62
13	1	0.28285 83482 3992	0	5
13	2	0.95232 60413 6462	0	5
13	3	2.01649 21385 7771	0	50
13	4	3.49235 40697 7799	0	50
13	5	5.40549 10200 1572	0	5
13	6	7.79281 39404 1242	0	5
13	7	10.70738 86889 8909	1	6
13	8	14.22715 23637 8996	1	10
13	9	18.47199 66342 2982	1	171
13	10	23.64178 37524 0122	1	279
13	11	30.12005 86261 0650	1	455
13	12	38.88928 43760 9550	1	76
14	1	0.26258 83981 7108	0	5
14	2	0.88355 03073 8774	0	5
14	3	1.86903 38151 9979	0	50
14	4	3.23241 86994 5120	0	50
14	5	4.99357 56070 7419	0	5
14	6	7.18061 04941 0079	0	5
14	7	9.83280 82510 2000	0	5
14	8	13.00562 24002 5865	1	85
14	9	16.77961 36411 7585	1	141
14	10	21.27791 17554 4831	1	226
14	11	26.70503 40989 2773	1	357
14	12	33.45278 49657 6721	1	559
14	13	42.52444 75660 1840	1	91
15	1	0.24503 30150 9310	0	5
15	2	0.82408 22200 4784	0	50
15	3	1.74187 24007 6449	0	50
15	4	3.00913 24590 1792	0	50
15	5	4.64163 40665 9064	0	50
15	6	6.66134 14966 2717	0	5
15	7	9.09830 27037 7053	0	5
15	8	11.99370 34331 8574	1	7
15	9	15.40498 80310 5145	1	119
15	10	19.41499 28281 6615	1	1890
15	11	24.14975 79520 2928	1	2920
15	12	29.81810 51196 7132	1	4460
15	13	36.81962 42569 9721	1	6760
15	14	46.17743 00169 8345	1	1060

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
16	1	0.22968 05054 2513	0	5
16	2	0.77214 49103 7539	0	5
16	3	1.63105 30990 6745	0	50
16	4	2.81514 45900 1225	0	50
16	5	4.33716 40773 3756	0	5
16	6	6.21464 27645 5924	0	5
16	7	8.47116 39813 4671	0	5
16	8	11.13833 19657 5080	1	6
16	9	14.25891 00216 2446	1	102
16	10	17.89205 34381 6953	1	1600
16	11	22.12262 01748 2668	1	2450
16	12	27.07931 14990 4742	1	3680
16	13	32.97497 35524 0162	1	5450
16	14	40.21658 37114 8568	1	8060
16	15	49.84622 17085 5745	1	1240
17	1	0.21614 03052 3945	0	5
17	2	0.72638 82432 5183	0	50
17	3	1.53359 31603 7353	0	50
17	4	2.64497 09986 1195	0	50
17	5	4.07097 81608 8018	0	50
17	6	5.82585 55151 0563	0	50
17	7	7.92850 41853 0666	0	5
17	8	10.40380 82899 5104	1	5
17	9	13.28466 10707 0697	1	88
17	10	16.61517 32168 6805	1	1380
17	11	20.45600 60200 3371	1	2090
17	12	24.89384 70253 3575	1	30900
17	13	30.05986 29201 6784	1	45000
17	14	36.17069 45436 3216	1	65600
17	15	43.64036 51841 5001	1	9540
17	16	53.52915 11602 5465	1	14300
18	1	0.20410 91085 7933	0	5
18	2	0.68576 75894 9453	0	5
18	3	1.44719 86793 8049	0	50
18	4	2.49443 08859 6248	0	50
18	5	3.83615 60319 9932	0	50
18	6	5.48411 54208 4047	0	5
18	7	7.45372 29495 8128	0	5
18	8	9.76498 43667 5180	0	5
18	9	12.44386 10519 8395	1	77
18	10	15.52434 30012 1457	1	1200
18	11	19.05171 87593 8568	1	1810
18	12	23.08800 07008 7174	1	26700
18	13	27.72155 63462 7824	1	38500
18	14	33.08585 98280 6407	1	54800
18	15	39.40115 37933 0861	1	77500
18	16	47.08820 96546 0141	1	1 11000
18	17	57.22481 18319 1492	1	16300

