Numerical Integration Formulas for use with Weight Functions x^2 and $x/\sqrt{1-x^2}$

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1. Introduction. Numerical integration formulas of the type

$$\int_{-1}^{1} f(x)\,dx \simeq C_n \sum_{i=1}^{n} f(x_{i,n}) \tag{1}$$

were investigated by Chebyshev [1]. It will be noted in correspondence (1) that all values of f used on the right hand side are given the same weight. This is advantageous when the values of f represent experimental measurements. Salzer [2] has given tables of $\{x_{i,n}\}$ for $n = 1, 2, 3, \cdots, 7$ and $n = 9$. Other values of n require values of $x_{i,n}$ outside of the interval $(-1, 1)$.

Chebyshev also investigated numerical integration formulas of the type

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} f(x)\,dx \simeq C_n \sum_{i=1}^{n} f(x_{i,n}). \tag{2}$$

Greenwood and Danford [3] investigated numerical integration formulas of the type

$$\int_{0}^{1} xf(x)\,dx \simeq C_n \sum_{i=1}^{n} f(x_{i,n}) \tag{3}$$

$$\int_{-1}^{1} xf(x)\,dx \simeq k_m \sum_{i=1}^{m} [f(x_{i,m}) - f(-x_{i,m})]. \tag{4}$$

These would appear to be of use when the first moment of a distribution is needed. The known usable cases for (3) are few in number, $n = 1, 2, 3$. Correspondence (4) may be used for $m = 1, 2, 3, 4$.

The usual procedure for getting as much "accuracy" as possible into the correspondences is to require that they be exact for as many of the functions 1, x, $x^2, \cdots, x^k, \cdots$ as possible. For correspondences (1), (2), (3), the requirement of equality for the function $f(x) = 1$ yields a value for C_n. Simple arguments suggest that if correspondences (1) and (3) are to be exact for polynomials of degree n, (i.e. linear combinations of $1, x, x^2, \cdots, x^n$), then there must be n abscissae used. Correspondence (4) requires a different approach, one such approach using continued fractions was suggested by Chebyshev [1]. Since the same abscissa value is used twice in correspondence (4), simple arguments suggest that correspondence (4) can be made exact for polynomials of degree $2m$.

2. Weight Function $x/\sqrt{1-x^2}$. Numerical integration formulas of the type

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} f(x)\,dx \simeq k_m \sum_{i=1}^{m} [f(x_{i,m}) - f(-x_{i,m})] \tag{5}$$

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TABLE 1. *Weights and Abscissae for* $x/\sqrt{1 - x^2}$

