Reviews and Descriptions of Tables and Books

1[A, B, C, D, F].—Tsuneta Yano, *Yano's Tables of Calculation*, The Tsuneta Yano Memorial Society, Dai-ichi-Seimei-Kan, Yurakucho, Tokyo, Japan, 1952, vi + 162 p., 18 cm. Price $2.50 + postage.

This handbook for computation contains tables of multiplication and division, reciprocals, powers and roots, trigonometrical functions, factoring, testing primality, logarithms, antilogarithms, cologarithms, conversion between common and natural logarithms, lists of formulas, and a conversion table of weights and measures. In addition to a preface written by his son, roughly half the book is devoted to an explanation of the functions and the use of the tables.

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2[B].—Arne Olander, 9^{9^9}, *Elementa*, v. 41, 1958, p. 202. (Swedish).

The author reports that there are exactly 3696 93100 significant decimal digits in this "largest number constructible conventionally with three digits"; the first twenty-two and last ten of these digits are given.

C. B. T.

Editorial Note: More extensive information regarding this number has appeared in MTAC, vol. II, p. 93–94, 224–225 (Notes 54 and 66, respectively).

3[F].—D. H. Lehmer, "Tables concerning the distribution of primes up to 37 millions," 1957, 17 p., 8½″ x 11″, mimeographed. Deposited in the UMT File.

These interesting tables concern the distribution functions $\pi(x)$, $\pi_k(x)$ and $\pi_{2,k}(x)$.

Let $\pi(x)$ be the number of primes $\leqq x$. This is listed for $x = 10^6(10^6)37\cdot 10^6$ together with $li(x) = \int_2^x (\log x)^{-1}\, dx$ and the difference and ratio (to 5D) of these functions. The difference ranges from 122 at $2\cdot 10^6$ up to 630 at $37\cdot 10^6$, and the ratio ranges from 1.00166 at 10^6 down to 1.00014 at $31\cdot 10^6$.

Let $\pi_k(x)$ be the number of primes $p \leqq x$ for which $p + k$ is also a prime (thus $\pi_2(x)$ is the number of pairs of twin primes $\leqq x$, $x + 2$). This is listed for $x = 10^6(10^6)37\cdot 10^6$ and $k = 2(2)70$ together with an appropriate multiple (depending on k) of $li_2(x) = \int_2^x (\log x)^{-2}\, dx$ (cf. [1]). The ratio (to 4D) is listed for $x = 10^6$, $5\cdot 10^6(5\cdot 10^6)35\cdot 10^6$ for each k. Also listed (to 4D) is $\pi_k(x)/\pi_2(x)$ for $x = 10^6$, $5\cdot 10^6(5\cdot 10^6)30\cdot 10^6$ for each k together with the expected (rational) ratio.

Let $\pi_{2,k}(x)$ be the number of primes $p \leqq x$ for which $p + 2$ and $p + k$ are also primes. This is listed for $x = 10^6(10^6)37\cdot 10^6$ and $k = 6(2)72$ with $3 \nmid k + 2$, together with a multiple (depending on k) of $li_3(x) = \int_2^x (\log x)^{-3}\, dx$. The ratio of these functions and the ratio $\pi_{2,k}(x)/\pi_{2,6}(x)$ are listed as in the case of $\pi_k(x)$.

Finally there is listed the least prime p_n with given difference $\Delta = p_{n+1} - p_n$ for $\Delta = 2(2)156$, 160, 164, 170, 180, 182, 210. The 23 primes whose differences exceed those of all smaller primes are starred, difference 180 occurring for 170 51707 and difference 210 occurring for 208 31323. These entries extend the earlier list given by A. E. Western [2].

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1. G. H. Hardy & J. E. Littlewood, "Some problems of partitio numerorum III," *Acta Math.*, v. 44, 1923, p. 1–70.
2. A. E. Western, "Note on the magnitude of the difference between successive primes," London Math. Soc., *Jn.*, v. 9, 1934, p. 276–278.

4[L].—NBS Applied Mathematics Series, No. 51, *Tables of the Exponential Integral for Complex Arguments*, U. S. Government Printing Office, Washington, D. C., 1958, xiv + 634 p., 26.8 cm. Price $4.50.

The exponential integral used in this volume is the function $E_1(z)$ defined by the integral

$$E_1(z) = \int_z^\infty (e^{-u}/u)\, du$$

taken along any path from $u = z = x + iy$ to $u = \infty + 0i$ avoiding a cut along the negative real axis; on the branch cut itself $E_1(z)$ has been defined for $z = -x < 0$ as $-Ei(x) - i\pi$, where, with the usual interpretation as a principal value,

$$Ei(x) = \int_{-\infty}^{x} (e^{u}/u)\, du.$$

The function $E_1(z)$ has a logarithmic singularity at the origin, $E_1(z) + \log_e z$ being equal to the sum of a power series in z.

Over 80 per cent of the tables relate to the first quadrant, the remainder to the second. As is remarked in the Introduction, more work remains to be done to bring the tabulation in the second quadrant up to the state of that in the first. The fact that so massive a volume does not exhaust the tabulation of one single function well illustrates the difficulty of making a thoroughly adequate table of a function of a complex variable. As is especially desirable in the case of tables with complex argument, the Introduction explains methods of interpolation, both for the analytic functions tabulated and for the non-analytic function $e^{x}E_1(z)$ given in Table III.