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
19	1	0.19334 77686 7901	0	5
19	2	0.64946 18107 9655	0	5
19	3	1.37007 54867 1706	0	5
19	4	2.36027 76257 7508	0	5
19	5	3.62737 58656 5560	0	5
19	6	5.18117 49562 2015	0	5
19	7	7.03443 06230 6566	0	5
19	8	9.20350 66842 8214	0	5
19	9	11.70932 47926 7879	1	69
19	10	14.57876 11306 9283	1	1060
19	11	17.84675 98274 1882	1	15900
19	12	21.55965 87534 0514	2	12
19	13	25.78070 10050 5240	2	17
19	14	30.59981 44865 1918	2	23
19	15	36.15265 02790 4671	2	33
19	16	42.66288 77797 8564	2	45
19	17	50.55778 33977 1778	2	64
19	18	60.93200 77265 0073	2	93
20	1	0.18366 51730 9616	0	50
20	2	0.61681 63821 2673	0	50
20	3	1.30079 94029 6677	0	500
20	4	2.23994 93990 0886	0	500
20	5	3.44047 19428 8150	0	500
20	6	4.91064 58796 7473	0	50
20	7	6.66116 06120 5821	0	50
20	8	8.70559 71771 2802	0	50
20	9	11.06110 86835 7575	1	612
20	10	13.74939 41200 8341	1	9450
20	11	16.79812 30059 5300	1	14100
20	12	20.24308 49633 0246	2	10
20	13	24.13156 72676 0267	2	15
20	14	28.52794 79713 1414	2	20
20	15	33.52361 76172 8890	2	28
20	16	39.25629 42907 8263	2	39
20	17	45.95295 06486 3834	2	53
20	18	54.04709 30247 0194	2	73
20	19	64.64971 24378 1605	2	105
21	1	0.17490 67523 8661	0	50
21	2	0.58730 30806 3812	0	50
21	3	1.23822 51018 3420	0	500
21	4	2.13139 62600 7712	0	500
21	5	3.27213 31335 1677	0	500
21	6	4.66749 44658 8837	0	500
21	7	6.32653 61976 7362	0	500
21	8	8.26067 09520 1371	0	50
21	9	10.48416 73812 0829	1	55
21	10	13.01484 87721 5828	1	8470
21	11	15.87508 70127 1499	1	1 26000
21	12	19.09325 19076 0624	1	18
21	13	22.70589 38881 7321	1	26
21	14	26.76117 02293 7952	1	357

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
21	15	31.32451 61370 0729	1	490
21	16	36.48870 33461 4824	1	666
21	17	42.39342 27457 7640	1	898
21	18	49.26881 38498 6849	1	1210
21	19	57.55442 09713 1555	1	1650
21	20	68.37703 78145 5231	1	235
22	1	0.16694 61390 1971	0	50
22	2	0.56049 08612 5406	0	50
22	3	1.18142 07583 5136	0	500
22	4	2.03295 74520 5904	0	500
22	5	3.11969 20402 1490	0	500
22	6	4.44769 50426 1758	0	500
22	7	6.02471 21681 1509	0	50
22	8	7.86043 77586 0933	0	50
22	9	9.96688 88709 4385	0	50
22	10	12.35891 86988 9685	1	765
22	11	15.05493 02175 9276	1	11300
22	12	18.07788 77154 4042	1	1 63000
22	13	21.45678 84662 6343	1	230
22	14	25.22887 63492 4893	1	318
22	15	29.44311 42999 0910	1	4310
22	16	34.16593 02974 6011	1	5830
22	17	39.49140 82536 2833	1	7810
22	18	45.56112 99787 8368	1	10400
22	19	52.60828 92435 3047	1	13900
22	20	61.07827 57571 7212	1	1860
22	21	72.11320 96312 7984	1	259
23	1	0.15967 90124 6359	0	50
23	2	0.53602 44507 2533	0	50
23	3	1.12962 03228 4296	0	500
23	4	1.94327 26194 9229	0	500
23	5	2.98097 42342 9980	0	500
23	6	4.24798 60975 3645	0	500
23	7	5.75098 94501 8811	0	500
23	8	7.49829 71042 4725	0	50
23	9	9.50013 87408 1857	0	5
23	10	11.76904 61305 3152	1	69
23	11	14.32037 80923 6802	1	1030
23	12	17.17304 70973 6204	1	14700
23	13	20.35054 65523 0563	2	10
23	14	23.88244 31429 7127	1	284
23	15	27.80661 96348 6417	1	3850
23	16	32.17279 15883 7417	1	5170
23	17	37.04832 56128 1149	1	68600
23	18	42.52855 84501 9839	1	90600
23	19	48.75689 07697 4619	1	1 19000
23	20	55.96946 93367 7959	1	15600
23	21	64.61735 31062 9634	1	20800
23	22	75.85754 84527 5272	1	2870

