

this calculation are shown in the following table. From these results it is seen that the conjecture holds for all primes less than 792,722, which amounts to the first 63,419 primes.

TABLE OF THE FUNCTION $P(i)$ FOR $0 < i < 95$

i	$P(i)$	$P(i) + i$	i	$P(i)$	$P(i) + i$	i	$P(i)$	$P(i) + i$
1	3	4	33	867	900	65	23266	23331
2	8	10	34	866	900	66	23265	23331
3	14	17	35	2180	2215	67	23264	23331
4	14	18	36	2179	2215	68	23263	23331
5	25	30	37	2178	2215	69	31500	31569
6	24	30	38	2177	2215	70	31499	31569
7	23	30	39	2771	2810	71	31498	31569
8	22	30	40	2770	2810	72	31497	31569
9	25	34	41	2769	2810	73	31528	31601
10	59	69	42	2768	2810	74	31527	31601
11	98	109	43	2767	2810	75	31526	31601
12	97	109	44	2766	2810	76	31526	31602
13	98	111	45	2765	2810	77	31528	31605
14	97	111	46	2764	2810	78	31527	31605
15	174	189	47	2763	2810	79	31526	31605
16	176	192	48	2763	2811	80	31526	31606
17	176	193	49	2763	2812	81	31536	31617
18	176	194	50	2763	2813	82	31535	31617
19	176	195	51	3366	3417	83	31534	31617
20	291	311	52	4208	4260	84	31533	31617
21	290	311	53	4207	4260	85	31532	31617
22	289	311	54	4206	4260	86	31531	31617
23	740	763	55	4205	4260	87	31538	31625
24	874	898	56	4204	4260	88	31537	31625
25	873	898	57	5943	6000	89	31536	31625
26	872	898	58	5944	6002	90	31535	31625
27	873	900	59	5943	6002	91	31534	31625
28	872	900	60	5942	6002	92	31535	31627
29	871	900	61	5941	6002	93	31534	31627
30	870	900	62	5940	6002	94	31533	31627
31	869	900	63	5940	6003	95	≥ 63324	≥ 63419
32	868	900	64	5940	6004	96	—	—

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Calculating the Coefficients of Certain Linear Predictors

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It is assumed that observations, x_j , have been made at the $n + 1$ points, $j = 0, 1, \dots, n$, which are equally spaced. It is desired to find a linear predictor

$$(1) \quad y_{n+1} = a_0x_0 + \dots + a_nx_n$$

Received 28 September 1958.

such that y_{n+1} is an estimate of the "next" observation, x_{n+1} , which has not yet been made, under the assumptions*

- (a) $\sigma_{x_0} = \cdots = \sigma_{x_n} = \sigma$
- (2) (b) $E(x_j)$ is a k^{th} degree polynomial in j
- (c) $E(y_{n+1}) = E(x_{n+1})$ for an arbitrary polynomial of degree k .

As an ancillary result, Karush and Wolfsohn [1] obtain expressions for the coefficients, a_j , which minimize $\sigma_{y_{n+1}}^2/\sigma^2$. These expressions involve determinants of order k . Contained herein is an expression for these coefficients which is more convenient for computational purposes. Also a table of some of these a_j is deposited in the unpublished mathematical tables repository. (See Reviews and Descriptions of Tables and Books.) By assumption (b) in (2), one can write

$$(3) \quad E(x_j) = \alpha_0 P_0(j) + \alpha_1 P_1(j) + \cdots + \alpha_k P_k(j); \quad j = 0, 1, \dots, n+1$$

where the α_i are constants and where the $P_i(j)$ are polynomials in j of degree i which are orthogonal over the set of points, $(0, 1, \dots, n)$, i.e.,

$$(4) \quad \sum_{j=0}^n P_s(j) P_t(j) = 0 \quad s \neq t.$$

Taking expected values on both sides of equation (1), one can write

$$(5) \quad E(x_{n+1}) = a_0 E(x_0) + \cdots + a_n E(x_n)$$

or, from (3),

$$(6) \quad \begin{aligned} & \alpha_0 P_0(n+1) + \cdots + \alpha_k P_k(n+1) \\ & = a_0[\alpha_0 P_0(0) + \cdots + \alpha_k P_k(0)] + \cdots + a_n[\alpha_0 P_0(n) + \cdots + \alpha_k P_k(n)]. \end{aligned}$$

From (6) one obtains the following set of equations which the coefficients must satisfy:

$$(7) \quad \begin{aligned} a_0 P_0(0) + \cdots + a_n P_0(n) & = P_0(n+1) \\ \cdot & \quad \cdot & \quad \cdot \\ a_0 P_k(0) + \cdots + a_n P_k(n) & = P_k(n+1). \end{aligned}$$

Since

$$(8) \quad \frac{\sigma_{y_{n+1}}^2}{\sigma^2} = a_0^2 + \cdots + a_n^2,$$

it is desired to minimize (8) under the constraints (7). Form the expression

$$(9) \quad \begin{aligned} F & = (a_0^2 + \cdots + a_n^2) + \lambda_0[a_0 P_0(0) + \cdots + a_n P_0(n)] \\ & \quad + \cdots + \lambda_k[a_0 P_k(0) + \cdots + a_n P_k(n)] \end{aligned}$$

where the λ_i are Lagrange multipliers. Setting $\partial F / \partial a_j = 0$, one obtains

$$(10) \quad 2a_j + \lambda_0 P_0(j) + \cdots + \lambda_k P_k(j) = 0 \quad j = 0, 1, \dots, n.$$

* The symbol, σ_{x_i} , refers to the standard deviation of x_i ; σ is a constant; and $E(x_i)$ refers to the expected value of x_i .

If these equations are all multiplied by $P_0(j)$ and then summed, and this process is repeated but with $P_1(j), \dots, P_k(j)$, one obtains the equations (by using (4) and (7)):

$$(11) \quad \begin{aligned} 2P_0(n+1) + \lambda_0 \sum_{j=0}^n P_0^2(j) &= 0 \\ \cdot &\cdot \\ \cdot &\cdot \\ 2P_k(n+1) + \lambda_k \sum_{j=0}^n P_k^2(j) &= 0. \end{aligned}$$

Therefore, by substituting for the λ_i in (10), one obtains

$$(12) \quad a_j = \frac{P_0(j)P_0(n+1)}{\sum_{j=0}^n P_0^2(j)} + \dots + \frac{P_k(j)P_k(n+1)}{\sum_{j=0}^n P_k^2(j)}; \quad j = 0, 1, \dots, n.$$

Values for a_j were obtained by using the following set of polynomials, which are orthogonal over any set of points symmetric about the origin [2]:

$$(13) \quad \begin{aligned} P_0(t) &= 1 \\ P_1(t) &= t \\ \cdot &\cdot \\ P_{i+1}(t) &= tP_i(t) - \beta_i P_{i-1}(t) \end{aligned}$$

where

$$(14) \quad \beta_i = \sum_{j=0}^n P_i^2(t_j) / \sum_{j=0}^n P_{i-1}^2(t_j).$$

By transforming the integer set $0, 1, \dots, n$ to the interval $(-2, 2)$ we derive [3] a simple expression for the β_i :

$$(15) \quad \beta_i = \frac{4i^2[(n+1)^2 - i^2]}{n^2(4i^2 - 1)}, \quad i = 1, 2, \dots, k-1.$$

It may be noted that $\sum_{j=0}^n a_j$ should be equal to one. For degrees one through eight this check sum for the coefficients is verified with a minimum of six significant figures. For degrees nine and ten the check sum is verified to a minimum of five significant figures.

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