As is usual with this series, the volume is a remarkably good value. It appears from the Preface that much the greater part of the work of computation was done by the New York organization under A. N. Lowan, presumably a decade or so ago.

Details of the functions tabulated follow. There are no differences.

Table I (p. 2–479). $E_1(z)$, 6D, $x = 0(.02)4$, $y = 0(.02)3(.05)10$.

Table II (p. 482–503). $E_1(z) + \log_e z$, 6D, $x = 0(.02)1$, $y = 0(.02)1$.

Table III (p. 506–587). $e^{x}E_1(z)$, 6D, $x = 4(.1)10$, $y = 0(.05)10$.

Table IV (p. 590–605). $E_1(z)$, 10D, $-x = 0(.1)3.1$, $y = 0(.1)3.1$;

$E_1(z) + \log_e z$, 10D, $-x + 0(.1)1$, $y = 0(.1)1$.

Table V (p. 608–633). $e^z E_1(z)$, 6D, $x = 0(1)20$, $y = 0(1)20$;
$-x = 0(.5)4.5$, $y = 0(.1)4(.5)10$;
$-x = 4.5(.5)10$, $y = 0(.5)10$;
$-x = 0(1)20$, $y = 0(1)20$.

Supplementary Table (p. 634). e^{-x}, 10D, $x = 4(.05)10$.

A. F.

5[L].—Akademia Nauk SSSR, *Tablit͡sy integral'noĭ pokazatel'noĭ funkt͡sii*, (Acad. Sci, USSR, *Tables of exponential integrals.*) Izdatel'stvo Akad. Nauk SSSR (Press of Acad. Sci, USSR), Moscow, 1954, 301 p., 27 cm. Price 27.75 rubles.

This volume belongs to the Mathematical Tables series of the Institute of Exact Mechanics and Computational Techniques.

The functions tabulated in the main table (pages 12–299) are the usual exponential integrals

$$Ei(x) = \int_{-\infty}^{x} e^t t^{-1}\, dt, \qquad -Ei(-x) = \int_{x}^{\infty} e^{-t} t^{-1}\, dt, \qquad x \geqq 0$$

the first integral being a principal value. Both integrals are tabulated to strictly $7S$ for $x = 0(.0001)1.3(.001)3(.0005)10(.1)15$. Leaving aside the single page of values for $x \geqq 10$ (where no differences are given), only on 21 pages is it necessary to provide either second differences for individual arguments or mean second differences for each column; over much the greater part of the table, linear interpolation suffices. Except for alternate arguments between $x = 3$ and $x = 10$, where the interval was halved by the Russian calculations, values to more figures may be found in NBS publications [1]. The Russian tables are said to be based on preliminary verification and correction of the NBS tables; the corrections are not stated, and it seems unlikely that many were found to be necessary.

On page 11 are tabulated the auxiliary functions $Ei(x) - ln\, x$ and $-Ei(-x) + ln\, x$ to 7D for $x - 0(.0001).0099$.

On page 300, and also on a loose card, are values of $\frac{1}{2}t(1 - t)$ to $4D$ for $t = 0(.001)1$. Second-difference interpolation (where needed) is also facilitated by a nomogram on another loose card.

A. F.

1. NBS MATHEMATICAL TABLES, *Tables of Sine, Cosine and Exponential Integrals*, v. 1, MT5; v. 2, MT6, U. S. Government Printing Office, Washington, D. C., 1940.

6[L].—L. N. Karmazina, *Tablitsy Polinomov Iakobi* (*Tables of Jacobi Polynomials*), Akad. Nauk, USSR, Moscow, 1954, 250 p., 26 cm. Price 27.9 rubles.

This book gives the coefficients, roots, and values of the Jacobi Polynomials $G_n(p, q, x)$ for $n = 1(1)5$; $p = 1.1(0.1)3.0$; $q = 0.1(0.1)1.0$; $x = 0(0.01)1.00$. The Legendre polynomials, which are the special case $p = 1$, $q = 1$, are also given separately, with the same arguments in n and x. The values and roots are given to 7D; the coefficients to $7S$. No comparable set of tables is known to the reviewer.

The polynomials G_n tabulated here can be expressed as

$$G_n(p, q, x) = \left\{\binom{2n + p - 1}{n}\right\}^{-1} P_n^{(p-q,q-1)}(2x - 1)$$

in terms of the polynomials P_n of Szegö [1], chapter 4. Note that the polynomials of this table differ by a normalizing factor from the polynomials $G_n(p, q, x)$ of Courant-Hilbert [2], p. 76. The polynomials of the table under review may be characterized by the properties that they are orthogonal on [0, 1] with weight function $x^{q-1}(1 - x)^{p-q}$, and are normalized so that the coefficient of the term of highest degree is one.