$m = 1$	$k_1 = 0.9068\ 9968$	$x_1 = 0.8660\ 2540$	$s_1 = 60°$	0′	0″
$m = 2$	${}_1k_2 = 0.4767\ 8538$	$x_1 = 0.6691\ 3061$	$s_1 = 42°$	0′	0″
		$x_2 = 0.9781\ 4760$	$s_2 = 78°$	0′	0″
	${}_2k_2 = 0.7714\ 5495$	$x_1 = 0.1045\ 2846$	$s_1 = 6°$	0′	0″
		$x_2 = 0.9135\ 4546$	$s_2 = 66°$	0′	0″
$m = 3$	${}_1k_3 = 0.3233\ 1355$	$x_1 = 0.5583\ 7888$	$s_1 = 33°$	56′	37.54″
		$x_2 = 0.8915\ 0871$	$s_2 = 63°$	3′	48.41″
		$x_3 = 0.9793\ 2755$	$s_3 = 78°$	19′	46.67″
	${}_2k_3 = 0.4369\ 1949$	$x_1 = 0.0754\ 1781$	$s_1 = 4°$	19′	30.83″
		$x_2 = 0.7439\ 8645$	$s_2 = 48°$	4′	19.62″
		$x_3 = 0.9781\ 7670$	$s_3 = 78°$	0′	28.88″
	${}_3k_3 = 0.6522\ 1071$	$x_1 = -0.1530\ 9943$	$s_1 = -8°$	48′	23.71″
		$x_2 = 0.4156\ 7465$	$s_2 = 24°$	33′	42.52″
		$x_3 = 0.9416\ 3401$	$s_3 = 70°$	19′	40.09″
	${}_4k_3 = 0.9613\ 1418$	$x_1 = -0.7896\ 4766$	$s_1 = -52°$	9′	9.35″
		$x_2 = 0.6842\ 8078$	$s_2 = 43°$	10′	44.65″
		$x_3 = 0.9223\ 7153$	$s_3 = 67°$	16′	31.02″
$m = 4$	${}_1k_4 = 0.2445\ 2340$	$x_1 = 0.4869\ 8884$	$s_1 = 29°$	8′	34.29″
		$x_2 = 0.8177\ 1937$	$s_2 = 54°$	51′	25.71″
		$x_3 = 0.9073\ 5870$	$s_3 = 65°$	8′	34.29″
		$x_4 = 0.9998\ 8810$	$s_4 = 89°$	8′	34.29″
	${}_2k_4 = 0.3049\ 1569$	$x_1 = 0.0598\ 0415$	$s_1 = 3°$	25′	42.86″
		$x_2 = 0.6351\ 1577$	$s_2 = 39°$	25′	42.86″
		$x_3 = 0.8943\ 7741$	$s_3 = 63°$	25′	42.86″
		$x_4 = 0.9864\ 9059$	$s_4 = 80°$	34′	17.14″
	${}_3k_4 = 0.3956\ 4717$	$x_1 = -0.1193\ 9422$	$s_1 = -6°$	51′	25.71″
		$x_2 = 0.3232\ 0966$	$s_2 = 18°$	51′	25.71″
		$x_3 = 0.8001\ 3355$	$s_3 = 53°$	8′	34.29″
		$x_4 = 0.9811\ 4838$	$s_4 = 78°$	51′	25.71″
	${}_4k_4 = 0.4933\ 6396$	$x_1 = -0.5383\ 5061$	$s_1 = -32°$	34′	17.14″
		$x_2 = 0.4606\ 4244$	$s_2 = 27°$	25′	42.86″
		$x_3 = 0.7017\ 9790$	$s_3 = 44°$	34′	17.14″
		$x_4 = 0.9678\ 3475$	$s_4 = 75°$	25′	42.86″
	${}_5k_4 = 0.5494\ 3910$	$x_1 = -0.3792\ 2546$	$s_1 = -22°$	17′	8.57″
		$x_2 = 0.2370\ 8038$	$s_2 = 13°$	42′	51.43″
		$x_3 = 0.6117\ 2430$	$s_3 = 37°$	42′	51.43″
		$x_4 = 0.9598\ 7524$	$s_4 = 73°$	42′	51.43″
	${}_6k_4 = 0.8890\ 1113$	$x_1 = -0.8506\ 8007$	$s_1 = -58°$	17′	8.57″
		$x_2 = 0.0299\ 1547$	$s_2 = 1°$	42′	51.43″
		$x_3 = 0.7628\ 2957$	$s_3 = 49°$	42′	51.43″
		$x_4 = 0.9413\ 8647$	$s_4 = 70°$	17′	8.57″

were studied by one of the co-authors of this paper in connection with the preparation of his master's thesis [4]. The substitution $x = \sin s$ changes correspondence (5) to the form

$$(6) \qquad \int_{-\pi/2}^{\pi/2} (\sin s) f(\sin s)\, ds \simeq k_m \sum_{i=1}^{m} [f(\sin s_{i,m}) - f(-\sin s_{i,m})].$$

A fairly sophisticated method for evaluation of the $x_{i,m}$ was used by Chebyshev [1]; that method was followed by the authors of this paper.

All calculations were performed on desk machines and carried to ten decimal places and then rounded off to eight decimal places. The results obtained are tabulated in Table 1. All values of $x_{i,m}$ and $s_{i,m}$ are tabulated with the corresponding value of k_m and hence with the corresponding value of m. In some cases more than one value of k_m was found. In some places certain subscripts not needed for clarity have been dropped. When more than one value of k was found a prescript was added to k_m.

The similarities in the $s_{i,4}$ values in the minutes and seconds columns is rather odd.

TABLE 2. *Weights and Abscissae for* x^2

$n = 1$	$c = 0.66666\ 66667$	$x_1 = 0.00000\ 00000$
$n = 2$	$c = 0.33333\ 33333$	$x_1 = 0.77459\ 66692 = -x_2$
$n = 3$	$c = 0.22222\ 22222$	$x_1 = 0.94868\ 32981 = -x_3$
		$x_2 = 0$
$n = 4$	$c = 0.16666\ 66667$	$x_1 = 0.92836\ 49436 = -x_4$
		$x_2 = 0.58149\ 68029 = -x_3$
$n = 6$	$c = 0.11111\ 11111$	$x_1 = 0.94100\ 74697 = -x_6$
		$x_2 = 0.81334\ 08269 = -x_5$
		$x_3 = 0.25298\ 16413 = -x_4$