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
24	1	0.15301 84895 9942	0	5
24	2	0.51360 83707 0681	0	5
24	3	1.08218 80942 3753	0	50
24	4	1.86121 64108 9897	0	50
24	5	2.85418 83431 3952	0	50
24	6	4.06569 47966 6047	0	50
24	7	5.50154 23910 2483	0	50
24	8	7.16891 89382 1722	0	5
24	9	9.07661 44660 1974	0	5
24	10	11.23531 54871 5978	1	63
24	11	13.65799 92756 6470	1	933
24	12	16.36046 86713 7067	1	13400
24	13	19.36209 01881 5387	1	1 88000
24	14	22.68683 57294 7187	1	2580
24	15	26.36479 43150 0290	1	3470
24	16	30.43444 26913 9403	1	46300
24	17	34.94620 46462 8518	1	61100
24	18	39.96833 92266 7395	1	80000
24	19	45.59738 43134 3094	1	1 04000
24	20	51.97849 53322 1339	1	1 36000
24	21	59.35068 02360 3490	1	1 77000
24	22	68.17050 51810 0240	1	23100
24	23	79.60945 44050 8424	1	3140
25	1	0.14689 16257 8146	0	5
25	2	0.49299 48461 2182	0	5
25	3	1.03859 20353 8078	0	50
25	4	1.78584 95467 3969	0	50
25	5	2.73784 45322 0507	0	50
25	6	3.89860 82901 4203	0	50
25	7	5.27322 15403 0644	0	50
25	8	6.86794 56382 7667	0	50
25	9	8.69039 61931 4098	0	5
25	10	10.74977 14241 0897	1	58
25	11	13.05715 36754 8216	1	852
25	12	15.62591 13581 6414	1	12200
25	13	18.47224 24144 4275	1	1 71000
25	14	21.61592 28735 8341	1	2340
25	15	25.08136 20017 9402	1	3140
25	16	28.89913 24031 5677	2	21
25	17	33.10826 73051 8389	2	27
25	18	37.75986 10435 6086	2	36
25	19	42.92302 51864 7370	2	46
25	20	48.69545 32211 6803	2	59
25	21	55.22399 77787 5012	2	76
25	22	62.75044 41754 7154	2	98
25	23	71.73671 59637 8361	2	129
25	24	83.36839 49267 2194	2	174
26	1	0.14123 67262 5809	0	5
26	2	0.47397 45378 8442	0	5
26	3	0.99838 34056 2151	0	50
26	4	1.71638 16871 9240	0	50