The tables were calculated at the Experimental Computation Laboratory of the Institute for Exact Mechanics and Computational Techniques of the USSR Academy of Sciences. There is a short introduction (in Russian) and four graphs showing the qualitative behavior of various families of the polynomials. The book is well printed, on good paper, and the numbers are easy to read. No differences are given, but the introduction has a section describing the accuracy obtainable from linear, quadratic, and cubic interpolation in p, q, and x. Nine recurrence formulas and two asymptotic formulas are also given to aid in extending the tables in n, p, and q.

According to the introduction, the tables were calculated for each p, q, and n by successively applying a Gaussian difference formula in x, starting with $x = 0$. The value obtained at $x = 1$ was then checked against an easily derived independent formula for $G_n(p, q, 1)$. The resulting tables were then checked by differencing. No mention was made of the type of machine used in the calculations.

In glancing through the table, one idiosyncrasy is immediately apparent. In many cases, the values of G_n are simple terminating fractions; in these cases, this table often leaves us with a string of four or five nines. For example $G_1(3.0, 0.1, 0.02)$ is given as $-0.005\ 0000$, while $G_1(3.0, 0.1, 0.03)$ is given as $+0.004\ 9999$. The positive values are given consistently in this way, and the internal evidence leads the reviewer to conjecture that the computation was performed to 8 or 9D on some type of tabulator with "nines complement" arithmetic in the counters, and the values were simply truncated to 7D, with no attempt to round anything off.

To check the accuracy of the tables, the reviewer coded a program on the Univac Scientific model 1103 digital computer at the University of Minnesota Scientific Computing Laboratory, which calculated the coefficients and values of G_n to 10D, for values of n, p, q, and x chosen "randomly" throughout the table by using a machine-generated sequence of pseudo-random numbers. The roots were not checked. In checking 100 values obtained in this way, no gross error was found, but the rounding conjecture above was substantiated. In 96 cases, the value given in the table was the truncated (unrounded) 7D approximation to the true value. The largest discrepancy found was certainly not serious: for $G_5(2.8, 0.1, 0.75)$ the true value rounds to $-0.001\ 865\ 205$, and the value given in the table is $-0.001\ 865\ 1$. The claim given in the introduction that the errors in the values and roots do not exceed two units in the seventh decimal place and the errors in the coefficients do not exceed one unit is probably correct, except for the ever-present spectre of the typographical error. There is one such error in the reviewer's copy on p. 13, the title page to the table of values; the exponent j of -1 is missing in the formula.

To sum up, this book will be extremely useful to anyone who has a need for Jacobi polynomials; the fact that the book is in Russian need deter no one, since 98 per cent of it is arabic numerals. The only criticism the reviewer has is a matter of esthetics; it is annoying to see .1249999 when $\frac{1}{8}$ is meant, particularly when it is so easy to round off, even with the most primitive of computational facilities.

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1. G. SZEGÖ, *Orthogonal Polynomials*, Amer. Math. Soc., Colloquium Publications, v. 23, 1939.
2. R. COURANT & D. HILBERT, *Methoden der Mathematischen Physik*, v. 1, Springer, Berlin, 1931.

7[W, X].—S. I. GASS, *Linear Programming: Methods and Applications*, McGraw-Hill Book Co., New York, 1958, 223 pages. Price $6.50.

A topic in pure mathematics has suddenly become applied, and so much so that classrooms are filled with non-mathematicians learning about linear programming. This is a rather thorough text book for a class of non-mathematicians—and provides welcome motivation for the mathematician. This book is taken from the author's notes for a one-semester, three-hour course, and, hence, should serve very well as a text. Figures, examples, and exercises are excellently delineated.

The subject matter of linear programming can be divided into three areas: mathematical concepts, computational techniques and algorithms, and applications. The author interlaces all three, always keeping in mind the fact that the reader is not a mathematician but one who desires to take advantage of linear programming in his own discipline. One chapter is devoted to the necessary mathematics of linear inequalities and convex sets. This provides a complete working background, even if lacking rigor (see, for example, the definition of a determinant on page 14).

The major portion of the book is given to theoretical and computational methods of solving the so-called general linear programming problem. As expected, a chapter is devoted to the simplex procedure, its justification and geometric interpretation. This is built upon the preceding chapter, which defines the problem and gives the fundamental theorems regarding existence and characteristics of solutions. Duality problems are next discussed, with emphasis upon the implications of symmetry. The next two chapters cover the revised simplex procedure and degeneracy problems.

The author obviously has had considerable experience solving linear problems on large electronic computers. Welcome comments allow the reader to benefit from this experience. An example is a discussion minimizing the importance of degeneracy techniques to practical problems and setting these procedures in a proper context. One chapter is devoted to preliminary analysis and computational devices which may be used to minimize computational work. Included is a survey of available digital computer codes. (We cannot pass the opportunity to note the error in the subtitle on page 130: "Available Digital-computer Coeds.")

The third part of the book deals with applications and includes one chapter on the transportation problem. One of the strong points of the book is a description by category of many typical applications. The categories include inter-industry problems, diet problems, industrial applications (by industry), economic analysis, military applications, and scheduling problems. An excellent attempt has been made to give the reader a grasp of the wide applicability of linear programming.