The derivation suggests that the numerical integration correspondence relation (5) should be exact for a polynomial of degree $2m + 1$ or less. Indeed, since the trivial identity $0 = 0$ results when $f(x)$ is chosen as an even function, one can consider the set of $m + 1$ functions $f_1(x) = x, f_3(x) = x^3, \cdots, f_{2m+1}(x) = x^{2m+1}$ and require that correspondence (5) be exact for this set of $m + 1$ functions. Thus the set of equations for $j = 0, 1, 2, \cdots, m$

$$\begin{aligned} \int_{-1}^{1} \frac{x}{\sqrt{1-x^2}}\, x^{2j+1}\, dx &= k_m \left[\sum_{i=1}^{m} x_{i,m}^{2j+1} - \sum_{i=1}^{m} (-x_{i,m})^{2j+1} \right] \\ &= 2k_m \left[\sum_{i=1}^{m} x_{i,m}^{2j+1} \right] \end{aligned} \tag{7}$$

could have been used to determine the parameters k_m and $x_{i,m}$, $i = 1, 2, \cdots, m$. However, this set of simultaneous non-linear equations is much more difficult to solve than the Chebyshev equations. Equations (7) may be used as a check set, and indeed the data of table 1 were checked by this method. Except for slight discrepancies of three or less units in the eighth decimal place the check equations above were verified. Such discrepancies are well within the possible round-off error.

3. Weight Function x^2. Numerical integration formulas of the type

$$\int_{-1}^{1} x^2 f(x)\, dx \simeq C_n \sum_{i=1}^{n} f(x_{i,n}) \tag{8}$$

were recently studied by another co-author of this paper in connection with the preparation of his master's thesis [5]. Again, a method suggested by Chebyshev [1] was used to outline the steps in the computation. Chebyshev's method required that the roots of an nth degree polynomial be obtained. A previously prepared program for determining roots of a polynomial was used on an automatic digital computer to get seven decimal place values for the roots. Hand methods and Newton's iteration were then used to get ten or eleven decimal place values. These values were rounded off to ten places, and are given in Table 2.

The polynomials for $n = 5, 7, 8$ had at least one pair of imaginary roots. Numerical quadrature formulas using imaginary abscissas are of doubtful utility and hence the roots are not tabulated for these values of n.

The derivation suggests that the numerical integration correspondence relation (8) should be exact for a polynomial of degree n or less. Consider, then, the set of $n + 1$ functions $f_0(x) = 1, f_1(x) = x, f_2(x) = x^2, \cdots, f_n(x) = x^n$, and require that correspondence (8) be exact for each of these $(n + 1)$ functions. The resulting equation for $f_0(x) = 1$ determines the value $C_n = 2/(3n)$. The other equations are of the type

$$\int_{-1}^{1} x^2 \cdot x^j \, dx = C_n \sum_{i=1}^{n} x_{i,n}^j \tag{9}$$

$j = 1, 2, \cdots, n$. With the value of C_n given above, these n nonlinear simultaneous equations could have been used to determine the n abscissas $x_1, x_2, \cdots, x_n$. In practice, equations (9) were used as a check set on the values obtained following the Chebyshev procedure. Except for slight discrepancies of two or less units in the tenth decimal place the check equations were all verified. Such discrepancies are well within the possible round-off error.

4. Conclusions. Comparison between the Newton-Cotes rule for numerical integration of $(x/\sqrt{1 - x^2})\, f(x)$ on $(-1, 1)$ and the Chebyshev method presented in **2** for $f(x)$ suggest that the Chebyshev method is faster and "more accurate" for a reasonably wide class of functions $f(x)$. Of course, this wide class of functions does not include $f(x) = (\sqrt{(1 - x^2)}/x)$ (polynomial in x) for which the Newton-Cotes method could be expected to be faster and more accurate.

Similar statements could be made when the weight function x^2 is used. Indeed, these numerical integration methods apply best when the value of the integral

$$\int_{-1}^{1} w(x) f(x) \, dx$$

is desired and when $f(x)$ is given empirically and where $w(x)$ is the given weight function. Thus, the abscissas given in table 2 would be quite useful in empirical determinations of second moments.

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1. P. L. Chebyshev, "Sur les Quadratures," *JOURNAL de MATHEMATIQUES PURES ET APPLIQUE* Series 2, vol. 19 (1874), p. 19–34, or *OEUVRES*, vol. II, pp. 165–180. (St. Petersberg, 1907). Many errors in the JOURNAL article have been corrected in the OEUVRES, so that when a choice is possible it is suggested that OEUVRES be used as a reference. (The Library of Congress has recently standardized the transliteration into Latin characters from Cyrillic characters. The above version Chebyshev would appear to be their preferred spelling.) The referee reports that Chebyshev's papers were reprinted in Russian in 1948, and that the above appears in Volume II of his works, pp. 49–62.
2. Herbert E. Salzer, "Tables for Facilitating Use of Chebyshev's Quadrature Formula," *Jn. Math. and Physics*, v. 26, 1947, p. 191–194.
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