Two Theorems on Inverses of Finite Segments of the Generalized Hilbert Matrix

By Richard B. Smith

1. Introduction. A quick check on the inverse S_n of finite segments of the generalized Hilbert matrix $H_n = (h_{ij})$, $h_{ij} = 1/(p + i + j - 1)$, $i, j = 1, 2, \cdots, n$, $p \neq -1, -2, \cdots, -(2n - 1)$, may be made by summation of the n^2 elements of the inverse. The summation requires complete accuracy.

2. Inversion of the Matrix. The inverse of this matrix has been derived by Savage and Lukacs [1] for $p = 0$ and by Collar [2] for nonnegative integer values of p. By using the method employed by Savage and Lukacs which is based on a formula in [4], the element in the ith row and jth column of S_n is

$$(1) \qquad S_n^{ij} = \frac{(-1)^{i+j}}{p + i + j - 1}\left[\frac{\prod_{k=0}^{n-1}(p + i + k)(p + j + k)}{(i-1)!(n-i)!(j-1)!(n-j)!}\right]$$

where $p \neq -1, -2, \cdots, -(2n - 1)$.

3. Summation Identity. Let binomial coefficients of the form $C_s^r = 0$ for $s < 0$ and $s > r$. Then

$$(2) \qquad \sum_{j=0}^{n} (-1)^j C_j^n C_{j+i}^{j+n} = (-1)^n C_{n+i}^n$$

follows from formula (27) in [5].*

4. Theorem I. Let S_n^{ij} be defined by (1), then

$$(3) \qquad \sum_{i,j=1}^{n} S_n^{ij} = n(p + n)$$

where $p \neq -1, -2, \cdots, -(2n - 1)$.

When $n = 1$ or 2 equation (3) is easily verified. By assuming (3) for n it remains to be shown that

$$\sum_{i,j=1}^{n+1} S_{n+1}^{ij} = (n + 1)(p + n + 1)$$
$$= n(p + n) + p + 2n + 1, \qquad \text{or}$$

$$(4) \qquad \sum_{i,j=1}^{n+1} S_{n+1}^{ij} - \sum_{i,j=1}^{n} S_n^{ij} = p + 2n + 1$$

where $p \neq -1, -2, \cdots, -[2(n + 1) - 1]$. By substitution of (1) for S_{n+1}^{ij} and S_n^{ij} in (4) we have

---

Received March 4, 1957; revised April 7, 1958.

* The author is indebted to the referee for this reference.

$$\sum_{i,j=1}^{n} \frac{(-1)^{i+j}}{p+i+j-1} \frac{\prod_{k=0}^{n-1}(p+i+k)(p+j+k)}{(i-1)!(n-i)!(j-1)!(n-j)!} \cdot \left[\frac{(p+n+i)(p+n+j)}{(n+1-i)(n+1-j)} - 1\right]$$

$$+ 2(-1)^{n+1} \frac{\prod_{k=1}^{n+1}(p+n+k)}{n!} \sum_{i=1}^{n} (-1)^i \frac{\prod_{k=0}^{n-1}(p+i+k)}{(i-1)!(n+1-i)!}$$

$$+ (p+2n+1)\left[\frac{\prod_{k=1}^{n}(p+n+k)}{n!}\right]^2 = (p+2n+1) \cdot \left[\sum_{i=1}^{n}(-1)^i \frac{\prod_{k=0}^{n-1}(p+i+k)}{(i-1)!(n+1-i)!} + (-1)^{n+1}\frac{\prod_{k=1}^{n}(p+n+k)}{n!}\right]^2.$$

Let $i - 1 = j$ and this becomes

$$(p+2n+1)\left[\sum_{j=0}^{n}(-1)^j \frac{\prod_{k=1}^{n}(p+j+k)}{j!(n-j)!}\right]^2. \tag{5}$$

Take the sum from (5) as the coefficient of x^{p+j} and consider

$$f(x) = \sum_{j=0}^{n}(-1)^j \frac{\prod_{k=1}^{n}(p+j+k)}{j!(n-j)!} x^{p+j} = \sum_{j=0}^{n}(-1)^j \frac{D^{(n)}x^{p+j+n}}{j!(n-j)!},$$

where $D = d/dx$. Differentiating x^{p+j+n} as $x^p \cdot x^{j+n}$ we have

$$f(x) = \sum_{j=0}^{n}(-1)^j \frac{1}{j!(n-j)!}\sum_{i=0}^{n} C_i^n\{D^{(i)}x^p\}\{D^{(n-i)}x^{j+n}\}$$

$$= \sum_{i=0}^{n} C_i^n\{D^{(i)}x^p\}\sum_{j=0}^{n}(-1)^j \frac{1}{j!(n-j)!}\{D^{(n-i)}x^{j+n}\}$$

$$= x^p\sum_{j=0}^{n}(-1)^j\frac{(j+n)!}{j!j!(n-j)!}x^j + \sum_{i=1}^{n} C_i^n \prod_{k=0}^{i-1}(p-k)x^{p-i}\sum_{j=0}^{n}(-1)^j\frac{(j+n)!}{j!(n-j)!(j+i)!}x^{j+i}.$$

Let $x = 1$. Then

$$f(1) = \sum_{j=0}^{n}(-1)^j C_j^n C_j^{j+n} + \sum_{i=1}^{n}\frac{\prod_{k=0}^{i-1}(p-k)}{i!}\sum_{j=0}^{n}(-1)^j C_j^n C_{j+i}^{j+n},$$

and considering (2) we have $f(1) = (-1)^n$. Thus (5) becomes $p + 2n + 1$.

5. Theorem II. Let S_n^{ij} be defined by (1), then

$$\sum_{i=1}^{n} S_n^{ij} = \sum_{i=1}^{n} S_n^{ji} = (-1)^{n+j} \frac{\prod_{k=0}^{n-1} (p + j + k)}{(j-1)!(n-j)!} \tag{6}$$

where $j = 1, 2, \cdots, n$ and $p \neq -1, -2, \cdots, -(2n-1)$.

The proof is similar to the one given for Theorem I.

Westinghouse Electric Corporation
Bettis Atomic Power Division
Pittsburgh, Pennsylvania

1. R. Savage & E. Lukacs, "Tables of Inverses of Finite Segments of the Hilbert Matrix," *Contributions to the Solution of Systems of Linear Equations and the Determination of Eigenvalues*, National Bureau of Standards, Applied Mathematics Series 39, 1954, p. 105–108.

2. A. R. Collar, "On the Reciprocation of Certain Matrices," Royal Soc. Edinburgh, *Proc.*, v. 59, 1939, p. 195–206.

3. A. R. Collar, "On the Reciprocal of a Segment of a Generalized Hilbert Matrix," Cambridge Phil. Soc., *Proc.*, v. 47, 1951, p. 11–17.

4. G. Pólya & G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, v. 2, Springer-Verlag, Berlin, 1954, p. 98.

5. E. Netto, *Lehrbuch der Combinatorik*, B. G. Teubner, Leipzig, 1901, p. 252.