The final chapter discusses the connection between linear programming and the theory of games. This is really independent of the rest of the book, but serves well to tie the main topic to other mathematical areas. It seems too bad that a little more has not been said about non-linear problems. Clearly, this is not the subject of the book but would serve to better place linear programming in the greater concept of optimizing problems.

An excellent bibliography is included.

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8[X].—L. Fox. *The use and construction of mathematical tables*. National Physical Lab., Math. Tables, v. 1, Her Majesty's Stationery Office, London, England, 1956, 60 p., 27.5 cm. Price 17s.6d.net.

The Mathematics Division of the National Physical Laboratory was formed in 1945 and part of its mission is the construction of new mathematical tables and the checking and extension of existing tables. In order to make the results of this work available to the public, a Mathematical Tables Series is planned. The present volume is a general introduction to the series, which will usually carry less fundamental tables than those which are in the volumes prepared for the Royal Society Mathematical Tables Committee, and in which the tabular material will be printed by photographic processes. The booklet, therefore, is addressed to the consumer of fairly special functions and will not be found easy reading by comparative novices. In order to avoid repetition, in this first volume, standard notation and standard methods of interpolation and table making are described so that future volumes, e.g.

2. L. Fox, *Tables of Everett Interpolation Coefficients*
3. G. F. Miller, *Tables of Generalized Exponential Integrals*

need only discuss special devices appropriate to the particular function.

This booklet is divided into three chapters, which we shall discuss separately: A. The use of mathematical tables; B. Derivation of formulae and analysis of error; C. The construction of mathematical tables. A list of references is given for A and B, and for C.

A. *The use of mathematical tables*. It is pointed out that volumes in the series will make use of auxiliary functions where these are convenient, and that changes in the dependent or independent variable will also be usual. Examples are given to show what we may expect.

Standard methods for direct univariate interpolation are discussed in some detail, and are compared.

i) Reduced derivatives, i.e., Taylor's Series.
ii) Bessel and Everett formula, with and without throw-backs of various kinds.

iii) Lagrangian methods.
iv) Chebyshev polynomials.

The first three methods are familiar. We describe the fourth briefly. We define the Chebyshev polynomials by

$$T_n(x) = \cos(n \operatorname{arc} \cos x), \qquad n = 0, 1, 2, \cdots$$
$$= 2^{n-1}x^n + \cdots$$

Since the leading coefficient of $T_n(x)$ does not vanish, it is possible to express x^n as a linear combination of T_n, T_{n-2}, T_{n-4}, $\cdots$. Accordingly, if we have an appropriate interpolating polynomial

$$f_p \doteqdot L(p) = a_0 + a_1p + a_2p + \cdots + a_{r-1}p^{r-1}$$

we can rearrange it as

$$f_p \doteqdot L(p) = b_0 + b_1T_1(p) + b_2T_2(p) + \cdots + b_{r-1}\, T_{r-1}(p)$$

In virtue of the minimum deviation property of the $T_n(x)$ in $-1 \leqq x \leqq 1$, it is possible that an adequate approximation will be obtained by fewer terms of the latter series and we may have

$$f_p \doteqdot L_1(p) = b_0 + b_1T_1(p) + b_2T_2(p) + \cdots + b_{m-1}T_{m-1}(p), \quad m < r.$$

We now expand the $T_n(p)$ as polynomials and collect terms to find an "economized polynomial"

$$f_p \doteqdot L_1(p) = c_0 + c_1p_1 + \cdots + c_{m-1}p^{m-1}.$$

Tabulation of the c_0, c_1, c_2, $\cdots$, c_{m-1} will enable interpolation to be done more conveniently than the tabulation of the a_0, a_1, a_2, $\cdots$, a_{r-1}. Various devices are used to improve the convenience of this method. For instance, c_0 is near f_0 and it is possible to use an expression

$$f_p \doteqdot L_2(p) = f_0 + d_1p + d_2p^2 + \cdots + d_{m-1}p^{m-1}.$$

There is a short section in inverse interpolation. The two-machine balancing method [1] is recommended when one is working in a table with differences. In case one is working in a table in which the coefficients of the (economized) interpolating polynomial are given, the problem is immediately that of the solution of an algebraic equation: the use of Newton's method is suggested.

There follows a discussion of methods of obtaining the first and second derivatives, and the integral of a tabulated function, which are appropriate when particular means of interpolation have been provided.

The final section of the first chapter deals with bivariate interpolation. Generally, the use of the Everett formulae would appear to be most satisfactory [2].

Throughout this chapter, there are compact worked examples, and statements of maximum errors, under various limitations.

B. *Derivation of formulae and analysis of error.* This chapter carries the justification of various material in the first chapter.

