TECHNICAL NOTES AND SHORT PAPERS

Table of Integers Not Exceeding 10 00000 That Are Not Expressible as the Sum of Four Tetrahedral Numbers

By Herbert E. Salzer and Norman Levine

For the past 21 years one of the authors has been concerned with empirical theorems expressing certain classes of positive integers as the sum of four tetrahedral numbers, or "tetrahedrals" $T_n \equiv n(n+1)(n+2)/6$, $n \geq 0$, the results being contained in short notes and abstracts, as well as unpublished tables [1]–[9]. We use Σ_p to denote a sum of p T_n's. Among the more interesting past findings were: every square $\leq$ 10 00000 a Σ_4 [7], [8], every $T_n \leq$ 10 00000 a Σ_4 other than the trivial decomposition $T_n = T_n + 0 + 0 + 0$ [3], [4], every multiple of 5 $\leq$ 1 00000 a Σ_4 [9] (see also [11]), and verification of Pollock's conjecture that every integer is a Σ_5 for the first 20000 integers [5], [10]. Incidentally all investigations in [1]–[9] were done by hand, employing at the most a desk calculator.

The exceptional numbers, which by definition are those not expressible as Σ_4, were tabulated previously only up to 2000. But even that far interesting features turned up, such as 1314 being the only exceptional number ending in 4, and the very few ending in 1 or 9 [4]. Then for numbers ending in 6 and $\leq$ 20000, 6186 turned out to be the only exceptional one [6]. Also there was no striking difference between the density of exceptional numbers in the first thousand and in the second thousand brackets, decreasing from around $4\frac{1}{2}\%$ to 3%, so that it was interesting to speculate upon the approximate density in the neighborhood of say 10 00000. Then finally, the verification of the conjectures that every integer is a Σ_5 and that every integer $m = 5r$ is a Σ_4 for the first 20000 cases, while every $10r + 6$, $r = 0, 1, 2, \cdots, 617$, is a Σ_4 until 6186, posed the question as to whether an exception might occur to either of the former two empirical theorems even after verification in those first 20000 cases. Thus it was felt that tabulation of the exceptional numbers $\leq$ 10 00000 would afford a much clearer picture as to their distribution and density, as well as stronger evidence for the truth of Pollock's conjecture and the author's (and Richmond's [11]) conjecture that every $m = 5r$ is a Σ_4.

This table of exceptional numbers $\leq$ 10 43999 presents a great surprise in its picture of their distribution which is entirely different from that envisioned from those $\leq$ 2000 [4]. Most strikingly unexpected is the decrease in the density of exceptional numbers from several percent in the neighborhood of 2000 to what appears to be practically zero near 10 00000. The scarcity of exceptional numbers in the higher ranges is in accordance with Hua's result that "almost all" positive integers are expressible as a Σ_4 [15], [16]. But even more, this table shows that the likelihood of a given number m being exceptional falls off so rapidly with increasing m that it appears to be a plausible conjecture that there might be some m_0 sufficiently large such that *every* $m > m_0$ is a Σ_4 (a conjecture unlikely to suggest itself from the exceptional numbers among only the first few thousand).

Received 14 January 1957.

Table of Exceptional Numbers $\leq$ 10 00000

17	1227	3183	9772	29157
27	1233	3218	9973	29487
33	1243	3263	10397	29938
52	1314	3463	10467	30298
73	1382	3512	10532	31973
82	1402	3887	10633	33183
83	1468	4003	10852	36262
103	1478	4307	11237	36913
107	1513	4317	11302	37798
137	1523	4563	11737	38453
153	1578	4832	11962	38707
162	1612	4923	12247	38807
217	1622	5013	12547	39693
219	1658	5142	12722	39913
227	1678	5238	12777	41278
237	1693	5283	12843	41322
247	1731	5483	12858	41433
258	1738	5508	13127	44833
268	1742	5538	13393	47627
271	1758	5563	13822	48043
282	1767	5618	14492	56467
283	1803	5647	15122	56842
302	1858	5707	15483	58613
303	1907	6022	15867	59077
313	1923	6057	16097	62158
358	1933	6067	16538	64752
383	2037	6186	16637	65253
432	2053	6213	16742	65567
437	2172	6263	17253	71157
443	2198	6343	17683	74687
447	2217	6462	17813	78003
502	2218	6863	17893	78787
548	2251	7067	18573	83603
557	2253	7278	18782	84023
558	2327	7377	19168	85993
647	2372	7387	19277	91128
662	2382	7423	20918	1 06277
667	2417	7497	21523	1 13062
709	2437	7542	22618	1 34038
713	2457	7662	22657	1 48437
718	2537	7793	23677	3 43867
722	2538	7873	24237	
842	2578	8223	24317	
863	2687	8307	24338	
898	2818	8322	25447	
953	2858	8973	25723	
1007	2898	9063	26007	
1117	2973	9488	27858	
1118	3138	9687	28617	
1153	3142	9753	28847	