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
26	5	2.63069 31145 8470	0	50
26	6	3.74487 77262 0273	0	50
26	7	5.06340 83123 3855	0	50
26	8	6.59177 56068 7320	0	50
26	9	8.33662 63598 0514	0	5
26	10	10.30594 30256 1368	1	5
26	11	12.50927 80113 1609	1	78
26	12	14.95806 12826 7794	1	11200
26	13	17.66600 89930 4484	1	1 56000
26	14	20.64967 47456 6110	1	2140
26	15	23.92920 78044 9273	2	14
26	16	27.52942 09021 3584	2	19
26	17	31.48133 78942 1101	2	25
26	18	35.82451 67628 4751	2	32
26	19	40.61069 00156 5943	2	414
26	20	45.90978 68582 2976	2	526
26	21	51.82061 58754 0514	2	672
26	22	58.49167 48142 7643	2	854
26	23	66.16744 93598 1048	2	110
26	24	75.31508 13581 0590	2	141
26	25	87.13389 48199 8148	2	189
27	1	0.13600 12570 4444	0	5
27	2	0.45636 93581 3000	0	5
27	3	0.96118 10273 0921	0	50
27	4	1.65214 28929 8375	0	50
27	5	2.53167 76794 9454	0	50
27	6	3.60294 55780 8908	0	50
27	7	4.86990 60826 2376	0	50
27	8	6.33740 31557 2642	0	50
27	9	8.01127 62543 8048	0	5
27	10	9.89850 34830 7608	0	5
27	11	12.00738 61381 8993	1	72
27	12	14.34778 78810 5506	1	10300
27	13	16.93144 76204 2762	1	143
27	14	19.77239 41164 6035	1	1950
27	15	22.88750 43962 7852	2	13
27	16	26.29727 10029 7123	2	17
27	17	30.02688 18438 7846	2	23
27	18	34.10778 47142 1721	2	292
27	19	38.58003 51906 4629	2	374
27	20	43.49597 58235 1797	2	472
27	21	48.92632 26588 5339	2	598
27	22	54.97096 30253 0608	2	758
27	23	61.77999 23077 3237	2	952
27	24	69.60052 54573 4751	2	1210
27	25	78.90479 29547 6498	2	156
27	26	90.90552 80995 1289	2	206
28	1	0.13114 02048 4340	0	5
28	2	0.44002 68410 2122	0	5
28	3	0.92665 89979 7975	0	5
28	4	1.59256 14353 9651	0	5

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
28	5	2.43989 88598 8214	0	5
28	6	3.47148 98767 8727	0	5
28	7	4.69085 73686 2107	0	5
28	8	6.10229 80663 5926	0	5
28	9	7.71097 27209 1486	0	5
28	10	9.52302 14214 8264	0	5
28	11	11.54571 07983 2683	1	7
28	12	13.78762 26083 0963	1	10
28	13	16.25889 71483 2334	1	132
28	14	18.97155 08191 9327	1	1800
28	15	21.93989 61709 5973	2	12
28	16	25.18110 69743 2692	2	16
28	17	28.71599 40224 8103	2	21
28	18	32.57009 65076 4767	2	266
28	19	36.77526 28337 6414	2	338
28	20	41.37202 26709 0563	2	430
28	21	46.41330 39270 2954	2	5380
28	22	51.97058 20538 3640	2	6760
28	23	58.14479 02456 7904	2	8480
28	24	65.08757 79811 6239	2	10600
28	25	73.04862 35108 1621	2	1330
28	26	82.50512 46756 5939	2	171
28	27	94.68291 12582 9295	2	224
29	1	0.12661 47772 6314	0	5
29	2	0.42481 56894 2803	0	5
29	3	0.89453 69967 2982	0	5
29	4	1.53714 63597 6714	0	5
29	5	2.35458 60074 0362	0	5
29	6	3.34938 08943 4731	0	5
29	7	4.52468 00068 0930	0	5
29	8	5.88431 36574 5907	0	5
29	9	7.43286 73847 3363	0	5
29	10	9.17577 57820 5631	0	5
29	11	11.11944 12276 9518	1	6
29	12	13.27138 44455 9570	1	9
29	13	15.64043 65527 0260	1	123
29	14	18.23698 62111 4301	1	1670
29	15	21.07330 14271 4752	2	11
29	16	24.16395 46279 8314	2	15
29	17	27.52639 39971 5279	2	19
29	18	31.18172 74331 4674	2	244
29	19	35.15582 50255 9695	2	308
29	20	39.48091 56524 1023	2	3900
29	21	44.19798 25434 6970	2	4880
29	22	49.36051 71053 0558	2	6100
29	23	55.04072 95577 5849	2	7580
29	24	61.34056 90221 6674	2	9440
29	25	68.41319 89167 3268	2	11700
29	26	76.51079 93357 6897	2	14700
29	27	86.11542 16992 4401	2	1860
29	28	98.46569 76629 3931	2	242