For instance, there is a detailed derivation of the coefficients a_i, b_i for the simultaneous throw back of the sixth and higher differences into the second and fourth:

$$\delta_m^2 f = \delta^2 f + a_1\delta^6 f + a_2\delta^8 f + \cdots$$
$$\delta_m^4 f = \delta^4 f + b_1\delta^6 f + b_2\delta^8 f + \cdots$$

for use in an Everett formula

$$f_p = (1 - p)f_0 + pf_1 + E_2\delta_m^2 f_0 + F_2\delta_m^2 f_1 + E_4\delta_m^4 f_0 + F_4\delta_m^4 f_1 .$$

Then there is a thorough discussion of the error incurred in the use of the above formula, or of the related ones:

$$f_p = L(1 - p)f_0 + pf_1 + E_2\delta_n^2 f_0 + F_2\delta_n^2 f_1 + M_4\gamma^4 f_0 + N_4\gamma^4 f_1$$

where

$\delta_n^2 f = \delta_m^2 f - 0.184\delta_m^4 f, \quad M_4 = 1000(E_4 + 0.184E_2), \quad N_4 = 1000(F_4 + 0.184F_2),$
$\gamma^4 f = \delta_m^4 f/1000;$
$f_p = (1 - p)f_0 + pf_1 + E_2\delta_n^2 f_0 + F_2\delta_n^2 f_1 + P_4\delta_n^4 f_0 + Q_4\delta_n^4 f_1$

where $P_4 = M_4/10$, $Q_4 = N_4/10$, $\delta_n^4 f = \delta_m^4 f/100$. It is concluded that the last arrangement is the most satisfactory and in order that this method should be convenient, the second volume in the series will contain a table of E_2, F_2 to 9D with P_4, Q_4 to 5D, at an interval of .0001 in p.

There follows an elaborate examination of the process of economization, including an explanation of its relation with the 'throw-back'. It follows from the Chebyshev theory, if $\pi(p)$ is the polynomial of degree n which satisfies $\pi(0) = f(0)$, $\pi(1) = f(1)$ and is the best (Chebyshev) approximation to $f(p)$ in $0 \leqq p \leqq 1$, then $e(p) = \pi(p) - f(p)$ has in [0, 1], n interior extrema, of the same absolute value, but alternating sign. This property of $e(p)$ is also possessed by

$$T_{n+1}\{(2p - 1) \cos (\pi/\overline{2n + 2})\}.$$

It is reasonable, therefore, to consider a representation of an approximating polynomial in the form:

$$y(p) = a_0 + 2\sum_{i=1}^{n} a_i T_i\{(2p - 1) \cos (\pi/2i)\}.$$

It can be shown that the Bessel coefficients can be expressed as linear combinations of the $T_i\{(2p - 1) \cos (\pi/2i)\}$, and so the Bessel expansion can be rearranged in the above form. We thus obtain a Bessel-Chebyshev interpolation formula:

$$f_p = (1 - p)f_0 + pf_1 + 2\phi_2\mu\rho^2 f_{1/2} + \phi_3\rho^3 f_{1/2} + 2\phi_4\mu\rho^4 f_{1/2} + \cdots$$

where the ϕ_n are, apart from a normalizing factor, $T_n\{(2p - 1) \cos (\pi/2n)\}$ and where the coefficients $\mu\rho^{2n} f_{1/2}$, $\rho^{2n+1} f_{1/2}$ can be represented as (infinite) series in the central differences of f. These formulae are "among the most efficient of all formulae". They have, however, some drawbacks, e.g., they involve differences of all orders. It is, however, possible to obtain from them an Everett-Bessel-Chebyshev formula in which only the even differences are involved. It is also possible to obtain directly, following Kopal, an Everett-Chebyshev formula:

$$f_p = \{q + \psi_3(q)\sigma^2 + \psi_5(q)\sigma^4 + \cdots\}f_0 + \{p + \psi_3(p)\sigma^2 + \psi_5(p)\sigma^4 + \cdots\}f_1 ,$$

where $q = 1 - p$, where $\psi_{2n+1}(x)$ is a constant multiple of $T_{2n+1}\{p \cos(\pi/\overline{4n + 2})\}$ and where each σ^{2n} is a series involving even central difference operators.

The relative merits of the three schemes are examined and it is concluded that, in practice, the Everett-Bessel-Chebyshev scheme is preferable.

C. *The construction of mathematical tables.* The author now states, correctly, that almost any technique in numerical analysis can be used in the production of the entries of a table. Accordingly he proceeds to give what is a rather scrappy syllabus for an introductory course in numerical analysis. Most of this chapter seems entirely out of place in the present booklet. It is a pity that an opportunity to give some account of the classical methods of British table makers has been missed.

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1. See e.g. Her Majesty's Nautical Almanac Office, *Interpolation and Allied Tables*, Her Majesty's Stationery Office, London, England, 1956.
2. Compare, T. H. SOUTHARD, "Everett's Formula for Bivariate Interpolation and Throwback of Fourth Differences," MTAC, v. 10, 1956, p. 216–223.

9[X].—I. P. NATANSON, *Konstruktive Funktionentheorie*, translated by K. Bogel, Akademie Verlag, Berlin, 1955. xiv + 515 pp. Price 36 DM.

The term "constructive theory of functions" is due to S. N. Bernstein. The subject derives from the work of Chebyshev and his pupils, for instance, Korkine, Zolotareff and the brothers Markoff, and was set up as an independent discipline by Bernstein, to whom and to whose pupils much of its later development is due [1, 2]. Basically the theory is concerned with the approximate representation of functions in terms of simpler ones: it is therefore likely to be of considerable practical value but it is also an intrinsically interesting area of mathematics.