A quick glance over this table is sufficient to verify Pollock's conjecture for each of the first million integers. Furthermore, from the absence of any exceptional numbers in the range 3 43867 to 10 00000 and from an upper bound to the magnitude of the first differences of tetrahedral numbers $\leq$ 2500 00000, it is easily shown that for any number m between 10 00000 and 2500 00000, it is always possible to find a T_n such that 3 $43867 < m - T_n < 10$ 00000. Thus *every* $m \leq 2500\ 00000$ *is a* Σ_5. (This fact and the extreme rarity of large exceptional numbers makes the empirical theorem of Pollock that every integer is an Σ_5 as overwhelmingly certain as one can be short of actual proof.) This table verifies that every $m = 5r$ is a Σ_4 for the first 2 00000 values of r, making it also a very plausible empirical theorem. In the class of the first 1 00000 values of both $m = 10r + 4$ and $m = 10r + 6$, 1314 and 6186 respectively are the sole exceptional numbers. Among the first 1 00000 values of $m = 10r + 1$, only 271, 1731 and 2251 are exceptional. Among the first 1 00000 values of $m = 10r + 9$, only 219 and 709 are exceptional. All exceptional numbers m, where $6186 < m < 10\ 00000$, are here seen to be of the form $10r + 2$, $10r + 3$, $10r + 7$ or $10r + 8$, and that appears to be another plausible conjecture for every exceptional $m > 6186$.

The method of computation was to obtain $\binom{i+2}{3}+\binom{j+2}{3}+\binom{k+2}{3}+\binom{l+2}{3}$ for every $i, j, k, l \geq 0$ until one found for every $m \leq 10\ 43999$ either a representation as a Σ_4 or that that m was exceptional. The calculation was begun upon the Univac Scientific Computer (ERA 1103) at the Convair Digital Computing Laboratory, the initial part re-run, and then those results were checked and continued upon the IBM 704 Digital Computer. At the start upon the ERA 1103, 500 words of high speed storage with 36 binary bits in each word permitted the investigation of 18000 numbers at a time. Then the IBM 704 had at its disposal 3000 words of high speed memory, each of the 36 binary bits in a word representing a number, so that 1 08000 numbers could be investigated at one time. The 1 08000 binary bits were filled with 1's at the start, and a 0 introduced into the binary position of the word which represented a non-exceptional m. After all possible combinations of i, j, k, l had been exhausted, each of the 3000 words was searched for binary bits that remained 1 and the exceptional numbers m corresponding to those bits were printed out. In choosing combinations of i, j, k, l, repetitions due to symmetry were avoided, as well as combinations yielding an m that was either too large or too small for the group of 1 08000 numbers under consideration.

Those interested in actual mathematical proofs (which appear to be rather involved) may consult Dickson [12], [13] for earlier work, and Watson [14] for the sharpest results to date. It is rather amazing that the proved p in $m = \Sigma_p$ for every m is $p = 8$, and no better than $p = 8$ for arbitrarily large m, while the actual p (according to the evidence in this table) may be only 5 for every m, and 4 for sufficiently large m, less by 3 and 4 respectively. Considering the difficulty of the existing proof for $p = 8$ [14], one may well wonder, should $p = 5$ or $p = 4$ be the truly minimum values, for every m, and m sufficiently large, respectively, how long the world must wait and how difficult and sharp the mathematical tools must be, until the desired proofs would be found.

References [15] and [16] were called to the author's attention by K. A. Hirsch.