TABLE II. *Roots of* W_n (ϵ/T)—Continued

n	l	$r_{n,l}$	C	$e_{n,l}$
30	1	0.12239 13636 8232	0	5
30	2	0.41062 22202 4781	0	5
30	3	0.86457 25660 9784	0	5
30	4	1.48547 36509 1174	0	5
30	5	2.27507 48206 5174	0	5
30	6	3.23564 71256 6498	0	5
30	7	4.37001 73777 7174	0	5
30	8	5.68161 56855 1702	0	5
30	9	7.17453 65183 2250	0	5
30	10	8.85361 57459 8089	0	5
30	11	10.72452 73673 4938	1	6
30	12	12.79390 50612 0659	1	8
30	13	15.06949 56117 3280	1	11
30	14	17.56035 39915 4328	1	1540
30	15	20.27709 38755 2512	2	10
30	16	23.23221 33320 6006	2	14
30	17	26.44052 46096 4200	2	18
30	18	29.91973 14194 9155	2	224
30	19	33.69122 07222 7026	2	284
30	20	37.78117 59363 7412	2	3580
30	21	42.22218 88741 5065	2	4440
30	22	47.05567 82251 9296	2	55200
30	23	52.33567 93797 4686	2	68400
30	24	58.13511 50396 8783	2	84800
30	25	64.55692 28253 1261	2	1 04000
30	26	71.75574 28866 9527	2	12900
30	27	79.98619 96989 1509	2	16000
30	28	89.73509 12115 8820	2	2020
30	29	102.25357 28564 7799	2	260

Legend:

$r_{n,l}$: the lth root of the n-th order polynomial, W_n. The estimated error bound is

$$| r_{n,l} - \text{root} | \leqq e_{n,l} \times 10^{-14}$$

The significance of the numbers in column C is:

0: the root was computed explicitly
1: the root was computed as $r_{n,l} = 10/r_{n,l}$
2: the root was computed as $r_{n,l} = 20/\bar{r}_{n,l}$

The formulae of equation (4) can be verified by induction based on the actual formal construction of successive derivatives of exp $(-\epsilon/T)$ [1]. After these relations have been established they can be used with a simple induction argument to establish the correctness of explicit formulae which are convenient for other purposes, albeit not for calculations

$$b_{n,l} = \frac{(-1)^{n-l} l (l+1)^2 \cdots (n-1)^2 n}{(n-l)!}. \tag{6}$$

The roots were located by the method of false position [2], which depends upon evaluation of the polynomials for successive approximations to the roots. The evaluations were computed by synthetic division using double precision arithmetic on the ORDVAC computer at the Aberdeen Proving Ground. The polynomials were calculated in one of three forms as described in the legend to Table II. This was necessary since the original form of the polynomial was relatively insensitive to changes in approximations for certain roots; that is, the remainder was too small

TABLE III. *Symmetric Function Check on Roots*

n	$b_{n,n-1}$	d_n
2	2	0
3	6	0
4	12	− 1
5	20	− 4
6	30	− 10
7	42	+ 11
8	56	− 6
9	72	+ 1
10	90	− 28
11	110	+ 5
12	132	+ 20
13	156	+ 47
14	182	+ 94
15	210	− 371
16	240	− 1263
17	272	−13059
18	306	+21289
19	342	+ 927
20	380	+ 27
21	420	− 6267
22	462	+39096
23	506	− 2411
24	552	−55267
25	600	− 5915
26	650	+23010
27	702	+ 1762
28	756	+ 223
29	812	− 4329
30	870	−18880

Legend: $b_{n,n-1} - \sum_{i=1}^{n} r_{n,i} = d_n \times 10^{-14}$

compared with the largest term in the synthetic division schema. When this occurred, one of the other forms of the polynomial was used in order to define the root more accurately.

The error estimates of Table II were based upon locating the root between two approximate values which gave remainders of opposite sign. Although this method of locating roots is inherently self-checking, a further numerical test was made by computing the symmetric function summations of Table III.

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2. F. B. HILDEBRAND, *Introduction to Numerical Analysis*, McGraw-Hill Book Co., New York, 1956, p. 446.