To indicate the content of the subject, we shall mention some of the results which are easy to present. The book is divided into three parts: I. Uniform approximation; II. Approximation in the mean; III. Interpolation and approximate quadrature.

I. The first part is concerned with the approximation of functions $f(x)$, continuous in a finite interval $[a,b]$ by polynomials, and that of continuous periodic functions $F(\theta)$, by trigonometrical polynomials. The fundamental results are Weierstrass' Approximation Theorems. Of the many proofs for the first of these, it is natural to choose that of Bernstein. What is proved is the following:

If $B_n(f, k) = \sum \binom{n}{k} x^k (1 - x)^{n-k} f(k/n)$, *then* $B_n(f, x) \to f(x)$, *uniformly in* $[0,1]$.

Of the many proofs for the corresponding result for $F(\theta)$ that of de la Vallée Poussin is chosen. It is:

If $V_n(F, \theta) = \dfrac{(2n)!!}{2\pi(2n - 1)!!} \displaystyle\int_{-\pi}^{+\pi} F(\phi) \cos^{2n} \tfrac{1}{2}(\phi - \theta)\, d\phi$, *then* $V_n(F,\theta) \to F(\theta)$, *uniformly for all* θ.

The actual equivalence of these two theorems is demonstrated.

The possibility of approximation established, the question of best approximation arises. This problem was solved in principle by Chebyshev, who gave a qualitative characterization of the polynomial of best approximation to $f(x)$. A clear account of the Chebyshev theory is presented. The properties of the Chebyshev polynomials $T_n(x) = \cos(n \arccos x)$ are discussed in detail. It is noted, for example, that $T_n(x)$, which is characterized, among polynomials of degree n and leading coefficient unity, as that which has minimum deviation from zero in $[-1, 1]$, has also the largest deviation from zero in $|x| > 1$. Specifically we have:

If $\pi_n(x)$ is a polynomial of degree not exceeding n, and if $M = \max_{-1 \leqq x \leqq 1} |\pi_n(x)|$, then for any real ξ, $|\xi| > 1$, we have $|\pi_n(\xi)| \leqq M|T_n(\xi)|$.

The next question to be discussed is how various local properties of $f(x)$, e.g. the behavior of its modulus of continuity, or its differential coefficients, affect the degree of approximation which is possible. In the opposite direction, there is a study of how the degree of approximation which is possible restricts the function. Let

$$E_n(f) = \min \max |f(x) - p_n(x)|$$

where the maximum is over all x, $a \leqq x \leqq b$, and the minimum over all polynomials $p_n(x)$ of degree n, and define $E_n(F)$ similarly. Then, in virtue of the Weierstrass Theorems, $E_n(f) \to 0$, $E_n(F) \to 0$. It might be expected that the rate of convergence of $\{E_n\}$ increases as the behavior of the function improves. This is indeed the case. For instance, Dunham Jackson showed that

$$E_n(F) \leqq 12\omega(n^{-1})$$

where $\omega(h)$ is the modulus of continuity of F, that is,

$$\omega(h) = \max_{|x-\xi| \leqq h} |F(x) - F(\xi)|,$$

while Bernstein showed that a necessary and sufficient condition for $F(x)$ to satisfy a Lipschitz condition of order α, $0 < \alpha < 1$, namely,

$$|F(x) - F(\xi)| \leqq M|x - \xi|^\alpha, \qquad M \text{ constant},$$

is that

$$|E_n(F)| \leqq An^{-\alpha}, \qquad A \text{ constant}.$$

In this area there belongs the theorems of the brothers Markoff. We quote that of A. A. Markoff.

If $\pi_n(x)$ is a polynomial of degree at most n such that

$$|\pi_n(x)| \leqq 1, \qquad -1 \leqq x \leqq 1,$$

then we have

$$|\pi_n'(x)| \leqq n^2, \qquad -1 \leqq x \leqq 1.$$

That this result is a best possible one is shown by the case of $T_n(x)$: it is easy to verify that

$$|T_n'(\pm 1)| = n^2.$$

The latter chapters of the first part are concerned with the best approximation to functions which are analytic on a segment of the real axis, and with the behavior of the Fourier-series and its Fejér and de la Vallée Poussin sums as approximations to periodic $F(\theta)$ and that of the Bernstein and Chebyshev polynomials to a general $f(x)$. Among the less familiar results is one, of a negative character, due to Woronowskaja.