Convair Astronautics
San Diego, California

1. H. E. SALZER, "Tetrahedral numbers," *MTAC*, v. 1, 1943, p. 95.

2. ——, "Table of first two hundred squares expressed as a sum of four tetrahedral numbers," Amer. Math. Soc., *Bull.*, v. 49, 1943, p. 688.

3. ——, "New tables and facts involving sums of four tetrahedral numbers," Amer. Math. Soc., *Bull.*, v. 50, 1944, p. 55.

4. ——, "On numbers expressible as the sum of four tetrahedral numbers," London Math. Soc., *Jn.*, v. 20, 1945, p. 3–4. This note includes a table of the exceptional numbers ≤ 1000 (erratum: omission of 107). That table is continued up to 2000 in *MTAC*, v. 3, 1949, p. 423–424.

5. ——, "Further empirical results on tetrahedral numbers," Amer. Math. Soc., *Bull.*, v. 52, 1946, p. 420.

6. ——, "An 'empirical theorem' which is true for the first 618 cases, but fails in the 619th," Amer. Math. Soc., *Bull.*, v. 53, 1947, p. 908 (errata on p. 1196).

7. ——, "Table expressing every square up to one million as a sum of four non-negative tetrahedral numbers," Amer. Math. Soc., *Bull.*, v. 54, 1948, p. 830.

8. ——, "Representation table for squares as sums of four tetrahedral numbers," *MTAC*, v. 3, 1948, p. 316.

9. ——, "Verification of the first twenty thousand cases of an empirical theorem with the aid of a device for mass computation," Amer. Math. Soc., *Bull.*, v. 55, 1949, p. 41.

Empirical Theorems of Other Authors:

10. F. POLLOCK, "On the extension of the principle of Fermat's Theorem of the polygonal numbers to the higher orders of series whose ultimate differences are constant. With a new theorem proposed, applicable to all the orders," Roy. Soc. London, *Proc.*, S A, v. 5, 1850, p. 922–924.

11. H. W. RICHMOND, "Notes on a problem of the 'Waring' type," London Math. Soc., *Jn.*, v. 19, 1944, p. 38–41.

Mathematical Papers:

12. L. E. DICKSON, *History of the Theory of Numbers*, v. 2, *Diophantine Analysis*, Carnegie Institute of Washington, publication no. 256, v. II (reprinted by G. E. Stechert, N. Y., 1934), p. iv–v and Chapter I, especially p. 4, 7, 22–23, 25, 39.

13. L. E. DICKSON, *Modern Elementary Theory of Numbers*, University of Chicago Press, Chicago, 1939, Chap. VII, "Sums of nine values of a cubic function," p. 130–146. This contains the proof of the theorem that every integer is the sum of nine tetrahedrals, with further references to the most advanced work up to that time, that of R. D. JAMES and L. K. HUA on the theorem that every sufficiently large integer is the sum of eight tetrahedrals.

14. G. L. WATSON, "Sums of eight values of a cubic polynomial," London Math. Soc., *Jn.*, v. 27, 1952, p. 217–224. This includes the proof of the theorem that every integer is the sum of eight tetrahedrals.

15. L. K. HUA, "Sur le problème de Waring relatif à un polynome du troisième degré," *Comptes Rendus*, Academie des Sciences, Paris, v. 210, 1940, p. 650–652.

16. L. K. HUA, "On Waring's problem with cubic polynomial summands," *Sci. Report Nat. Tsing Hua Univ.*, (A) v. 4, 1940, p. 55–83.

GROUPS OF PRIMES HAVING MAXIMUM DENSITY

By John Leech

The following lists give groups of six or more primes which minimize the difference between first and last, the lists being complete for the range 50 to 100 00000. Four numbers out of nine can be prime, such as 191, 193, 197, 199. There are 897 such groups of four in the range. Five numbers out of thirteen can be prime; there are 318 such groups in the range. Six numbers out of seventeen can be prime, such as 97, 101, 103, 107, 109, 113; there are seventeen such groups in the range, centered on:

Received February 7, 1955. Due to misfiling in the *MTAC* office, this paper is appearing later than was scheduled; see *MTAC*, Review 110, v. 11, 1957, p. 274. Some of the results have meanwhile appeared in "On a generalization of the prime pair problem," by Herschel F. Smith, *MTAC*, v. 11, 1957, p. 274.