If $f(x)$ is a bounded function which has a finite second derivative $f''(\xi)$ at a point ξ, then

$$B_n(f, \xi) - f(\xi) = \frac{f''(\xi)\cdot\xi(1-\xi)}{2n} + 0\left(\frac{1}{n}\right)$$

II. The second part contains material which is likely to be more familiar, essentially the L^2-theorem of orthogonal systems, in particular polynomials. There is an account of the moment problem both in the finite and infinite case. Among the less familiar results which are discussed is the following problem of Zolotareff and Korkine. Consider, for all polynomials $\bar{\pi}_n(x)$ of degree n and leading coefficient unity, the integral

$$\int_{-1}^{+1} |\bar{\pi}_n(x)|\, dx.$$

For what polynomial $\bar{\pi}_n(x)$ is this minimal? The extremal function can be shown, by comparatively elementary means, to be the Chebyshev polynomial of the second kind:

$$\tilde{U}_n(x) = 2^{-n}\frac{\sin(\overline{n+1}\text{ arc cos } x)}{\sin(\text{arc cos } x)}$$

III. The third part begins with a discussion of Lagrangian interpolation, in the case of distinct nodes and in the case of multiple nodes (Hermite). The question of the convergence of a sequence of Lagrangian interpolations, given sequences of nodes: $x_1^{(1)}$; $x_1^{(2)}$; $x_2^{(2)}$; $\cdots$ is then considered. Suppose $L_n(f, x)$ interpolates f at $x_1^{(n)}, x_2^{(n)}, \cdots, x_n^{(n)}$; what can be said about the convergence of $\{L_n(f, x)\}$ for a fixed x? Clearly if f is a polynomial (or an entire function), the sequence is uniformly convergent, no matter how the nodes are distributed. However, given any set of nodes $\{x_j^{(i)}\}$, it is possible to construct a continuous function such that $\{L_n(f, x)\}$ is not uniformly convergent to $f(x)$. The special case, when $f(x) = |x|$ and when the $\{x_i^{(n)}\}$, are equally spaced in $[-1, 1]$ for each n, is discussed: here there is convergence at the points 0, ± 1, only.

The remainder of the book is concerned with approximate quadrature and includes discussion of the quadratures connected with orthogonal polynomials and the convergence of quadrature schemes. Suppose given two triangular matrices: one of abscissae $x_1^{(1)}$; $x_1^{(2)}, x_2^{(2)}$; $\cdots$ (for which $a \leqq x_i^{(j)} \leqq b$ always holds) and one of weights $\lambda_1^{(1)}$; $\lambda_1^{(2)}, \lambda_2^{(2)}$; $\cdots$. Then we consider the sequence of quadratures

$$Q_n(f) = \sum_{i=1}^{n} \lambda_i^{(n)} f(x_i^{(n)})$$

and can ask whether

$$\text{(x)} \qquad \lim_{n=\infty} Q_n(f) = \int_a^b f(x)\, dx?$$

It is shown, for instance, that a necessary and sufficient condition that (x) should be true of every continuous function, is that it should be true for polynomials and that $\sum_{i=1}^{n} | \lambda_i^{(n)} | \leqq K, n = 1, 2, \cdots$.

The book appears to be written clearly and the printing makes reading pleasant. In the early part of the book the demands on the reader are light, only the facts of classical real variable theory being assumed. The author has decided to forego complex variable methods, at the expense of brevity in some proofs and the omission of some material, and defends this point of view in the preface. Exposition seems excellent—although a rather unmotivated account of the Bernstein proof of the Weierstrass theorem is a rather tough beginning.

There is a great deal to be said in favor of including material of the kind covered in this book in regular courses in advanced calculus or real variable theory, and in courses on the theoretical aspects of numerical analysis, where the sharpness and elegance of many of the results will compensate for some of the unavoidable vaguenesses and crudities of practical numerical analysis. This point of view has been elaborated by the reviewer elsewhere [3].

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1. S. N. BERNSTEIN, "Constructive theory of functions as a development of Chebyshev's ideas," in Russian. *Izv. Akad. Nauk.* 9, 1945, 145–158.
2. N. I. AHIEZER, Academian S. N. Bernstein and his work on the constructive theory of functions, in Russian. *Isdat. Hav'kov.* Gos. Univ., Kharkov, 1955.
3. JOHN TODD, Special polynomials in numerical analysis, *Proceedings, Symposium on Numerical Approximations*, University of Wisconsin, 1958.

10[X, Z].—MATHEMATISCHES LABOR DER TECHNISCHEN HOCHSCHULE WIEN, *MTW Mitteilungen*, vol. 3, 1956, 312 pp., vol. 4, 1957, 420 pp., Price DM 10. per volume.

The subtitle of this bi-monthly periodical reads, translated, "A journal dedicated to relations among mathematics, technology and economy." Its pages are divided between contributed articles, progress reports from the Mathematical Laboratory of the Technische Hochschule of Vienna, book reviews, contents of journals, literature surveys, information items, communications of the Gesellschaft für Angewandte Mathematik und Mechanik (GAMM), "communications" of commercial companies—IBM, Bull, Remington-Rand—and advertising.

The contributed articles account for about half of the total number of pages. Only a few of them are new research contributions; some are reports about past meetings or about newly established organizations, and there are also many worthwhile expository articles. Subjects covered include numerical mathematics, scientific, engineering and management applications of digital computers, automation and cybernetics, analog machines and their applications. Few of the articles are specifically slanted toward the most modern large computing machines. Among the new contributions of general mathematical interest we mention (all titles translated from the German): *A. Hirschleber* and *G. Exner*, Approximate evaluation of Stieltjes integrals by means of selected ordinates; *Hj. Kolder* and *N. Untersteiner*, Physical processes during an explosive decompression; *H. Scholz*, Contributions to the Krylov-Bogolyubov approximation method for nonlinear ordinary differential equations; *W. Spindelberger*, Determination of real roots of algebraic equations on small electronic computers.

The progress reports of the Mathematical Laboratory consist mainly in lists of titles of completed tasks, without details about objective or methods. The Laboratory operates an IBM 604 and an analog installation.

Under the heading "Periodicals service" the tables of contents of about 20 technical journals are reprinted. The "Literature reports" give references to new publications arranged by subject matter. Examples of subject matter fields covered are: algebraic equations; structure of matter; automation; eigenvalue problems; solid state physics; financial arithmetic; cybernetics; mathematical methods in biology; and about twenty others, similarly diversified.

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11[Z].—R. L. COSGRIFF, *Nonlinear Control Systems*, McGraw-Hill Book Company, 1958, viii + 327 p. Illus. Price $9.00.

The field of nonlinear control systems has received little treatment in the published textbook literature, a rather unusual situation particularly in view of the fact that almost two dozen texts have been written on linear control theory and that control systems are in general nonlinear. The fact that control systems can often be treated satisfactorily by linear methods does not, of course, make a textbook on non-linear control theory less overdue.

The book consists of eleven chapters arranged as follows. The first three chapters are devoted to introductory and linear theory aspects. Chapters four and five are devoted to small-signal and piecewise-linearization techniques for the analysis of nonlinear systems. The following three chapters are devoted to phase-plane and describing-function techniques. It is interesting to note, however, that the author steadfastly uses the term "nonlinear gain" rather than the more universally used term "describing function". The discussion on nonlinear gain is marked by a fair amount of attention to asymmetrical systems in which a d.c. component must be considered as well as harmonics of the excitation frequency.

The following chapter is devoted to linear equations with periodic coefficients. This chapter is introduced because the frequency response of a nonlinear system can often be determined by reducing the nonlinear differential equation to a linear equation with periodic coefficients. This chapter ends with a discussion of sampled data systems treated as linear systems with time-varying parameters.

The last two chapters are devoted to the analysis of nonlinear control systems with random processes as inputs, and to the use of logic circuits in nonlinear control system actuation.

This book represents an introductory rather than a comprehensive treatment. For instance, in the discussion of phase plane-techniques, the Lienard construction method, acceleration-plane method, phase-space methods, etc., are not treated. Systems with several nonlinearities distributed throughout the system are not treated by describing-function methods. The significant works of Feldbaum, Bose, Kuba and Kazda, etc., are not mentioned. Also no discussion is presented of the powerful techniques of time-domain synthesis evolved by the Dynamic Analysis and Control Laboratory. However, as an introductory treatment which may serve as a stepping stone to some of these more powerful techniques, the book can be recommended.

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12[Z].—L. LANDON GOODMAN, *Man and Automation*, Penguin Books, Ltd., London, 1957, 286 pp. illus. Price $0.85.

An appropriate subtitle might be, "a strictly British view of automation." While Mr. Goodman's book is an interesting review of some of the problems facing British industry today, it is not likely to advance the technical or general knowledge of the American reader who possesses even a nodding acquaintance with modern industrial technology.

Mr. Goodman, a highly respected British industrial engineer, begins with the accepted definition of "automation" that ties the word to a closed-loop feedback control system not involving routine human intervention. He briefly describes the development of automatic and automated processes through history, identifies areas in which automation could fruitfully be employed, and then proceeds to extended discussion of these major premises:

1. Automation offers the only way in which the British standard of productivity can be raised sufficiently to enable Britain to compete in the world market, particularly with the U. S., Germany, and Russia.

2. British trade union policies are largely restricting introduction of automation, particularly owing to: a) insistence on full employment at any cost; b) misunderstanding of the effects and potential benefits of automation; and c) inflexible and outmoded attitudes of crafts unions (for example, in Britain a plumber must walk to a household repair job; he may not even ride a bicycle).

3. British management must awaken to new responsibilities in: a) development and research in new products and processes; b) improving industrial relations; and c) developing management skills through in-service and university training.

4. The influx of new capital to British industry must be stepped up considerably to provide the tools for automation. He reveals that industrial Britain, once a world leader, now has the lowest capital investment per worker, the lowest energy available per worker, and the lowest productivity per worker of any of the major industrial nations.

As a revelation of the problems Great Britain faces today, the book is most interesting. With respect to the fetish of Labour for full employment, Mr. Goodman quotes a major Labour leader as publicly saying: "Who wants to work? We don't care if you work or not, so long as you are getting full pay either way." It is hard to conceive of an attitude that could be more destructive of national morale.

Management in Britain is not let off lightly. Mr. Goodman is highly critical of the prevalence of reliance on tradition (called "common sense") rather than reason in making management decisions, and also of management's failure to recognize the value of education and directed development in preparing for management positions.

Mr. Goodman concludes: "Only the companies which enthusiastically and sincerely adopt the newest ideas will flourish. The others will fall by the wayside." For Labour's benefit he adds these words: "The real threat to full employment is failure to adopt the benefits of scientific and technological advances."

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